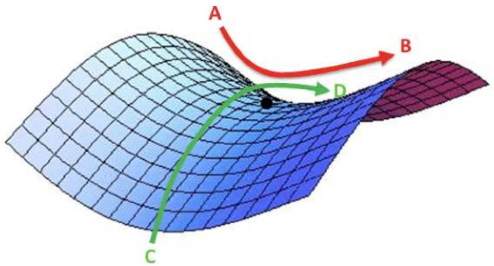
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 1

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Dipartimento di Ingegneria dell'Informazione, Elettronica e Telecomunicazioni

A.y. 2023-2024

Motivation and Overview of the Course

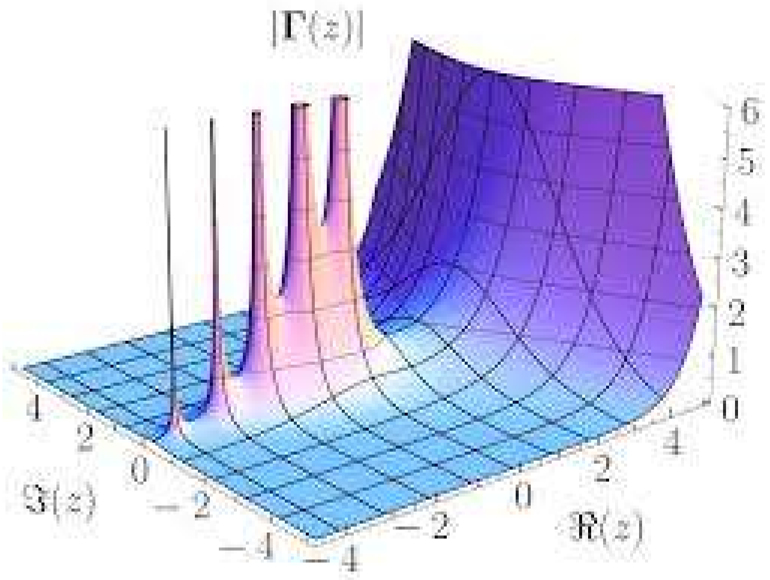
Analytical vs. Numerical

The aim of numerical analysis is to find ***algorithms*** for solving a mathematical problem with the *minimum time* and with the *maximum accuracy*

The aim of analytical models is to gain **physical insight** into the involved wave processes

Leitmotiv

This course will provide information on important analytical tools for the analysis of waves (not necessarily electromagnetic) with a unifying theme: **complex analysis**.



Inspirational Quotes…

…entre deux vérités du domaine réel, le chemin le plus facile et le plus court passe bien souvent par le domaine complexe.

Paul Painlevé

Paul Painlevé, *Analyse des travaux scientifiques* (Gauthier-Villars, 1900; reprinted in Librairie Scientifique et Technique, Albert Blanchard, Paris, 1967, pp. 1-2; reproduced in *Oeuvres de Paul Painlevé*, Éditions du CNRS, Paris, 1972-1975, vol. 1, pp. 72-73.

Cited by Jacques Hadamard in J. Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field* (Princeton U. Press, 1945; Dover, 1954; Princeton U. Press, as *The Mathematician's Mind*, 1996))

Inspirational Quotes…

One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.

Julian Schwinger

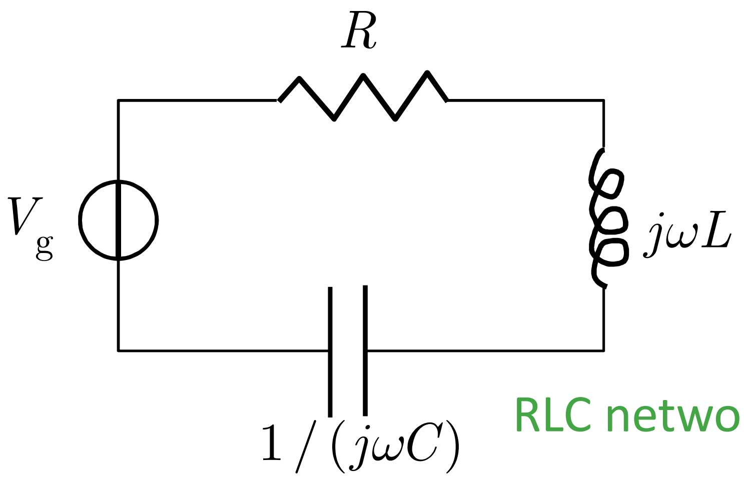
Julian Schwinger, *Particles, sources, and fields*. Vol. 1, Reading, MA: Addison Wesley, 1970.

Representation of Time-Harmonic Quantities

Complex scalars or vectors can be used to conveniently represent time-harmonic quantities:

- Scalar phasors

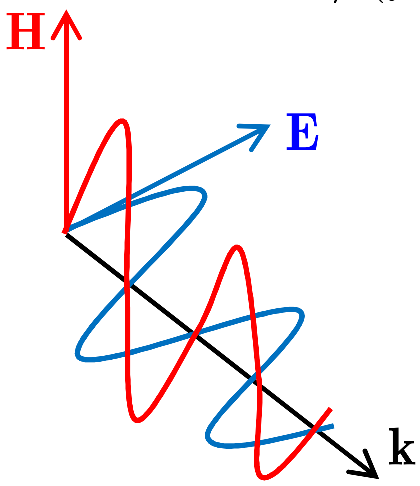
$$\mathcal{A}(t) = \mathrm{Re}\left[A\, e^{j\omega t}\right]$$



RLC network

- Vector phasors

$$\mathcal{A}(t) = \mathrm{Re}\left[\mathbf{A}\, e^{j\omega t}\right]$$



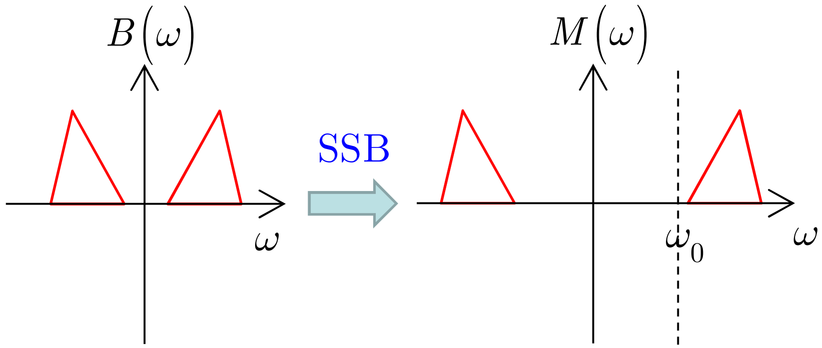
EM wave

Modulated and Transient Signals

Complex functions for the description of modulated or transient signals:

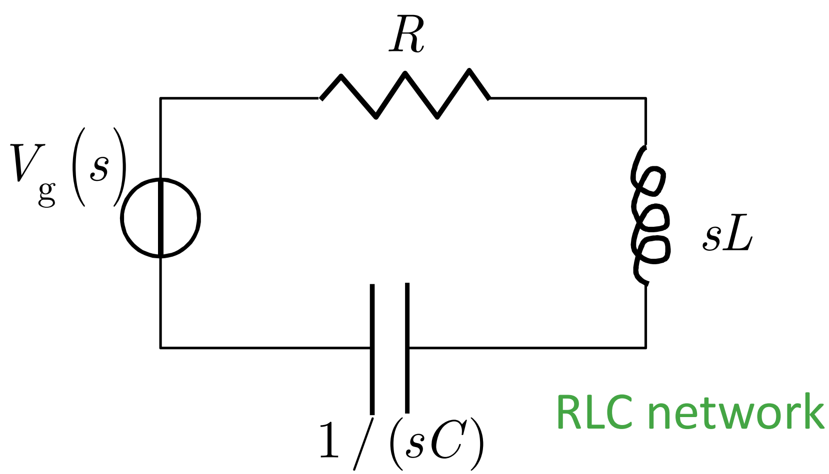
- Analytic signal

$$t \in \mathbb{R} \to \mathcal{A}^+(t) = \frac{1}{2\pi} \int_0^{+\infty} A(\omega) e^{j\omega t} \mathrm{d}\omega \in \mathbb{C}$$



- Laplace domain

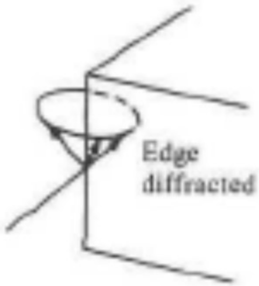
$$s \in \mathbb{C} \to A(s) = \int_0^{+\infty} \mathcal{A}(t) e^{-st} \mathrm{d}t \in \mathbb{C}$$



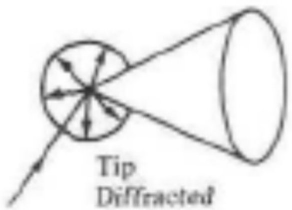
Complex Methods for the Analysis of Wave Objects

Probably less well known:

- Complex analysis allows for **rigorously defining wave objects**, especially (but not only) in high-frequency asymptotic regimes

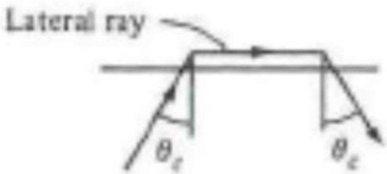


edge

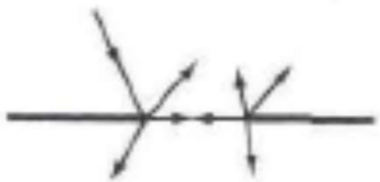


tip or corner

curvature



interface between
two media



multiply diffracted rays
for a slit

- Complex methods allow for gaining **physical insight** and deriving **compact representations** of otherwise complicated wave phenomena

Course Syllabus

1. Fundamentals of complex function theory (September 19 and 21)

1.1 Elementary holomorphic functions, Cauchy-Riemann equations, elementary Riemann surfaces.

1.2 Complex integration, Cauchy theorem and consequences, residue calculus.

2. Asymptotic expansions and ray optics (September 26 and 28)

2.1 Introduction, asymptotic sequences, and elementary examples.

2.2 The Luneburg-Kline asymptotic expansion: Ray optics.

3. Asymptotic evaluation of integrals (October 3 and 5)

3.1 Integration by parts, Watson lemma, Laplace method, stationary-phase method.

3.2 The method of steepest descents (saddle-point method).

4. Applications: Time-harmonic waves in layered media (October 10 and 12)

4.1 Point source above a single interface: space waves, plasmon waves, Zenneck waves.

4.2 Point source above a grounded slab: lateral waves, surface waves, leaky waves.

5. Applications: Plane-wave scattering from half planes (October 17 and 19)

5.1 PEC half plane: elementary solution and Wiener-Hopf approach.

5.2 Resistive half plane: Wiener-Hopf solution and uniform asymptotic evaluation of the field.

6. Applications: Scattering from spheres (October 24 and 26)

6.1 Spherical wave functions; dipole on a PEC sphere, Watson transformation, creeping waves.

6.2 Plane-wave scattering from PEC and dielectric spheres; the rainbow and the glory.

Course Schedule and Teacher’s Contacts

The course will be held **from 19 September to 26 October 2023** in the **seminar room** at the second floor of the DIET department, Via Eudossiana 18, 00184 Rome, Italy, with the following schedule:

Tuesday 10:00-13:00
Thursday 10:00-13:00

Classes will also be held on Google Meet at the link: https://meet.google.com/hmw-agon-ihm

Teacher’s Contacts:

Paolo Burghignoli
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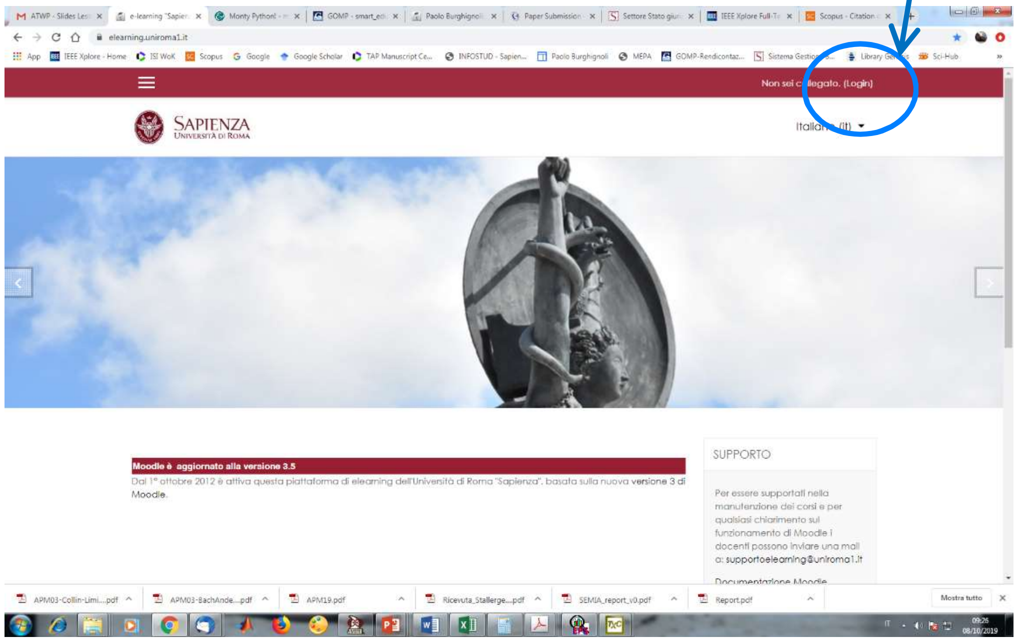
E-mail: paolo.burghignoli@uniroma1.it
Website: https://sites.google.com/a/uniroma1.it/paoloburghignoli-eng

ATWP on E-Learning Sapienza

https://elearning.uniroma1.it/

login



Register to the Moodle (if you have not an account yet)

Find the course and try to sign up…

Fundamentals on Complex Functions:
Elementary Properties of Analytic Functions

Complex Differentiation

We shall be concerned with **complex functions of one complex variable** whose fundamental property is that of being **differentiable**.

We shall see that such a simple assumption, in contrast with the case of real functions of one real variabe, produces **a wealth of extraordinary consequences**.

Definition:

Let Ω be an **open** set of the complex plane. If $z_0 \in \Omega$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, we denote this limit by $f'(z_0)$ and call it the **derivative** of f at z_0.

Jusqu'ici tout va bien... (M. Kassovitz, *La Haine*, 1995)

Complex Differentiation (cont'd)

The power of complex differentiability stems from the fact that the limit occurring in the definition has to be done **in the metric of the plane**: in simple terms, this means that z can approach z_0 following an arbitrary path in the 2D complex plane.

For instance, by letting $f(z) = f(x+jy) = u(x,y) + jv(x,y)$

real increment:

$$\lim_{z\to z_0} \left.\frac{f(z)-f(z_0)}{z-z_0}\right|_{z-z_0\in\mathbb{R}} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

imaginary increment:

$$\lim_{z\to z_0} \left.\frac{f(z)-f(z_0)}{z-z_0}\right|_{z-z_0\in\mathbb{I}} = \frac{\partial f}{j\partial y} = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since these expressions must be equal, one has

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

The Cauchy-Riemann Equations

We have thus proved the easy part of the following:

Theorem (Looman-Menchoff):

Let Ω be an open set and f a continuous function in Ω. Let the partial derivatives

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist everywhere but a countable set in Ω. Then

$f\left(z\right) = f\left(x+jy\right) = u\left(x,y\right) + jv\left(x,y\right)$ is holomorphic in Ω if and only if it satisfies

in Ω the **Cauchy-Riemann equations**:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

The Class of Holomorphic (or Analytic) Functions

Definition:

If $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is **holomorphic** (or **analytic**) in Ω.

The class of all holomorphic functions in Ω will be denoted by $H(\Omega)$.

If $f \in H(\Omega)$ and $g \in H(\Omega)$, then also $f + g \in H(\Omega)$, $fg \in H(\Omega)$, so that $H(\Omega)$ is a ring; the usual differentiation rules apply.

Superpositions of holomorphic functions are holomorphic: If $f \in H(\Omega)$, $f(\Omega) \subset \Omega_1$, $g \in H(\Omega_1)$ and if $h = g \circ f$, then $h \in H(\Omega)$ and the usual chain rule applies:

$$h'(z_0) = g'(f(z_0))f'(z_0)$$

Elementary Analytic Functions

- For $n=0,1,2,\ldots,$ z^n is holomorphic in the whole plane (such functions are called **entire**); hence the same is true for any **polynomial** in z:

$$f(z) = P(z) = a_n z^n + a_{n-1} z^{n-1} \ldots + a_1 z + a_0 \qquad \text{polynomial}$$

- The function $1/z$ is holomorphic in $\{z : z \neq 0\}$; hence, taking $g(w) = 1/w$ in the chain rule, if $f_{1,2} \in H(\Omega)$ and f_2 has no zero in $\Omega_1 \subset \Omega$, then $f_1 / f_2 \in H(\Omega_1)$

$$f(z) = R(z) = \frac{a_n z^n + a_{n-1} z^{n-1} \ldots + a_1 z + a_0}{b_m z^m + a_{m-1} z^{m-1} \ldots + b_1 z + b_0} \qquad \text{rational function}$$

Power Series in the Complex Plane

To achieve more variety, one must take limits…

Power series in the complex plane are a means for obtaining holomorphic functions, as shown next. Let us first recall some basics:

Definition:

To each power series $\sum_{n=0}^{+\infty} c_n (z-a)^n$ there corresponds a unique number $R \in [0, +\infty]$ such that

- The series **converges absolutely and uniformly** in $\bar{D}(a;r)$ for every $r < R$.
- The series **diverges** if $z \notin \bar{D}(a;R)$

The '*radius of convergence*' R is given by the root test: $\frac{1}{R} = \limsup_{n \to +\infty} |c_n|^{1/n}$

Example: the Geometric Series

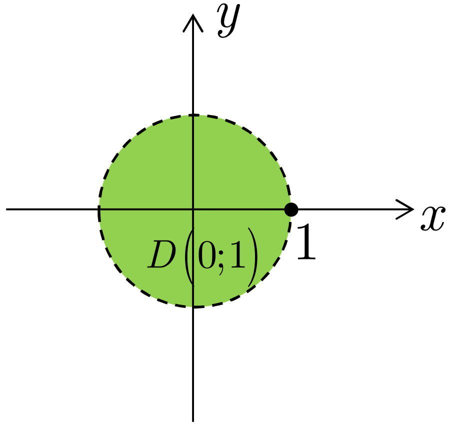
Definition: $\sum_{n=0}^{+\infty} z^n$ (i.e., $a=0$ and $c_n=1$)

The root test gives: $\frac{1}{R} = \limsup_{n\to+\infty} 1^{1/n} = 1$ ⇒ $R = 1$ radius of convergence

hence the series **converges** (absolutely) if $|z| < 1$

and its (well known) sum is $f(z) = \frac{1}{1-z}$, holomorphic in $\Omega = \{z : z \neq 1\}$

The series is **not convergent** if $|z| \geq 1$



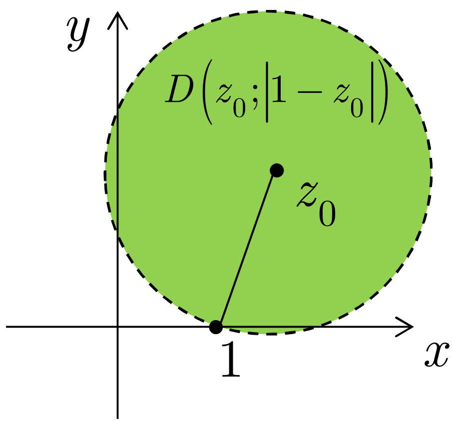
Example: the Geometric Series (cont'd)

Note that the sum of the geometric series can be represented as a sum of a power series also **outside** the disk $|z| < 1$:

$$f(z) = \frac{1}{1-z} = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{1-z_0}\frac{1}{1-\frac{z-z_0}{1-z_0}} = \frac{1}{1-z_0}\sum_{n=0}^{+\infty}\left(\frac{z-z_0}{1-z_0}\right)^n$$

the series being again a geometric series, now convergent in the disk $D(z_0; |1-z_0|)$

i.e., in $|z-z_0| < |1-z_0|$



What limits the radius of convergence is the point $z=1$, where the function $f(z)$ is **not** holomorphic.

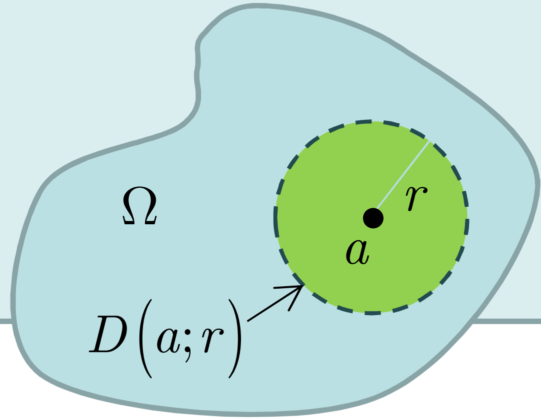
Power Series in the Complex Plane (cont'd)

Definition:

A function f defined in Ω is **representable by power series** in Ω if to every disc $D\left(a;r\right)\subset\Omega$ there corresponds a series

$$\sum_{n=0}^{+\infty} c_n\left(z-a\right)^n$$

which converges to $f\left(z\right)$ for all $z\in D\left(a;r\right)$.



Theorem:

If f is representable by power series in Ω then $f\in H\left(\Omega\right)$ and f' is also representable by power series in Ω :

$$f\left(z\right)=\sum_{n=0}^{+\infty} c_n\left(z-a\right)^n,\ z\in D\left(a;r\right) \Rightarrow f'\left(z\right)=\sum_{n=1}^{+\infty} nc_n\left(z-a\right)^{n-1},\ z\in D\left(a;r\right)$$

Power Series in the Complex Plane (cont'd)

Remark 1:

Since f' satisfies the same hypothesis as f does, the theorem can be applied to f'.

Hence f **has derivative of all orders**, all representable by power series:

$$f^{(k)}(z) = \sum_{n=k}^{+\infty} n(n-1)\dots(n-k+1)c_n(z-a)^{n-k},\ z \in D(a;r)$$

Remark 2:

$$k!\,c_k = f^{(k)}(a) \quad (k = 0, 1, 2, \dots)$$

This means that the coefficients c_k are **uniquely determined** for each $a \in \Omega$ and that *each power series in the complex plane is the Taylor series of its sum*.

The Exponential Function

This is the most important function in mathematics. It is given for every complex number z by

$$\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

The radius of convergence is infinite, hence $\exp$ is an **entire** function.

The absolute convergence allows for writing

$$\sum_{k=0}^{+\infty} \frac{a^k}{k!} \sum_{m=0}^{+\infty} \frac{b^m}{m!} = \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} = \sum_{n=0}^{+\infty} \frac{(a+b)^n}{n!}$$

hence $\exp(a)\exp(b) = \exp(a+b)$ (addition formula)

Note that $\exp(0) = 1$.
Furthermore, letting $e = \exp(1)$, we will write $\exp(z) = e^z$.

The Exponential Function: Exercise

Prove that the following assertions are true:

1) For every complex number z we have $e^z \neq 0$

2) $\exp$ is its own derivative: $\exp'(z) = \exp(z)$

3) The restriction of $\exp$ to the real axis is a monotonically increasing positive function with

$$\lim_{x \to +\infty} e^x = +\infty, \quad \lim_{x \to -\infty} e^x = 0$$

4) There exists a positive number π such that $e^{j\frac{\pi}{2}} = j$ and $e^z = 1$ if and only if $z/(2\pi j)$ is an integer.

5) $\exp$ is a periodic function with period $2\pi j$

6) $t \to e^{jt}$ maps the real axis onto the unit circle

7) If w is a complex number and $w \neq 0$ then $w = e^z$ for some z.

The Logarithm

By definition, $z = \log w$ is a root of the equation $w = e^z$.

First of all, since $e^z \neq 0$, the number 0 has no logarithm.

For $w \neq 0$ the equation $w = e^{x+jy}$ is equivalent to $e^x = |w|, \; e^{jy} = \frac{w}{|w|}$.

- The first equation has the unique solution $x = \log|w|$ (*real logarithm*)

- As mentioned (cf. previous slide, point 6), the second equation has a unique solution $0 \leq y < 2\pi$ and is also satisfied by any y that differs from this solution by an integer multiple of 2π (cf. previous slide, point 5).

⇨ Any nonzero complex number has **infinitely many** logarithms, that differ from each other by multiples of $2\pi j$.

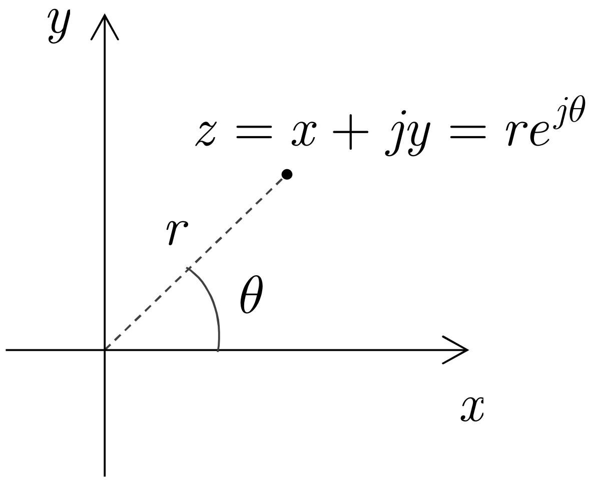
The Argument of a Complex Number

The imaginary part of the logarithm is called the **argument** of w :

$$\log w = \log\left|w\right| + j\arg w$$

$$z = e^{\log z} = e^{\log\left|z\right| + j\arg z} = e^{\log\left|z\right|}e^{j\arg z}$$
$$= \left|z\right| e^{j\arg z} = re^{j\theta}$$

polar representation



The argument of z can be interpreted as the **angle**, in radians, between the positive real axis and the half line from 0 through z.

Cosine and Sine Functions

The $\cos$ and $\sin$ functions are defined by

$$\cos z = \frac{e^{jz} + e^{-jz}}{2} \qquad \sin z = \frac{e^{jz} - e^{-jz}}{2j}$$

Hence $\cos$ and $\sin$ are **entire** functions, which reduce to the ordinary trigonometric functions for real arguments (*cf., by exercise, their power series representation*).
Furthermore:

$$e^{jz} = \cos z + j \sin z \qquad \text{(Euler's identity)}$$

$$\cos^2 z + \sin^2 z = 1$$

$$\cos' z = -\sin z, \;\; \sin' z = \cos z$$

The other trigonometric functions, $\tan$, $\cot$, etc. are defined in the customary way. All of them are *rational functions* of e^{jz}.

Inverse Cosine and Sine Functions

The inverse cosine function $\arccos$ is obtained by solving the equation

$$\cos z = \frac{e^{jz} + e^{-jz}}{2} = w \to e^{jz} = w \pm \sqrt{w^2 - 1}$$

hence

$$z = \arccos w = -j \log\left(w \pm \sqrt{w^2 - 1}\right) = \pm j \log\left(w + \sqrt{w^2 - 1}\right)$$

The inverse sine function $\arcsin$ is most easily defined by

$$\arcsin w = \frac{\pi}{2} - \arccos w$$

In the theory of complex analytic functions all elementary transcendental functions can thus be expressed in terms of e^z and its inverse $\log z$. In other words, **there is essentially only one elementary transcendental function**.

Complex Powers, N-th Roots

The symbol a^b, where a and b are arbitrary complex numbers with $a \neq 0$, means

$$a^b = e^{b \log a}$$

Therefore, a^b has in general **infinitely many values** which differ by the factor

$$e^{2\pi jnb} \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

There will be a **single value** if and only if b is an **integer** n (hence a^b is a power of a or a^{-1}).

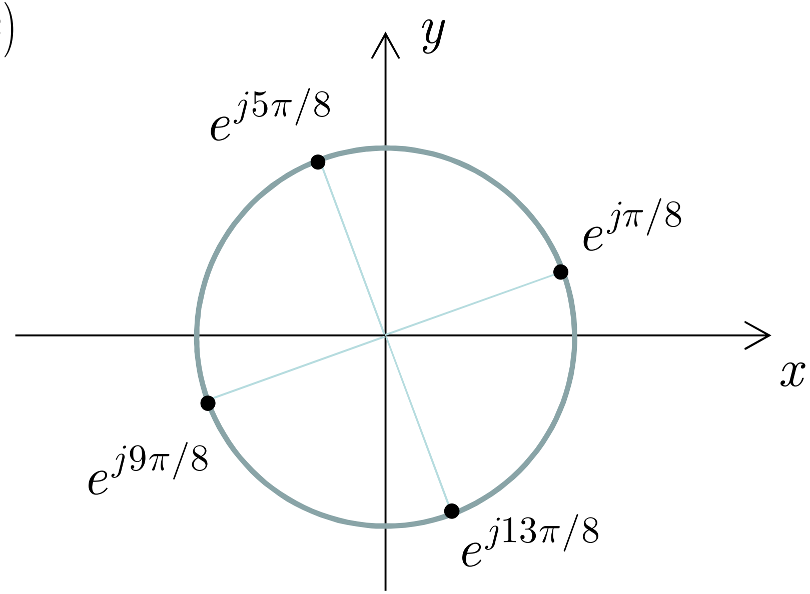
Complex Powers, N-th Roots

There will be a **finite number of values** if and only if b is a **rational number**; if the *reduced form* of b is p/q, then a^b has **exactly q values** and can be represented as

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} \qquad \text{(}q\text{-th square roots)}$$

<u>Example:</u>

$$\sqrt[4]{j} = \left(e^{j\pi/2}\right)^{1/4} = e^{j\left(\pi/8 + n\pi/2\right)}$$



Analytic Functions as Mappings

Consequences of the Cauchy-Riemann Equations

1) Four equivalent expressions for the derivative of f:

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}$$
$$= \frac{\partial v}{\partial y} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}$$

2) Squared absolute value of the derivative:

$$\left|f'(z)\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$
$$= \frac{\partial(u,v)}{\partial(x,y)} = J(u,v)$$

Jacobian determinant

Consequences of the Cauchy-Riemann Equations

In the next lesson we will see that the derivative of an analytic function is itself analytic.

By this fact u and v will have continuous partial derivatives of all orders and hence their mixed derivatives will be equal.

We then have

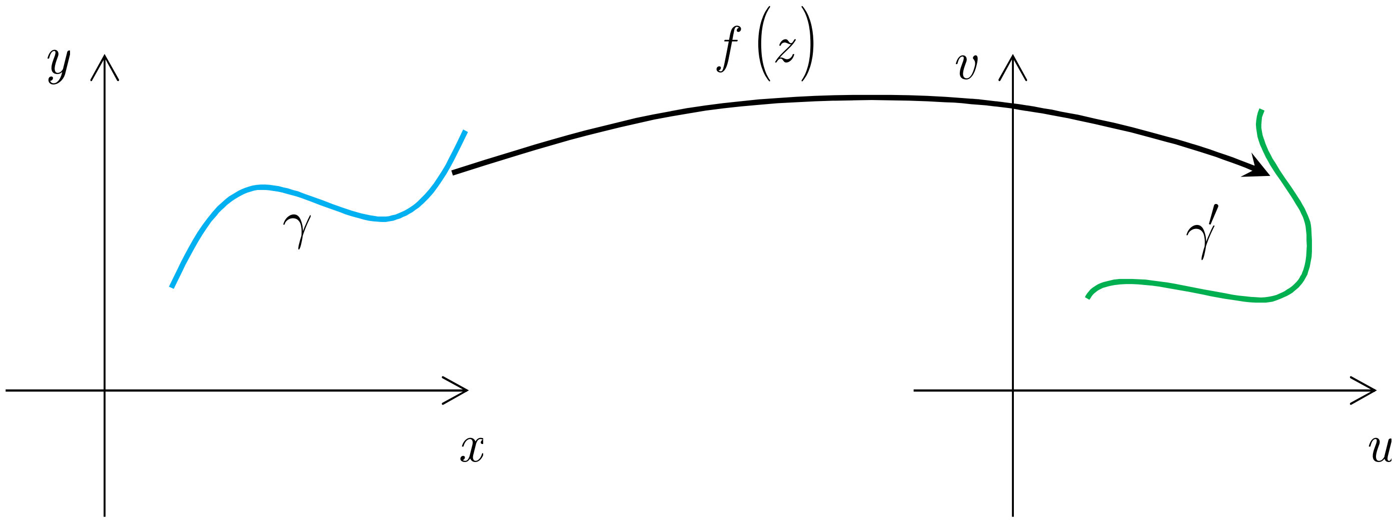
3) $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

i.e., u and v are **harmonic functions**.

Since they are the real and imaginary parts of an analytic functions, each of them is said to be the **harmonic conjugate** of the other.

Analytic Functions as Mappings: Lengths

Let $\gamma : z = z(t)$ be a smooth curve and $\gamma' : w = f(z(t))$ its image under the analytical map f :



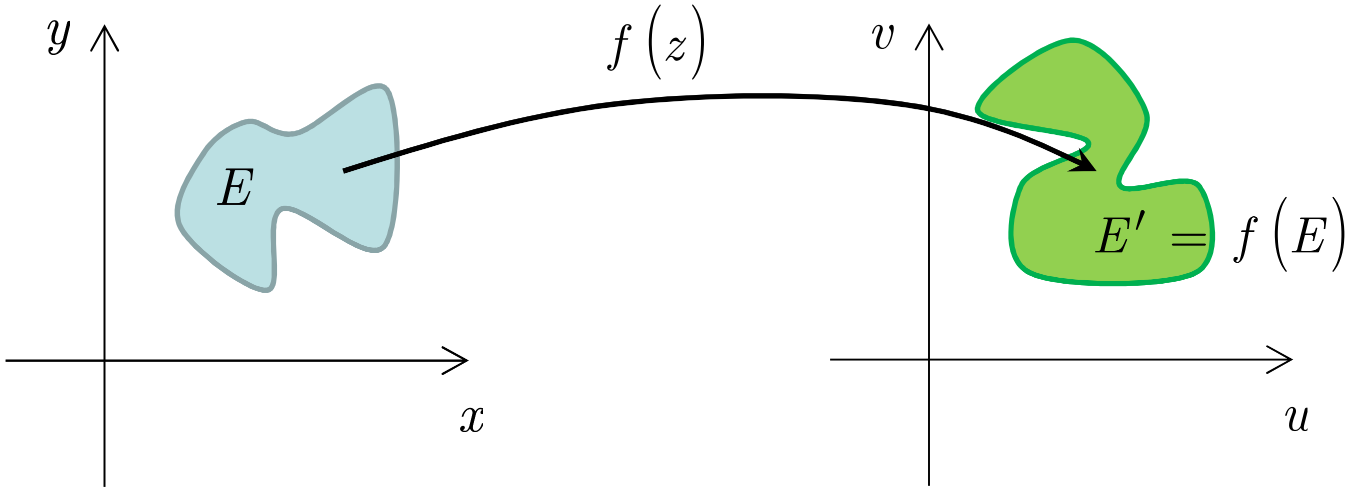
$$L(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t = \int_a^b |z'(t)|\,\mathrm{d}t$$

$$L(\gamma') = \int_a^b |w'(t)|\,\mathrm{d}t = \int_a^b |f'(z(t))||z'(t)|\,\mathrm{d}t$$

Hence $|f'(z)|$ provides the scaling factor for elementary lengths under the mapping $f(z)$.

Analytic Functions as Mappings: Areas

Let $f\left(z\right) = f\left(x + jy\right) = u\left(x, y\right) + jv\left(x, y\right)$ be a bijective analytic map:



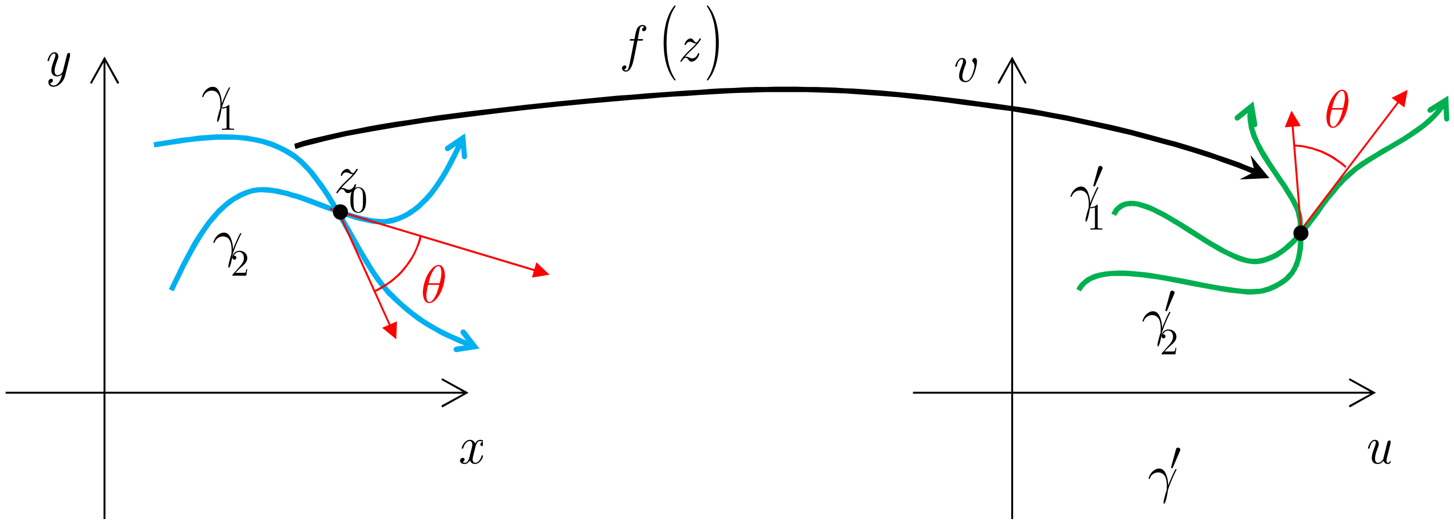
$$A\left(E\right) = \int_E \mathrm{d}x\,\mathrm{d}y$$

$$A\left(E'\right) = \int_E \mathrm{J}\left(u, v\right)\mathrm{d}x\,\mathrm{d}y = \int_E \left|f'\left(z\right)\right|^2 \mathrm{d}x\,\mathrm{d}y$$

Hence $\left|f'\left(z\right)\right|^2$ provides the scaling factor for elementary areas under the mapping $f(z)$.

Analytic Functions as Mappings: Angles

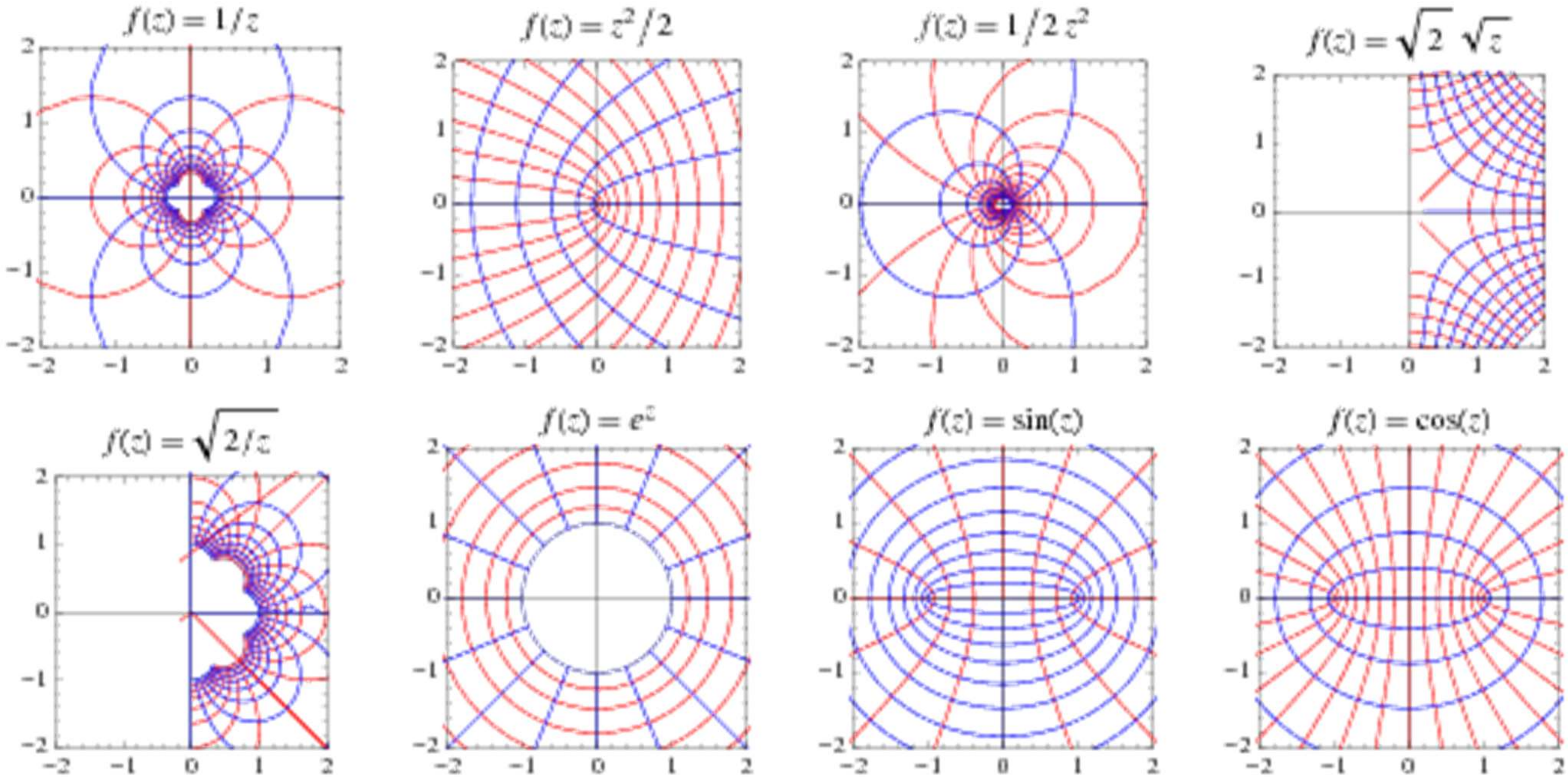
Let $\gamma_{1,2} : z = z_{1,2}(t)$ be two arbitrary smooth curves through $z_0 = z_{1,2}(t_0)$ and $\gamma'_{1,2} : w = f\left(z_{1,2}(t)\right)$ their images under the analytical map f such that $f'(z_0) \neq 0$



$$w'(t) = f'\left(z(t)\right)z'(t) \quad \Rightarrow \quad \arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0)$$

Two curves which form an angle at z_0 are mapped upon curves forming **the same angle**: the mapping $w=f(z)$ is said to be **conformal** at all points where $f'(z) \neq 0$

Examples of Conformal Maps



Invertible Maps: the Condition $f'(z) \neq 0$

If f is analytic on Ω and the mapping $f : \Omega \to \Omega' = f(\Omega)$ is one-to-one (i.e., bijective) with continuous inverse, then the inverse map is also analytic.

In fact, it can be shown that under the stated assumptons it results $f'(z) \neq 0$ everywhere in Ω, hence the derivative of the inverse function is $1 / f'(z)$.

Conversely, assuming $f'(z_0) \neq 0$ allows for concluding that the mapping is bijective with continuous inverse *only in some neighborhood* of z_0 (i.e., **locally**).

In fact, since

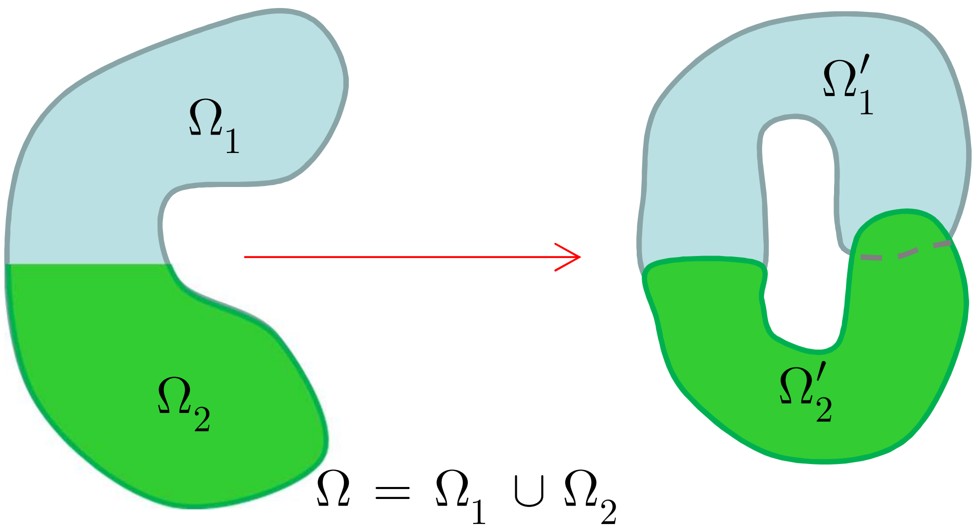
$$\left| f'\left(z_0 = u_0 + j v_0\right) \right|^2 = J\left(u_0, v_0\right) \quad ,$$

the conclusion follows from the standard Implicit Function Theorem.

Global Invertibility: What Can Go Wrong

But even if $f'(z) \neq 0$ in all Ω, we **cannot** assert that the mapping is bijective with continuous inverse *in the whole region* (i.e., **globally**).

In fact, what can happen is depicted in the figure:



The mappings of the subregions Ω_1, Ω_2 are one-to-one, but the images **overlap**.

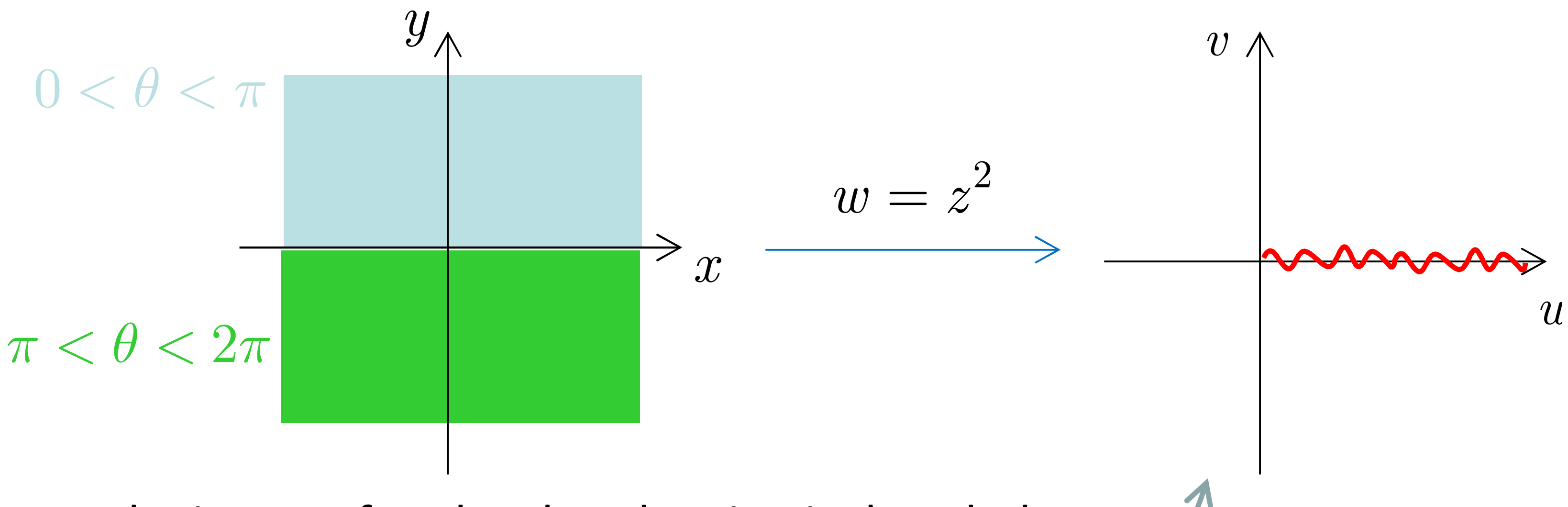
It is helpful to think the image of the whole region as a transparent film which partly covers itself. This simple idea was used by Riemann for introducing the concept of Riemann surfaces…

Example: the Function $w = z^2$

The simplest Riemann surface is connected with the mapping $w = z^n$, $n>1$.

Let us first consider the case $n=2$:

$$w = z^2 = \left(re^{j\theta}\right)^2 = r^2 e^{j2\theta}$$

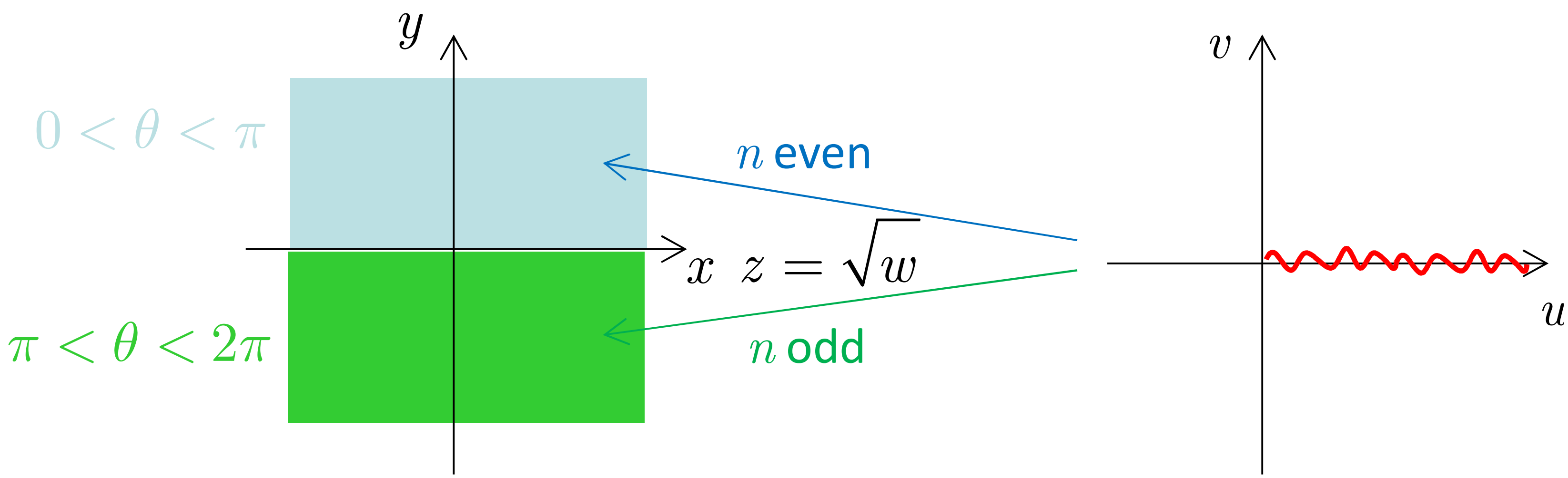


The image of each colored region is the whole w plane 'cut' along the real positive axis

A region which is mapped in a one-to-one manner onto the whole plane, except for one or more cuts, is called a **fundamental region**.

The Square-Root Function $z = \sqrt{w}$

$$w = z^2 = r' e^{j\left(\theta' + 2n\pi\right)} \quad \Rightarrow \quad z = \sqrt{w} = \sqrt{r'} e^{j\left(\frac{\theta'}{2} + n\pi\right)}$$



The cut is called a **branch cut**, as it allows for defining *single-valued branches* of the square-root function.

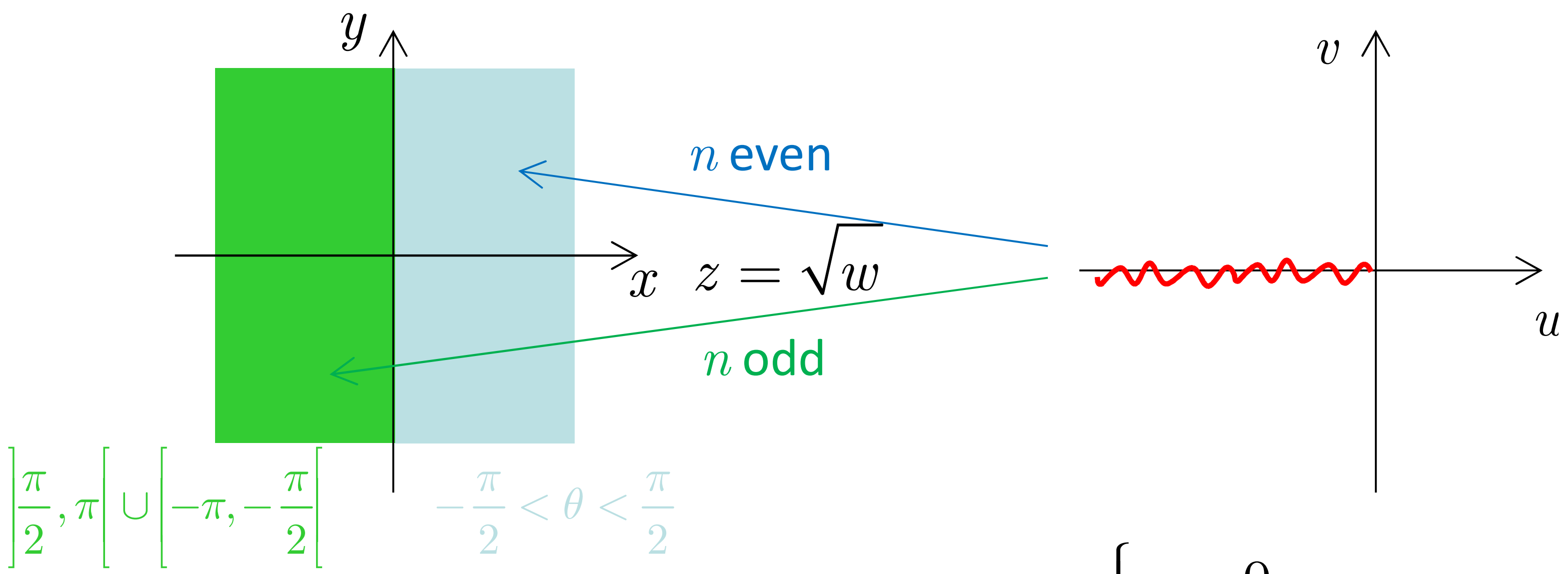
The Square-Root Function $z = \sqrt{w}$

<u>REMARK</u>

Of course, there is nothing special in the positive real axis: the branch cut can be made along **any line** joining 0 and infinity.

This is equivalent to choosing *different* fundamental regions.

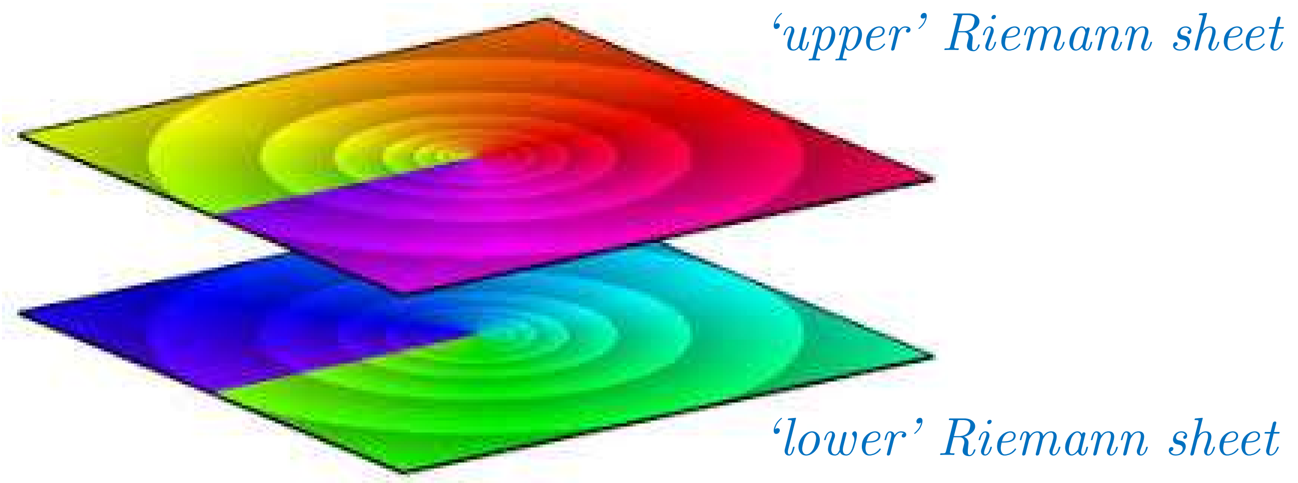
For example:



principal branch: $x > 0 \cup \begin{cases} x = 0 \\ y \geq 0 \end{cases}$

The Square-Root Function: Riemann Surface

The square-root function is two-valued, **but it can be considered one-valued if its domain is made of two copies of the complex plane**, both cut along the chosen branch cut and *glued in such a way that the resulting function is continuous*…

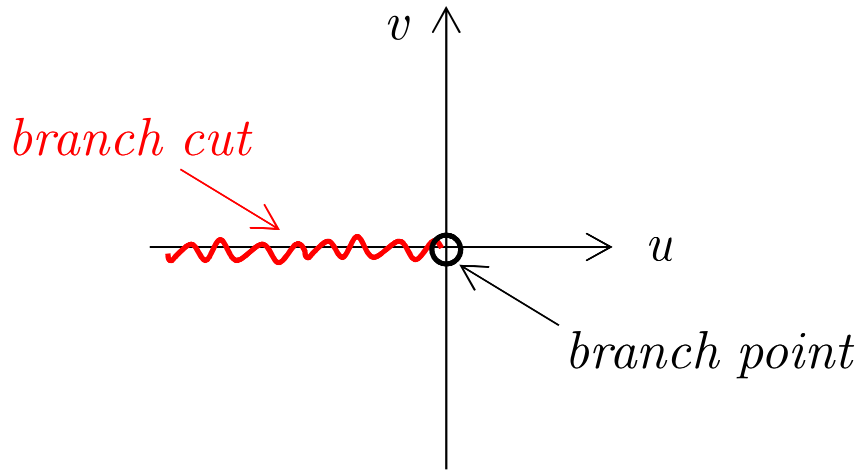


…the resulting domain of the square-root function is the **Riemann surface** associated with the considered map.

The Square-Root Function: Branch Points

The point $w = 0$ is special: it connects all the copies of the complex plane (technically, the **Riemann sheets**) that constitute the Riemann surface, and a closed curve must wind twice around it before it closes.

Such a point is called a **branch point**.



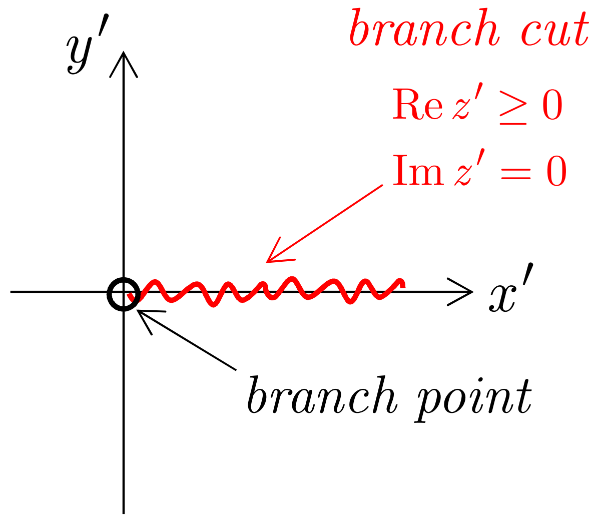
REMARK

In more general cases a branch point need not connect all sheets: if it connects h sheets it is a branch point of order $h-1$.

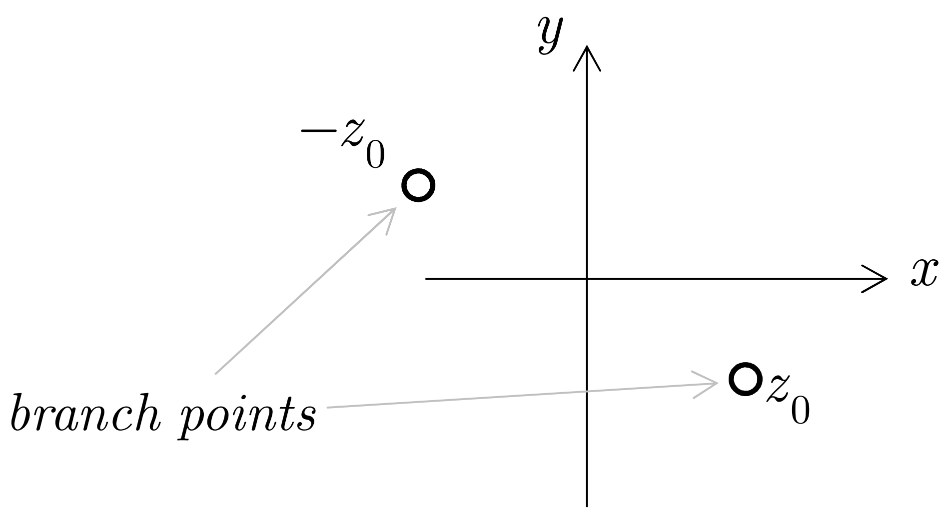
The Function $w = \sqrt{z_0^2 - z^2}$

$$w = \sqrt{z'}$$

$$z' = z_0^2 - z^2 \rightarrow z = \sqrt{z' + z_0^2}$$



The branch point $z' = 0$ is mapped to the **pair of branch points** $z = \pm z_0$

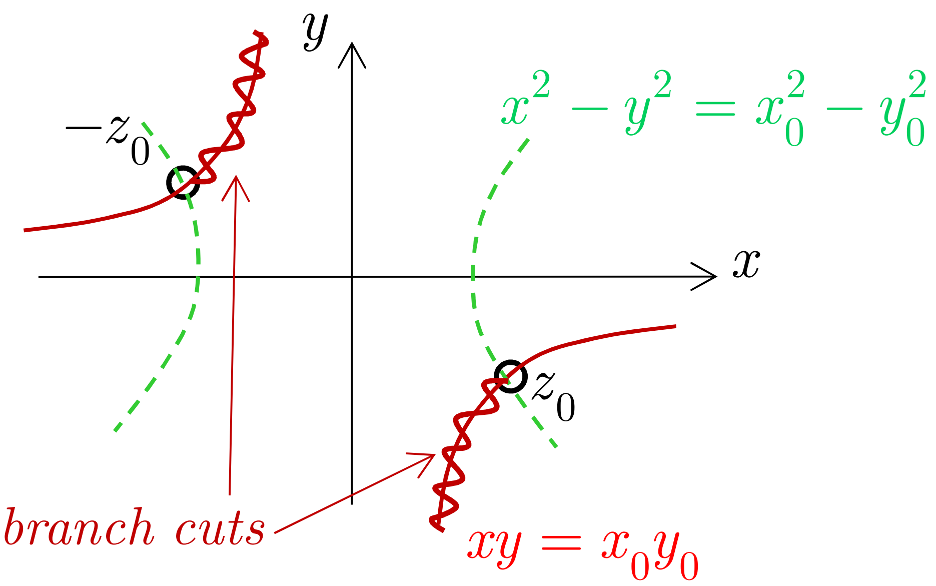


The Function $w = \sqrt{z_0^2 - z^2}$

The branch cut along the *positive real axis* is mapped to **a pair of hyperbolic branch cuts**:

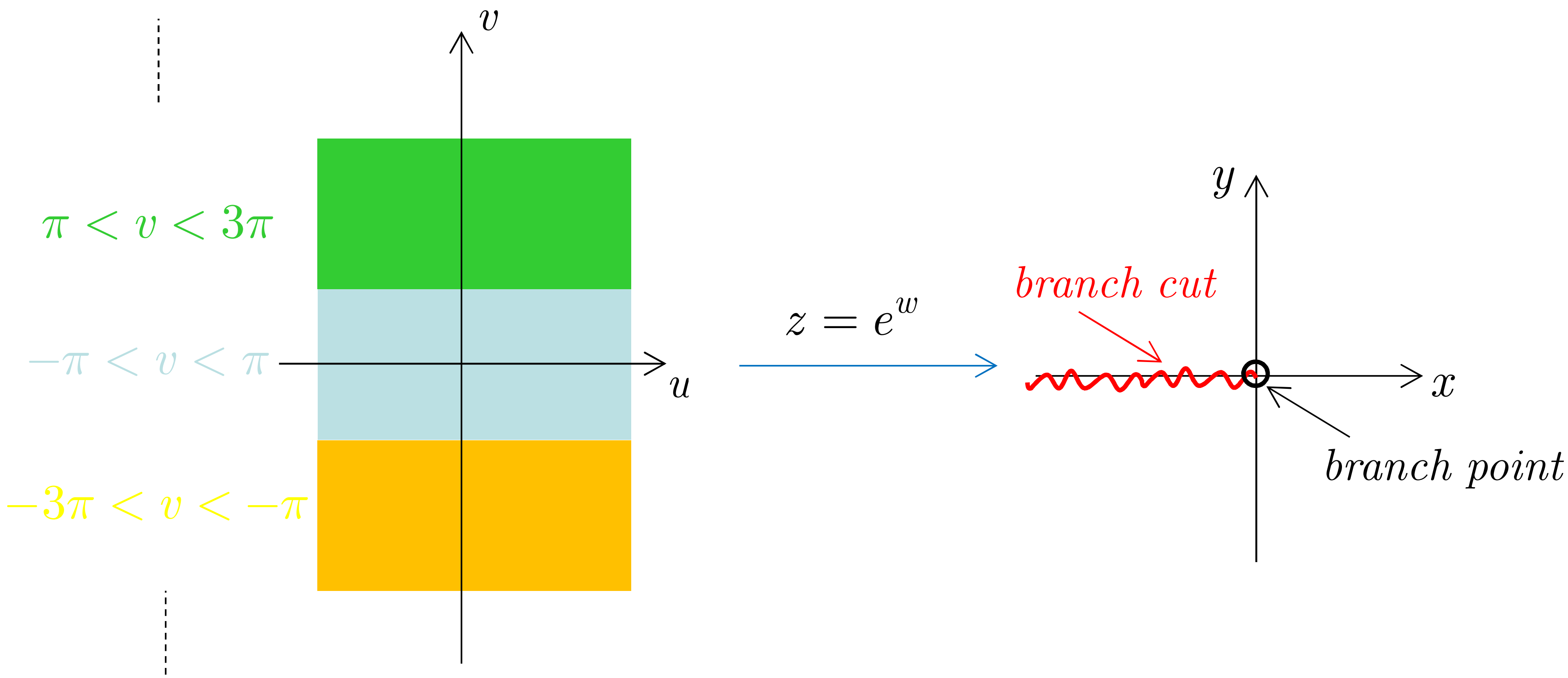
$$z' = z_0^2 - z^2 = (x_0 + jy_0)^2 - (x + jy)^2 = x_0^2 - y_0^2 + x^2 - y^2 + 2j(x_0 y_0 - xy)$$

$$\begin{aligned} \operatorname{Re} z' &\geq 0 \\ \operatorname{Im} z' &= 0 \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} x_0^2 - y_0^2 - x^2 + y^2 &\geq 0 \\ x_0 y_0 - xy &= 0 \end{aligned}$$



The Function $z = e^w$

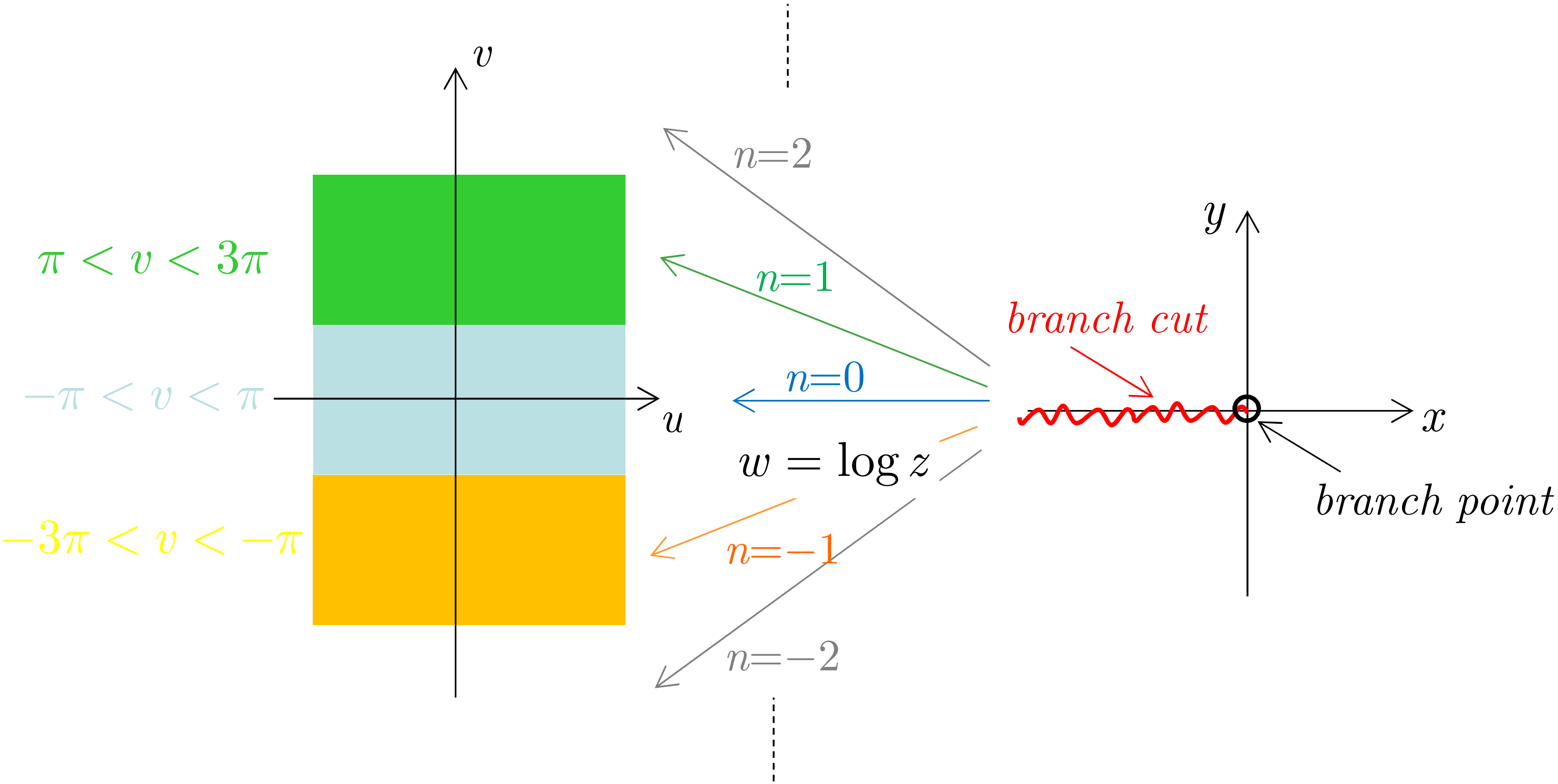
$$z = re^{j\theta} = e^w = e^u e^{jv} \rightarrow r = e^u,\ \theta = v$$



fundamental regions

The Function $w = \log z$

$$w = \log z = \log r + j\left(\theta + 2n\pi\right)$$

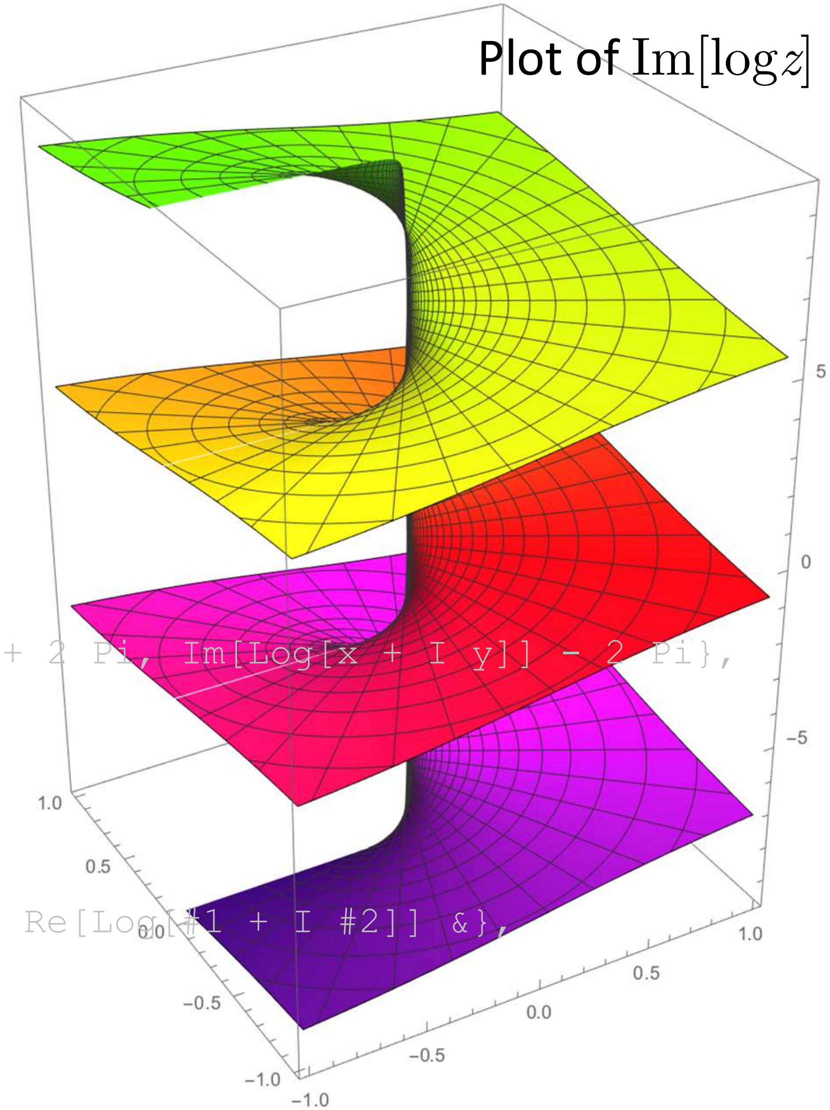


principal branch: $-\pi < v < \pi$

The Function $w = \log z$

The Riemann surface has now an **infinite number of sheets**.

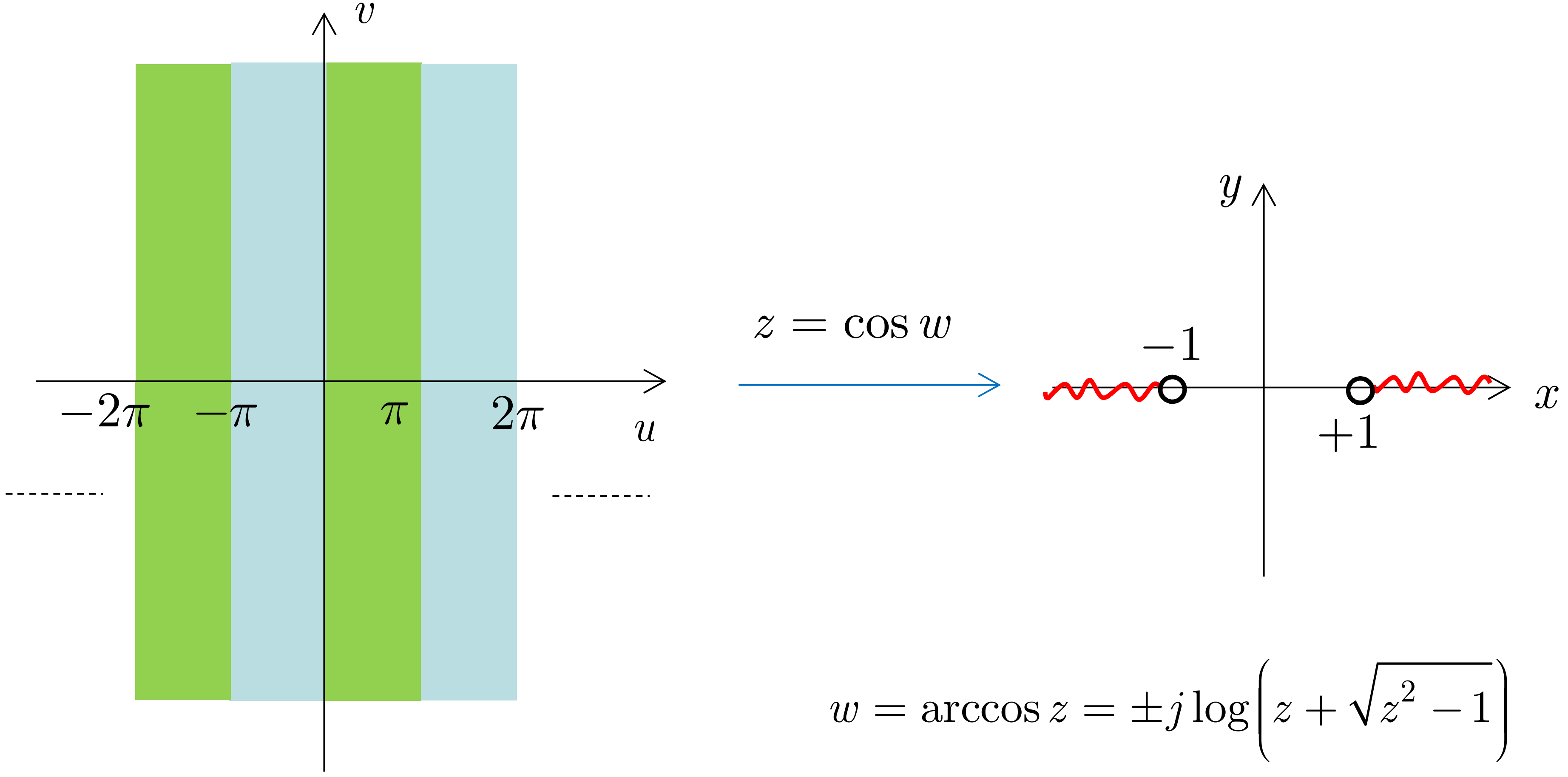
In this case the **branch point** $w{=}0$ **does not belong** to the Riemann surface.




```
Plot3D[{Im[Log[x + I y]], Im[Log[x + I y]] + 2 Pi, Im[Log[x + I y]] - 2 Pi},
    {x, -range, range}, {y, -range, range},
    BoxRatios -> {1, 1, 1.5},
    PlotRange -> All,
    PlotPoints -> 50,
    Mesh -> 30,
    MeshFunctions -> {Im[Log[#1 + I #2]] &, Re[Log[#1 + I #2]] &},
    ImageSize -> Large,
    ColorFunction -> mycolor
 ]
```

The Function $w = \arccos z$

$$z = \cos w = \cos u \cosh v - j \sin u \sinh v$$



$$w = \arccos z = \pm j \log\left(z + \sqrt{z^2 - 1}\right)$$

fundamental regions

References

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J. B. Conway, *Functions of one complex variable*. New York, NY: Springer-Verlag, 1995 (2nd ed.)