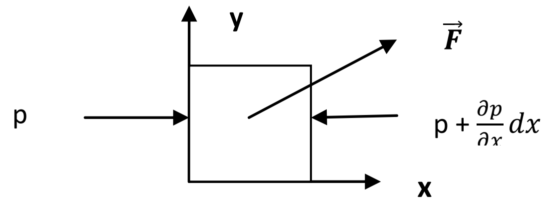
FLUIDSTATICS.

In a fluid at rest in any reference system, also non-inertial, they are not shear stresses. The **principle of Pascal** applies:

$$\sigma_{ij}(x,y,z) = -p(\mathrm{x,y,z})\delta_{\mathrm{ij}} \qquad (1)$$

where $\boldsymbol{\delta}_{ij}$ =1 if i=j and $\boldsymbol{\delta}_{ij}$ =0 if i≠j. The pressure p coincides with the thermodynamic pressure
The balance of forces applied to a volume dxdydz around the point P



$$\left[p - \left(p + \frac{\partial p}{\partial x}dx\right)\right] dy\, dz = \rho\, F_x dx\, dy\, dz$$
$$\frac{\partial p}{\partial y} = \rho F_y$$
$$\frac{\partial p}{\partial z} = \rho F_z$$

where $\vec{F}$ is the resultant of the force of mass per unit of mass; its dimensions are those of an acceleration (m/s^2)
Shortly:

$$\rho\overrightarrow{F} = \text{grad p} = \nabla\text{p} \qquad (2)$$

which could be obtained also from the Navier-Stokes equation or from the Euler equation by zeroing all the velocity terms.
This implies that for the static equilibrium of a fluid the mass forces must be conservative:

$$\vec{F} = \nabla U \qquad (3)$$

that is

$$\rho\nabla U = \nabla p \qquad (4)$$

In a fluid at rest the **isobaric surfaces** (p = const) coincide with the **equipotential surfaces** of mass forces (U = const) and coincide with the **isochoric surfaces** (ρ = const). **The static equilibrium is possible only for a barotropic [**ρ = ρ(p)] **fluid subjected to conservative forces.**
In the case that the force is weight, the isobaric equipotential isochoric surfaces are the **concentric spheres with the earth**, or, in the local approximation, the **horizontal planes**. In this latter case the eq. (4) becames

$$\frac{dp}{dz} + \rho g = \frac{dp}{dz} + \gamma = 0 \qquad (5)$$

were z is a vertical axis directed upward, g is the gravity acceleration (g = 9,81 m/s^2), and γ = ρg is the specific weight. (for the water γ= ~ 9810 N/m^3)

HYDROSTATICS.

For the water γ = const = 9810 N/m^3 approximately. The (5) integrated gives the Stevin law

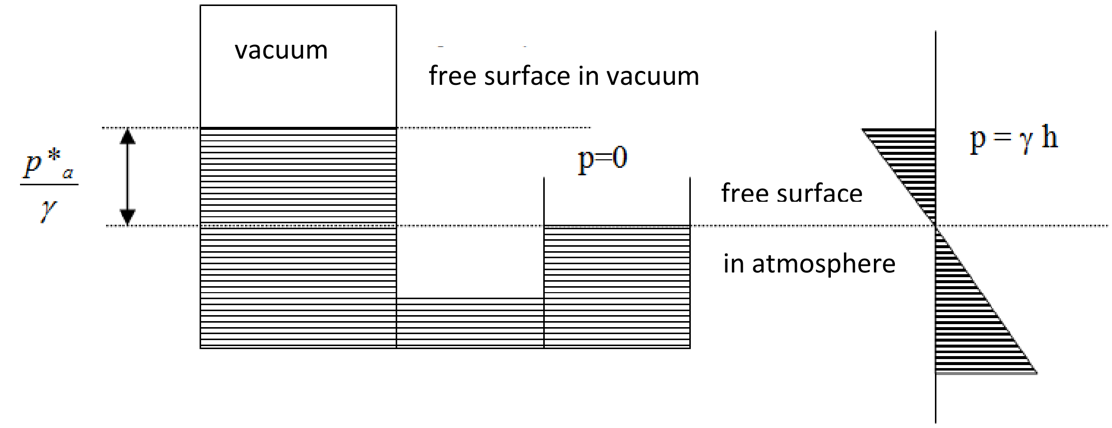
$$z + \frac{p}{\gamma} = \zeta = const \quad (6)$$

"z" is said *elevation head* "p/γ", is said *pressure head* and "ζ" *piezometric head.* The Stevin law can be enunciated: "*in a fluid at rest the piezometric head is constant*".
The ordinary pressure is sometimes called **absolute pressure**, in order to distinguish from **gauge pressure,** which is defined as the absolute pressure minus the atmospheric pressure:

$$p_{gauge} = p - p_{atm} \quad ; \quad p_{atm} = 1{,}013 \times 10^5 \, {}^{N}/_{m^2} = 1{,}013 \times 10^5 Pa = 1{,}013 \, bar \qquad (7)$$

In a liquid (ρ= const) at rest the pressure increases linearly with the depth h=ζ-z from the surface in contact with the atmosphere

$$p = \gamma(\zeta - z) = \gamma h \quad (8)$$

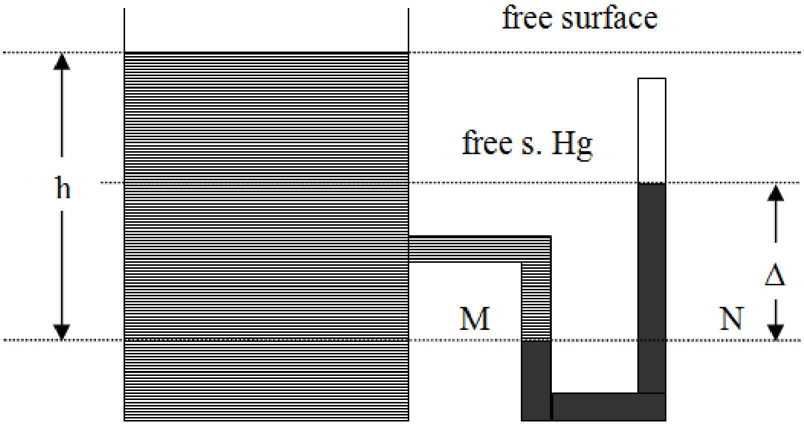


The **Torricelli barometer** measure the vertical distance between the liquid free surface in contact with the vacuum and the free surface in contact with the atmosphere (p_a*/γ)
If the liquid is water the column is about 10,33 m high, using mercury ($\rho_{Hg} = 13600 kg/m3$) its height is reduced to about 76 mm.

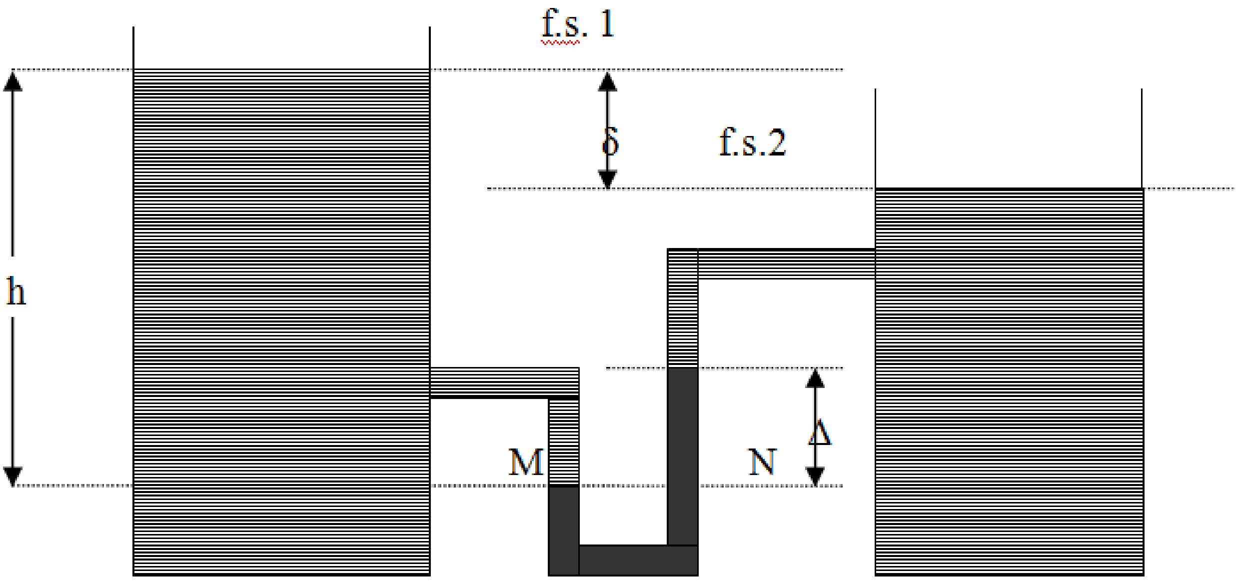
A **piezometer** is a simple trasparent tube to measure the piezometric head.
A **manometer** or **pressure gauge** uses a fluid with a different specific weight, generally higher, from the one in question.

$$p_M = \gamma h = p_n = \gamma_m \Delta$$
$$h = \frac{\gamma_m}{\gamma}\Delta \quad (9)$$

where γ is the specific weight of fluid in question and γ_m is that of manometric fluid, usually mercury.



A **differential pressure gauge** determines the difference in pressure at two points:

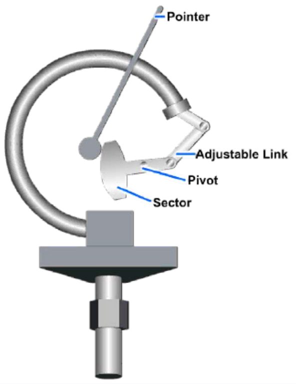


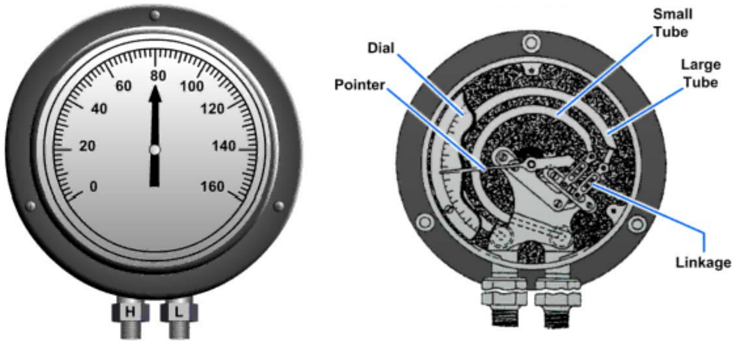
$$\boldsymbol{p_M = \gamma h = p_N = \gamma_m \Delta + \gamma(h - \Delta - \delta)}$$

$$\boldsymbol{\delta = \frac{\gamma_m - \gamma}{\gamma}\Delta = \left(\frac{\gamma_m}{\gamma} - 1\right)\Delta \quad (10)}$$

in the case of water and mercury $\delta \approx 12{,}6\ \Delta$; in the case of water and air ($\gamma \approx 800\gamma_m$) the U must be reversed and in this case $\delta \approx \Delta$

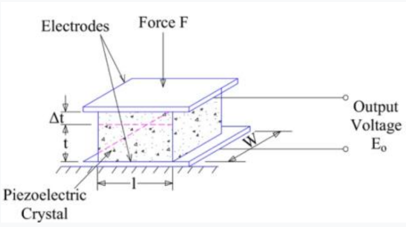
Bourdon tube pressure gauge. The Bourdon pressure gauge uses the principle that a flattened tube tends to straighten or regain its circular form in cross-section when pressurized.

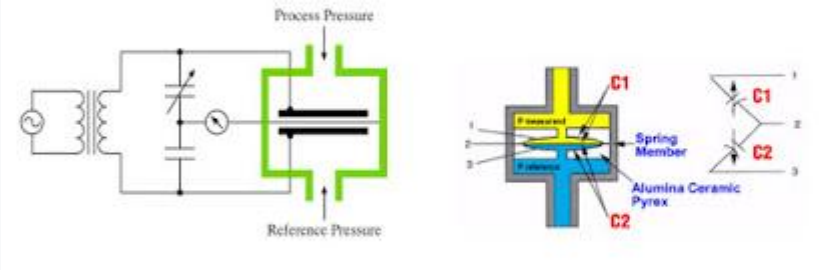




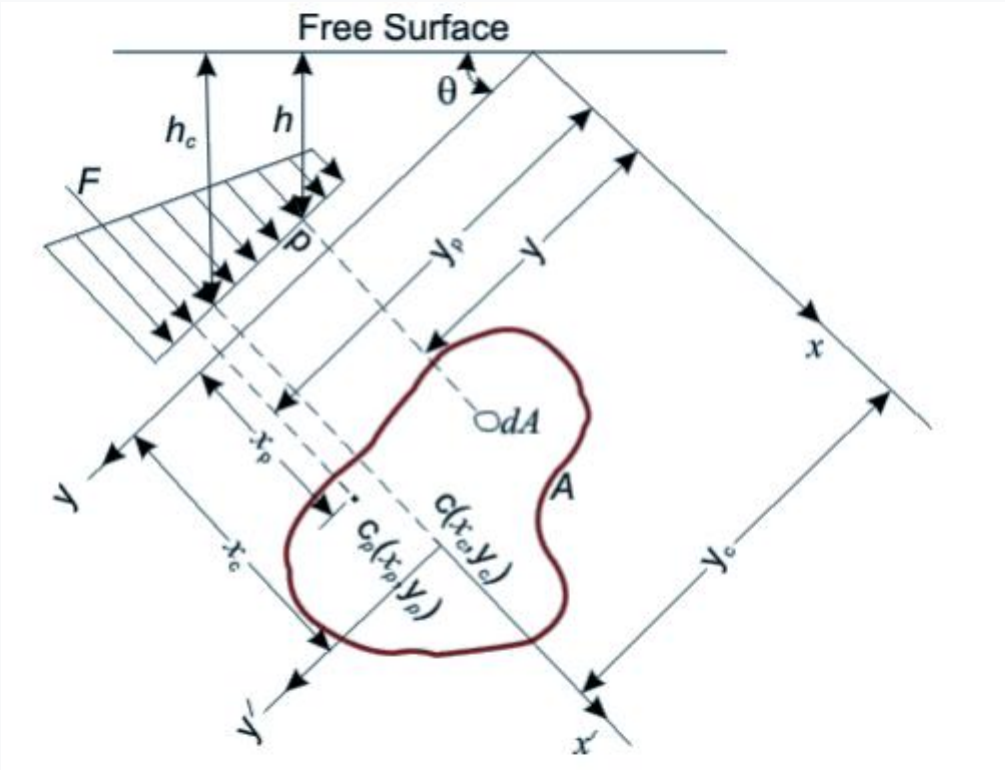
Bourdon-tube differential pressure gauge.

Electric pressure transducers are increasingly used today of various types: piezoresistive, capacitive, electromagnetic, piezoelectic etc.





Hydrostatic thrusts on submerged plane surfaces. Consider a plane surface of arbitrary shape wholly submerged in a liquid so that the plane of the surface makes an angle θ with the free surface of the liquid. We will assume the case where the surface shown in the figure below is subjected to hydrostatic pressure on one side and atmospheric pressure on the other side.



Let p denotes the gauge pressure on an elemental area dA. The resultant force F on the area A is therefore

$$F = \iint_A pdA = \rho g \iint_A hdA = \gamma h_C A = p_C A \quad (11)$$

hydrostatic thrust on a plane surface is equal at its centroid sinking times the total area of the surface.

The point of action of the resultant force on the plane surface is called the centre of pressure C_p. Let x_p and y_p be the distances of the centre of pressure from the y and x axes respectively. Equating the moment of the resultant force about the x axis to the summation of the moments of the component forces, we have

$$x_p F = x_p \gamma h_c A = \iint_A pxdA = \iint_A \gamma hxdA = \gamma \sin\theta \iint_A xy\, dA$$

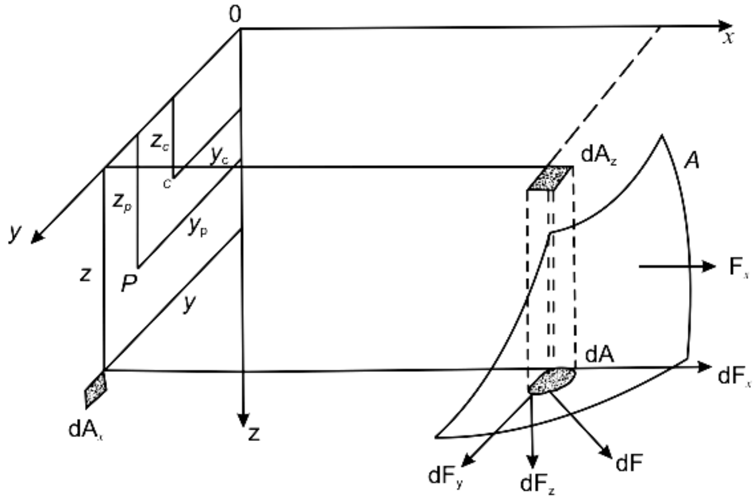
$$y_p F = y_p \gamma h_c A = \iint_A pydA = \iint_A \gamma hydA = \gamma \sin\theta \iint_A y^2 dA$$

$$x_p = \frac{\iint_A xydA}{\iint_A ydA} = \frac{I_{xy}}{Ay_c} \quad (12)$$

$$y_p = \frac{\iint_A y^2 dA}{\iint_A ydA} = \frac{\iint_A [(y-y_c)+y_c]^2 dA}{\iint_A ydA} = \frac{I_{xx}}{Ay_c} + y_c \quad (13)$$

The two double integrals in the numerators of Eqs (12) and (13) are the product of inertia about the x-axis I_{xy} and the moment of inertia I_{xx} about area respectively. By applying the theorem of parallel axis we obtain the last expression of Eq (13). The first term on the right hand side of the Eq. (13) is always positive. *Hence, the centre of pressure is always at a higher depth from the free surface than that at which the centre of area lies.* This is obvious because of the typical variation of hydrostatic pressure with the depth from the free surface. When the plane area is symmetrical about the y axis I_{xy}=0 and x_p=x_c.

Hydrostatic thrusts on submerged curved surfaces. On a curved surface, the direction of the normal changes from point to point, and hence the pressure forces on individual elemental surfaces differ in their directions. Therefore, a scalar summation of them cannot be made. Instead, the resultant thrusts in certain directions are to be determined and these forces may then be combined vectorially. An arbitrary submerged curved surface is shown in figure below. A rectangular Cartesian coordinate system is introduced whose xy plane coincides with the free surface of the liquid and z-axis is vertical directed downward below the x - y plane.



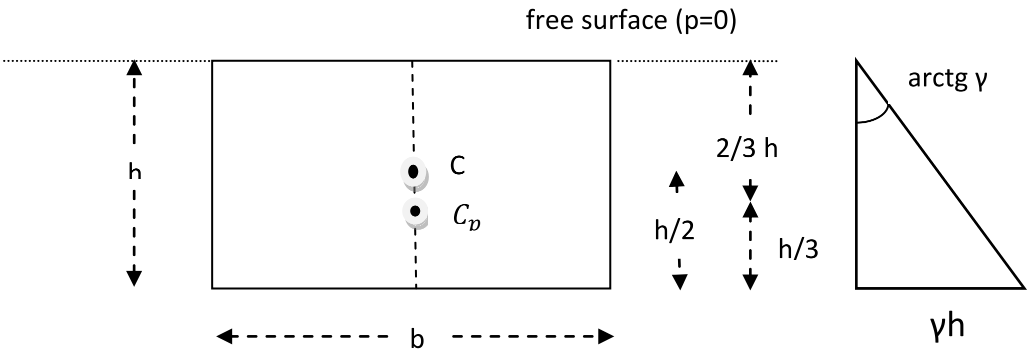
Consider an elemental area dA at a depth z from the surface of the liquid. The hydrostatic force on the elemental area dA is $dF = \gamma gz\, dA$, and the force acts in a direction normal to the area dA. The components of the force dF in x, y and z directions are:
$dF_x = \gamma z \cos nx\, dA = \gamma z\, dA_x;\;\; dF_y = \gamma z \cos ny\, dA = \gamma z\, dA_y; F_z = \gamma z \cos nz\, dA = \gamma z\, dA_z$
Therefore, the components of the total hydrostatic force along the coordinate axes are:

$$F_x = \iint_A \gamma \mathrm{z}\, \mathrm{d}A_x = \gamma z_{cx} A_x \quad ; \quad F_y = \iint_A \gamma \mathrm{z}\, \mathrm{d}A_y = \gamma z_{cy} A_y \quad ; \quad F_z = \iint_A \gamma \mathrm{z}\, \mathrm{d}A_z = \gamma W \qquad (14)$$

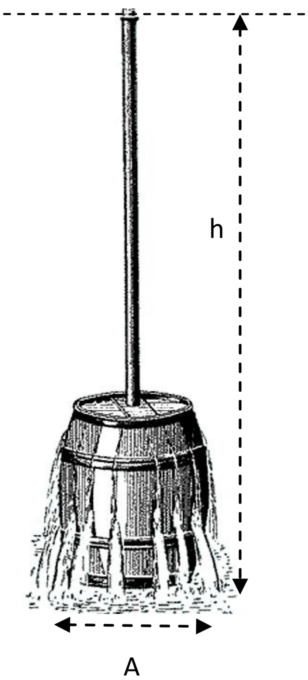
where z_{cx} and z_{cz} are the z coordinate of the centroid of area A_x and A_y (the projected areas of curved surface on yz and xz plane respectively), and W is the volume of the body of liquid within the region extending vertically above the submerged surface to the free surface of the liquid.
We can conclude that for a curved surface, *the component of hydrostatic force in a horizontal direction is equal to the hydrostatic force on the projected plane surface perpendicular to that direction and acts through the centre of pressure of the projected area.*
The vertical component of hydrostatic force on a submerged curved surface is equal to the weight of the liquid volume vertically above the solid surface of the liquid and acts through the center of gravity of the liquid in that volume. While for the horizontal components of the thrust all the considerations relative to the thrusts on flat surfaces are valid.
Buoyancy. The Eq. (4) can be integrated into a volume W (boundary A, normal n), using Green' formula

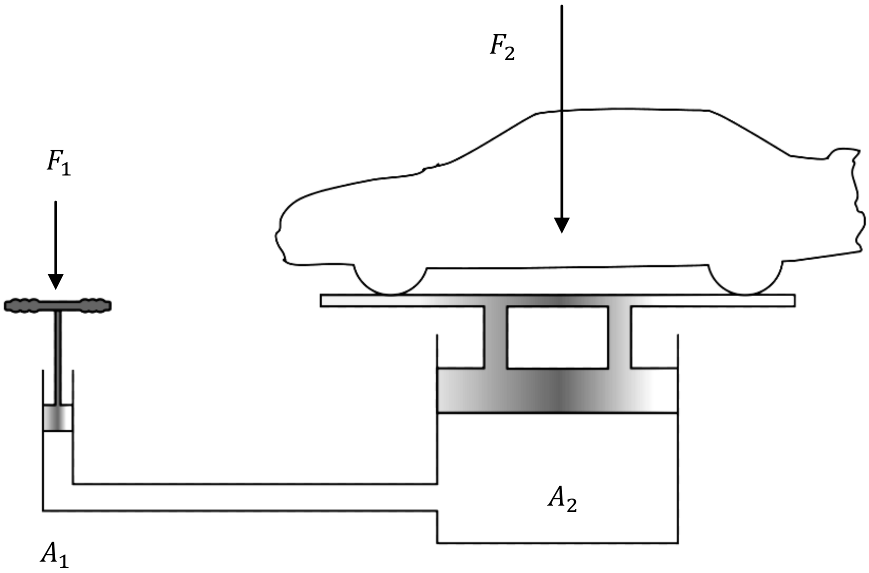
$$\iiint_W \rho \vec{g} dW - \iiint_W \nabla p\, dW = \iiint_W \rho \vec{g} dW + \iint_A p\vec{n}\, dA = \vec{G} + \vec{\Pi} = 0 \qquad (15)$$

The **Archimedes principle** states that the buoyant force on a submerged body is equal to the weight of liquid displaced by the body, and acts vertically upward through the centroid of the displaced volume.
Examples. Calculation of the thrust on a vertical wall



the thrust on the vertical surface bh is equal to $\frac{1}{2}\gamma bh$. The wall centroid lies at h/2, while the center of thrust is in the center of gravity of the triangular thrust diagram, that is at 2/3 h.
Pascal's barrel. In the bottom of the barrel area A acts *the vertical trust γhA equal to the weight of a volume of water hA, while* the volume of water actually present is much smaller, approximately h_bA. The difference trust $\gamma(h-h_b)A$ is a vertical upward *thrust applied to the barrel lid*.
This is an important paradox of hydrostatics

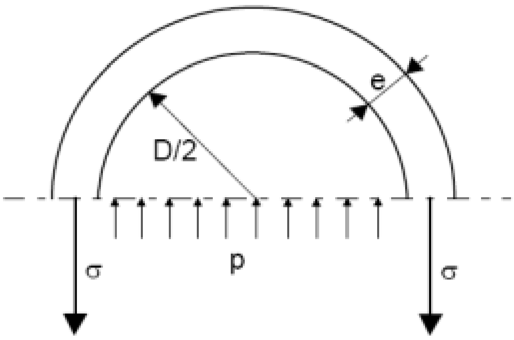




Hydraulic lifting and pressing devices.

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad\qquad F_2 = \frac{A_2}{A_1} F_1$$

Mariotte's formula for thin tubes.



If the pipe is subjected to high pressure, the pressure varies little with altitude and we can assume it constant, which is equivalent to neglecting the weight of the fluid in the global equilibrium eq. (4).
In this approximation the thrust that acts on the curved surface of the half-pipe is the same as on the surface of one of its diametrical planes.
Taken a clip of long pipe dL of very small thickness e (at least one fiftieth of the diameter) so as to be able to retain the tensile stress $\boldsymbol{\sigma}$ uniformly distributed over this thickness, with the symbols in the figure,

$$2\boldsymbol{\sigma e}\, \boldsymbol{dL} = \boldsymbol{pD}\, \boldsymbol{dL} \qquad\qquad \boldsymbol{e} = \frac{\boldsymbol{pD}}{\boldsymbol{2\sigma}} \qquad\qquad (16)$$

it can be interpreted as the minimum thickness to be given to the tube subject at the maximum allowable stress.

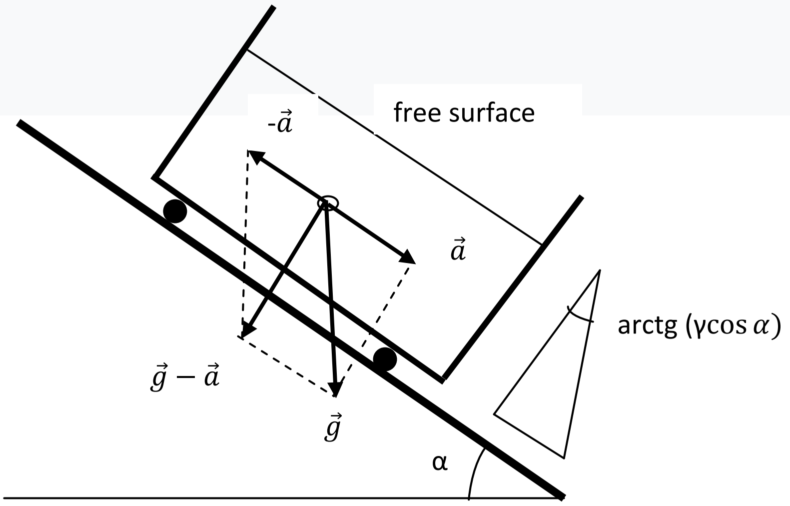
Fluids in Rigid-Body Motion. This is the case of a quiescent liquid in a non-inertial reference. The fluid is subjected not only to its own weight but also to forces (per unit of volume) of inertia $\rho\vec{a}$ due to dragging, the static equilibrium (2) therefore become

$$\rho\vec{g} - \vec{a} = \nabla p \qquad \text{possible if} \qquad \vec{a} = \nabla U$$

from which $$\nabla\left(z + \frac{p}{\gamma} + \frac{U}{g}\right) = 0 \qquad\qquad z + \frac{p}{\gamma} + \frac{U}{g} = const \qquad\qquad (17)$$

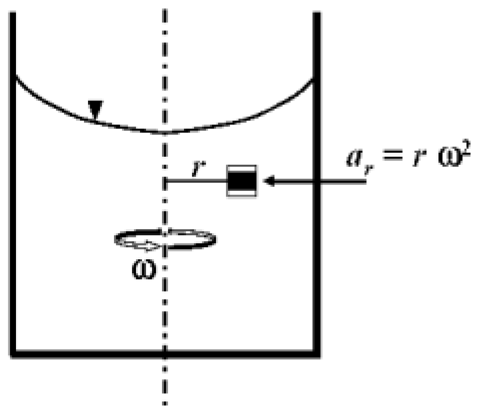
Tank falling on an inclined plane in the absence of friction. Gravity g is replaced by the force of constant modulus $|\vec{g}-\vec{a}| = g\cos\alpha$, the isobaric surfaces are the planes parallel to the inclined plane, the pressure varies linearly along the normal to the inclined plane with the law

$$p = \gamma h \cos\alpha \qquad (18)$$



for 90 ° the vessel falls freely and the pressure is zero. In a free jet in vacuum the absolute pressure is zero. *In a free jet in the atmosphere the gauge pressure is zero.*

Centrifuge



with the symbols in the figure

$$U = -\frac{1}{2}\omega^2 r^2 + \ const$$

$$z + \frac{p}{\gamma} + \frac{\omega^2 r^2}{2g} + const$$

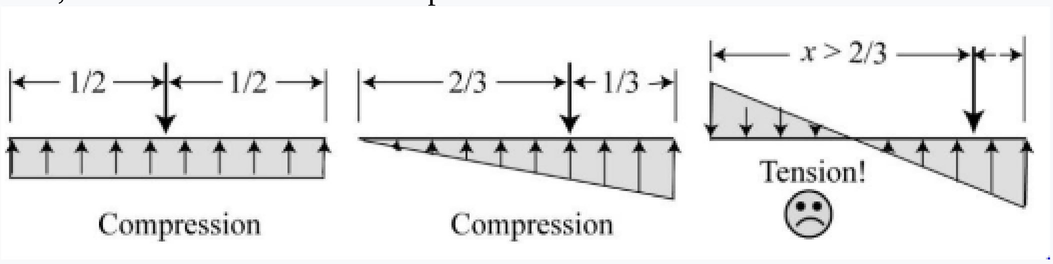
the isobaric surfaces

$$z + \frac{\omega^2 r^2}{2g} = const = z_0$$

are paraboloids of revolution. Along the verticals the pressure varies hydrostatically, it is minimal in the center and increases with the square of the radius in the periphery.
At each point there is a mass force normal to the isobaric surfaces equal to the resultant of the weight and of the centrifugal force. Force that increases with the distance from the axis and with the square of the rotation speed.
On this principle the centrifuges work for the rapid separation of elements of different densities.
In the case of closed containers there can be depressions in the center with high rotational speed until we reach the absolute pressure zero with the consequent rupture of the fluid vein and the start of **cavitation**. Phenomenon to be kept in mind in all cases of rapidly rotating fluids such as centrifugal pumps, turbines, propellers, etc.

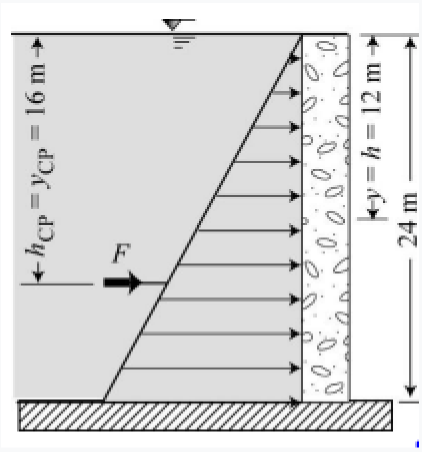
Dam stability.
The main concept for the stability of gravity dams involves the hydrostatic force and the weight of concrete. There are at least two main concepts regarding dam stability, and it is imperative to

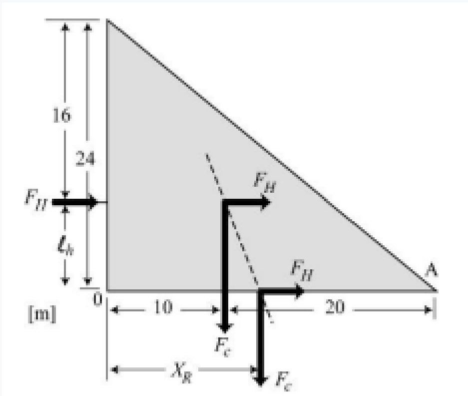
prevent: (1) overturning or toppling of the dam; and (2) tension cracks at the base of concrete dams. In the first case, the resultant force must pass to through the base of the dam. The second criterion is more restrictive, to prevent tension in the base of a solid concrete dam without reinforcement, the resultant force must pass through the central third of the base. When the force is outside the central third, tension cracks could develop at the base.



In the first figure, find the force F and its point of the application on a vertical (y=h) rectangular plate of height = 24 m and unit width,

$A = 24m^2$ $\quad\quad F = \gamma \frac{Ah}{2} = 9810 \times 24 \ \times 12 = 2825kN$ $\quad\quad y_{cp} = \frac{2}{3}h = 16m$





In the second figure will tension develop at the base of the following dam configuration

$$F_c = \gamma_c W A_c = 24 \frac{kN}{m^3}(1m)\left(\frac{30m \times 24m}{2}\right) = 109000kN$$

the clockwise sum of moments about 0 is$F_c l_c + F_H l_h = 8640 + 2825 \times 8 = 109000kN \times m$
At the base $M_0 = F_c X_R + F_H \times 0$ from which $X_R = \frac{M_0}{F_c}$ = 109000/8640 = 12,6 m.
The force passes through the central third, i.e. 10 < XR < 20 m, and there is no tension.

Curved surface
For the dam sketched with a unit width and determine the following:
(a) the constant k of the parabolic equation $y = kx^2$, with 24=k(12)^2, k=1/6
(b) the horizontal hydrostatic force $F_H = \gamma A \frac{h}{2} = 9800 \times 24 \times 12N = 2825kN$
(c) the weight of water above the dam is $A_{above} = 12 \times 24 - \int_0^{12} kx^2 dx = 192m^2$

$F_w = \gamma A_{above} = 1883\ kN$

(d) Divide the concrete part into three segments and determine the weight and CG for each segment

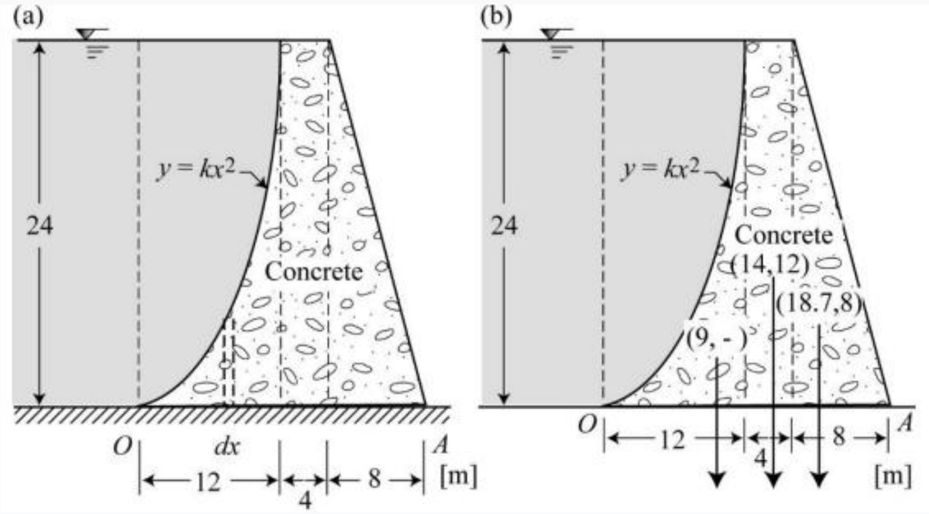
$$W_{c1} = \gamma_c A_{c1} = 24 \times 96 = 2304kN \quad W_{c2} = \gamma_c A_{c2} = 24 \times 4 \times 24 = 2304kN$$
$$W_{c3} = \gamma_c A_{c3} = 24 \times 0{,}5 \times 8 \times 24 = 2304kN$$

e) calculate the sum of horizontal $F_H = 2825kN$ and vertical forces
$F_V = F_w + W_{c1} + W_{c2} + W_{c3} = 1883 + 3 \times 2304 = 8795kN$
g) calculate the sum of moments about O, for the weight of water

$$x_G = \frac{1}{A}\iint_A xdA = \frac{1}{A}\int_0^{12} x(h-kx^2)dx = \frac{1}{192}\int_0^{12}\left(24x - \frac{1}{6}x^3\right)dx = \frac{1726-864}{192} = 4{,}5m$$

we can find the area under the curved surface as Ac = 96 m^2 and the position of the centroid is

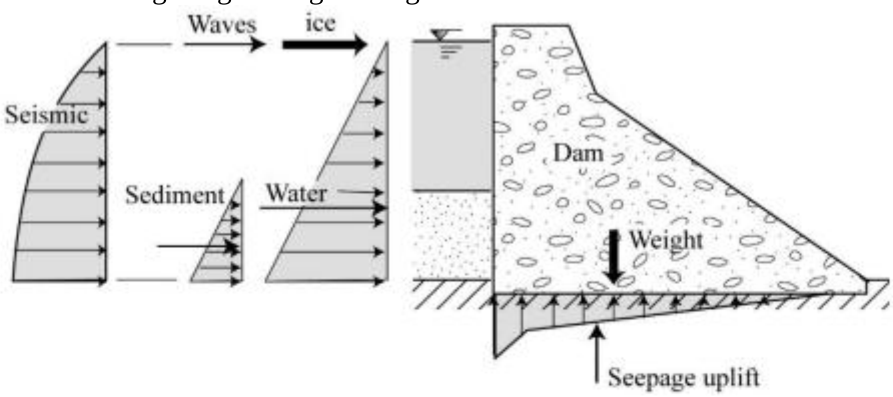
$$x_c = \int_0^{12} xkx^2dx \Big/ \int_0^{12} kxdx = 9m$$



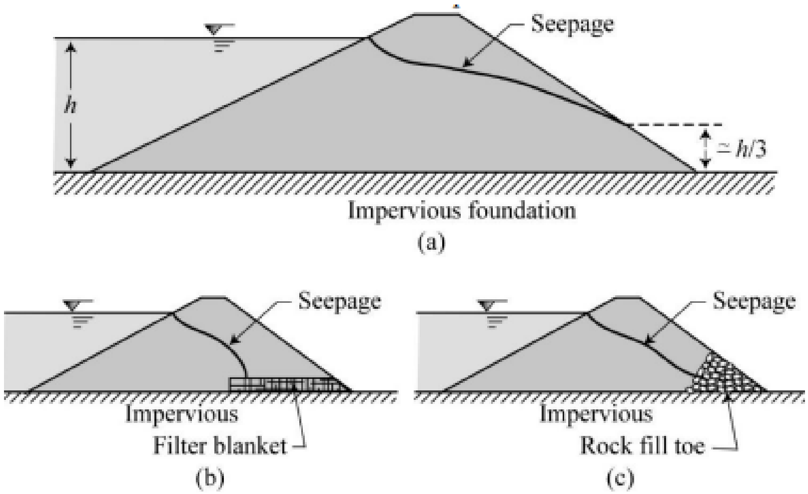
$$M_{res} = F_H \times 8 + F_w \times 4{,}5 + W_{c3} \times (9 + 14 + 18{,}67 = 127100kN\ m$$

the resultant is distant from O $X_R = \frac{M_{res}}{F_V} = \frac{127100}{8795} = 14{,}5m$ the resultant force is passing to the left of point A and therefore, *the dam is safe against overturning.*

Gravity Dams. Besides the hydrostatic fluid force and weight of concrete, the detailed analysis of gravity dams involves more force components sketched in Figure 1.9 including: (1) uplift pore pressure at the base of the dam; (2) active sediment force at the base of the dam; (3) seismic loading and inertia of the dam; and (4) wave and ice forces near the surface. Accordingly, the design of dams requires knowledge of hydraulics, structures and geological engineering.



The analysis of each force component can become quite complex. For instance, the seismic force often becomes dominant in deep reservoirs of depth H , because the pressure increases with the square root of the water depth h , from Westergaard formula $p_{sism} = 0{,}875ga_{sism}\sqrt{hH}$.



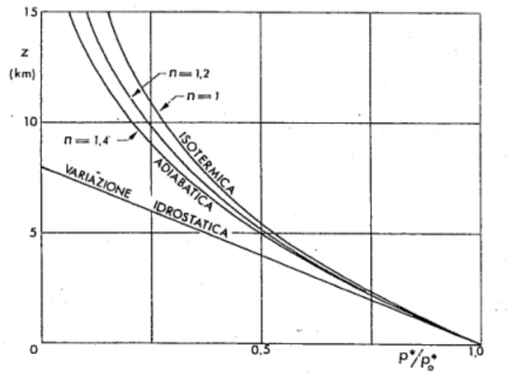
Rock and earth-fill dams are permeable and allow seepage. On impervious foundations, this can be done with a rock fill toe or a drainage blanket. Levees and permeable dams are also sensitive to failure from a sudden drawdown in water level.

Standard Atmosphere.

In a dry static atmosphere, the ideal gas law can be applied. Indicating with the subscript "0" the different quantities at the altitude z = 0 of the sea level

$$\frac{p}{p_0} = \frac{\rho}{\rho_0}\frac{T}{T_0} \qquad p_0 = 1{,}013bar \qquad \rho_0 = 1{,}225kgm^{-3} \qquad T_0 = 288{,}15°K \qquad z_0 = \frac{p_0}{\rho_0 g} = 8400m$$

which together with the eq. (5) of static equilibrium is now not sufficient to resolve the problem as it is introduced a new variable the *absolute temperature* T. The energy balance, under the assumption of the absence of exchanges of heat, leads to the *adiabatic* equation. The experimental results are better suited to a more general transformation called *polytropic* transformation with a more generic exponent n $\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^n$ substituting in the eq. (5) and integrating



$$T = T_0\left[1 - \frac{n-1}{n}\frac{\rho_0 g}{p_0} z\right] \qquad (19)$$

$$\rho = \rho_0\left[1 - \frac{n-1}{n}\frac{\rho_0 g}{p_0} z\right]^{\frac{1}{n-1}} \qquad (20)$$

$$T = T_0\left[1 - \frac{n-1}{n}\frac{\rho_0 g}{p_0} z\right]^{\frac{n}{n-1}} \qquad (21)$$

$$\frac{\partial T}{\partial z} = -\frac{n-1}{n}\frac{T_0}{z_o} \qquad (22)$$

- for n tending to infinity, a *hydrostatic atmosphere* would be obtained. A "sea of air" 8400 m deep which envelops the earth;
- n = 1,4 corresponds at an *adiabatic atmosphere* with a lapse rate of about 10°K /km and with a height about 28 km;
- the measured *lapse rate* is -6,5°K/km which correspond to n = 1,2. We obtain the *polytropic atmosphere* assumed as the **standard atmosphere** of the troposphere by the **International Civil Aviation Organization.** is height is 48 km, well beyond its application (11km);
- for n = 1 we obtain the *isothermal atmosphere* unlimited at the top, which better simulates the upper layers of the atmosphere, where, however, the continuum scheme is no longer applicable;
- finally, for n less than 1 the phenomenon of *thermal inversion* occurs.