Extract from on-line books of lecture notes on solid mechanics of prof. P. Kelly – Department of Engineering Science, University of Auckland
The complete text is available at
http://homepages.engineering.auckland.ac.nz/~pkel015/SolidMechanicsBooks/index.html

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1 Introduction

This brief chapter introduces the subject of Solid Mechanics and the contents of this book (Book I) and the books which follow (Books II-V).

1.1 What is Solid Mechanics?

Solid mechanics is the study of the deformation and motion of solid materials under the action of forces. It is one of the fundamental applied engineering sciences, in the sense that it is used to describe, explain and predict many of the physical phenomena around us.

Here are some of the wide-ranging questions which solid mechanics tries to answer:

When will this cliff collapse?

1

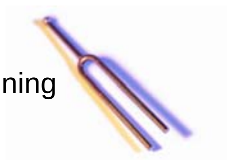
2

How does the heart contract and expand as it is pumped?

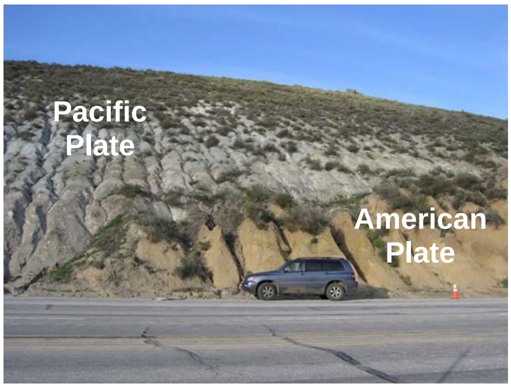
3

When will these gears wear out?

How long will a tuning fork vibrate for?



How will the San Andreas fault in California progress? How will the ground move during an earthquake?



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how do you build a bridge which will not collapse?

5

6

why does nature use the materials it does?

Solid mechanics is a vast subject. One reason for this is the wide range of materials which falls under its ambit: steel, wood, foam, plastic, foodstuffs, textiles, concrete, biological materials, and so on. Another reason is the wide range of applications in which these materials occur. For example, the hot metal being slowly forged during the manufacture of an aircraft component will behave very differently to the metal of an automobile which crashes into a wall at high speed on a cold day.

Here are some examples of Solid Mechanics of the cold, hot, slow and fast …



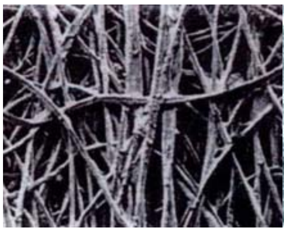
7


8

how did this Antarctic ice fracture?
what materials can withstand extreme heat?

how much will this glacier move in one year?
what damage will occur during a car crash?

Here are some examples of Solid Mechanics of the small, large, fragile and strong …



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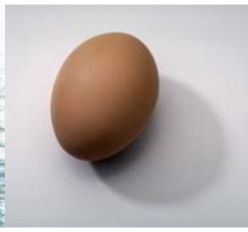


what affects the quality of paper? (shown are fibers 0.02mm thick)
how will a ship withstand wave slamming?

how strong is an eggshell and what prevents it from cracking?
how thick should a dam be to withstand the water pressure?

1.1.1 Aspects of Solid Mechanics

The theory of Solid Mechanics starts with the **rigid body**, an ideal material in which the distance between any two particles remains fixed, a good approximation in some applications. Rigid body mechanics is usually subdivided into

- **statics**, the mechanics of materials at rest, for example of a bridge taking the weight a car
- **dynamics**, the study of bodies which are changing speed, for example of an accelerating and decelerating elevator

Following on from statics and dynamics usually comes the topic of **Mechanics of Materials** (or **Strength of Materials**). This is the study of some elementary but very relevant deformable materials and structures, for example beams and pressure vessels. **Elasticity theory** is used, in which a material is assumed to undergo *small* deformations when loaded and, when unloaded, *returns to its original shape*. The theory well approximates the behaviour of most real solid materials at low loads, and the behaviour of the "engineering materials", for example steel and concrete, right up to fairly high loads.

More advanced theories of deformable solid materials include

- **plasticity theory**, which is used to model the behaviour of materials which undergo *permanent* deformations, which means pretty much anything loaded high enough
- **viscoelasticity theory**, which models well materials which exhibit many "fluid-like" properties, for example plastics, skin, wood and foam
- **viscoplasticity theory**, which is a combination of viscoelasticity and plasticity

Some other topics embraced by Solid Mechanics, are

- **rods, beams, shells and membranes**, the study of material components which can be approximated by various model geometries, such as "very thin"
- **vibrations of solids and structures**
- **composite materials**, the study of components made up of more than one material, for example fibre-glass reinforced plastics
- **geomechanics**, the study of materials such as rock, soil and ice
- **contact mechanics**, the study of materials in contact, for example a set of gears
- **fracture and damage mechanics**, the mechanics of crack-growth and damage in materials
- **stability of structures**
- **large deformation mechanics**, the study of materials such as rubber and muscle tissue, which stretch fairly easily
- **biomechanics**, the study of biological materials, such as bone and heart tissue
- **variational formulations and computational mechanics**, the study of the numerical (approximate) solution of the mathematical equations which arise in the various branches of solid mechanics, including the Finite Element Method
- **dynamical systems and chaos**, the study of mechanical systems which are highly sensitive to their initial position
- **experimental mechanics**
- **thermomechanics**, the analysis of materials using a formulation based on the principles of thermodynamics

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Images used:

1. http://www.allposters.com/-sp/Haute-Ville-on-Cliff-Edge-Bonifacio-South-Corsica-Corsica-France-Mediterranean-Europe-Posters_i8943529_.htm
2. http://www.natureworldnews.com/articles/1662/20130430/uk-begin-clinical-trials-gene-therapy-treat-heart-failure.htm
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7. http://allabout.co.jp/gm/gl/16461/
8. http://www.autoblog.com/2007/06/22/brilliance-bs6s-adac-crash-test-is-anything-but/
9. http://www.worldwideflood.com/ark/anti_broaching/anti-broaching.htm
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13. http://edition.cnn.com/2008/US/07/06/nose.cone/index.html?iref=mpstoryview
14. http://rsna.kneadle.com/secure/Radiology/Modalities/CT/Clinical/MDCT-Peds-Skull-Frac.html

1.2 What is in this Book?

This book is divided into five "sub-books":

I. An Introduction to Solid Mechanics
II. Engineering Sold Mechanics
III. The Finite Element Method
IV. Foundations of Continuum Solid Mechanics
V. Material Models in Continuum Solid Mechanics

One can take a "bottom up" approach or a "top down" approach to the subject. In the former, one looks at the particular – a restricted set of ideal geometries and materials, and a restricted set of models and equations. One then builds upon this knowledge incrementally, upwards and outwards. This would be the approach taken if one began at the start of Book I and worked through all the books more or less sequentially. This is the course taken by most engineering students, who would work through (a subset of the) material over three to four years. Alternatively, one can begin very generally – consider all the relevant equations and only later simplify these to the particular application under study. This approach would involve starting at the beginning of Book IV and then working through Book V.

The principal advantage of the bottom-up approach is that one can begin the journey with only a limited mathematical knowledge. One can also develop a very physical feel for the subject, and over 95% of applications in the real world can probably be attacked using material from the first three books. The advantage of the top-down approach is that it gives a lofty perspective of the subject at the outset, although the mathematics required is not easy.

The aim of Book I is to cover the essential concepts involved in solid mechanics, and the basic material models. It is primarily aimed at the Engineering or Science undergraduate student who has, perhaps, though not necessarily, completed some introductory courses on mechanics and strength of materials. Apart from giving a student a good grounding in the fundamentals, it should act as a stepping stone for further study into Books II to V and into some of the more specialised topics mentioned in §1.1. The philosophy adopted in Book I is as follows:

- The mathematics is kept at a fairly low level; in particular, there are few differential equations, very little partial differentiation and there is no tensor mathematics
- The critical concepts – the ones which make what follows intelligible, and which students often "miss" – are highlighted
- The physics involved, and not just the theory, is given attention
- A wide range of material models are considered, not just the standard Linear Elasticity

The outline of Book I is as follows: Chapter 2 covers the essential material from a typical introductory course on mechanics; it serves as a brief review for those who have seen the material before, and serves as an introduction for those who are new to the subject. Chapters 3-8 cover much of the material typical of that included in a Strength of Materials or Mechanics of Materials course, and includes the elementary beam theory and energy methods. The latter part of the book, Chapters 10-12 cover the more advanced material models, namely viscoelasticity, plasticity and viscoplasticity.

In Book II, differential equilibrium and strain is introduced, allowing for more complex problems to be tackled, including problems of contact mechanics, fracture mechanics and elastodynamics, the study of wave propagation and vibrations, and more complex problems of plasticity theory and viscoplasticity.

In Book III, the Finite Element Method, the standard method of obtaining approximate/numerical solutions to the equations of Solid Mechanics, is examined.

In Book IV, tensor mathematics is introduced, allowing one to analyse the mechanics of solid materials without making any approximations, for example regarding the strain in materials.

Finally, in Book V, material models are described.

2 Statics of Rigid Bodies

Statics is the study of materials at *rest*. The actions of all external forces acting on such materials are exactly counterbalanced and there is a zero net force effect on the material: such materials are said to be in a state of **static equilibrium**.

In much of this book (Chapters 6-8), **static elasticity** will be examined. This is the study of materials which, when loaded by external forces, deform by a small amount from some initial configuration, and which then take up the state of static equilibrium. An example might be that of floor boards deforming to take the weight of furniture. In this chapter, as an introduction to this subject, **rigid bodies** are considered. These are ideal materials which do not deform at all.

The chapter begins with the fundamental concepts and principles of mechanics – **Newton's laws of motion**. Then the mechanics of the **particle**, that is, of a very small amount of matter which is assumed to occupy a single point in space, is examined. Finally, an analysis is made of the mechanics of the rigid body.

The material in this chapter covers the essential material from a typical introductory course on statics. Although the concepts presented in this chapter serve mainly as an introduction for the later chapters, the ideas are very useful and important in themselves, for example in the design of machinery and in structural engineering.

2.1 The Fundamental Concepts and Principles of Mechanics

2.1.1 The Fundamental Concepts

The four fundamental concepts used in mechanics are **space**, **time**, **mass** and **force**[1]. It is not easy to define what these concepts are. Rather, one "knows" what they are, and they take on precise meaning when they appear in the principles and equations of mechanics discussed further below.

The concept of space is associated with the idea of the position of a point, which is described using coordinates (x, y, z) relative to an origin o as illustrated in Fig. 2.1.1.

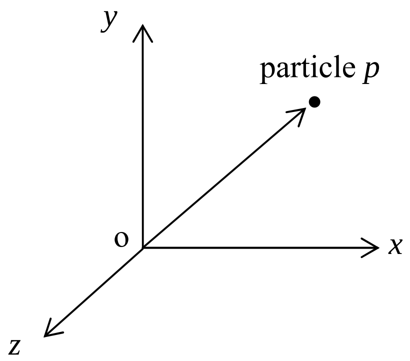

Figure 2.1.1: a particle in space

The time at which events occur must be recorded if a material is in motion. The concept of mass enters Newton's laws (see below) and in that way is used to characterize the relationship between the acceleration of a body and the forces acting on that body. Finally, a force is something that causes matter to accelerate; it represents the action of one body on another.

2.1.2 The Fundamental Principles

The fundamental laws of mechanics are Newton's three laws of motion. These are:

Newton's First Law:
if the resultant force acting on a particle is zero, the particle remains at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion)

By **resultant force**, one means the sum of the individual forces which act; the resultant is obtained by drawing the individual forces end-to-end, in what is known as the **vector**

[1] or at least the only ones needed outside more "advanced topics"

polygon law; this is illustrated in Fig. 2.1.2, in which three forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ act on a single particle, leading to a non-zero resultant force[2] $\mathbf{F}$.

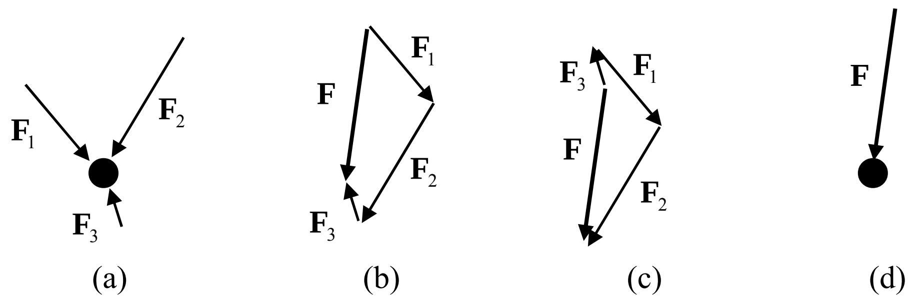

Figure 2.1.2: the resultant of a system of forces acting on a particle; (a) three forces acting on a particle, (b) construction of the resultant F, (c) an alternative construction, showing that the order in which the forces are drawn is immaterial, (d) the resultant force acting on the particle

Example (illustrating Newton's First Law)

In Fig. 2.1.3 is shown a floating boat. It can be assumed that there are two forces acting on the boat. The first is the boat's **weight** $\mathbf{F}_g$, that is its mass times the acceleration due to gravity *g*. There is also an upward buoyancy force $\mathbf{F}_b$ exerted *by* the water *on* the boat. Assuming the boat is not moving up or down, these two forces must be equal and opposite, so that their resultant is zero.

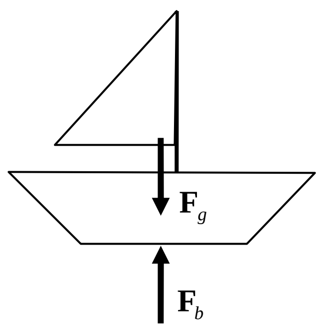

Figure 2.1.3: a zero resultant force acting on a boat

■

The resultant force acting on the particle of Fig. 2.1.2 is non-zero, and in that case one applies Newton's second law:

[2] the construction of the resultant force can be regarded also as a principle of mechanics, in that it is not proved or derived, but is taken as "given" and is borne out by experiment

Newton's Second Law:
if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant force and in the direction of this resultant force:

$$\mathbf{F} = m\mathbf{a} \tag{2.1.1}$$

where[3] $\mathbf{F}$ is the resultant force, $\mathbf{a}$ is the acceleration and m is the mass of the particle. The units of the force are the Newton (N), the units of acceleration are metres per second squared (m/s^2), and those of mass are the kilogram (kg); a force of 1 N gives a mass of 1 kg an acceleration of 1 m/s^2.

If the water were removed from beneath the boat of Fig. 2.1.3, a non-zero resultant force would act, and the boat would accelerate at g m/s^2 in the direction of $\mathbf{F}_g$.

Newton's Third Law:
each force (of "action") has an equal and opposite force (of "reaction")

Again, considering the boat of Fig. 2.1.3, the water exerts an upward buoyancy force *on* the boat, and the boat exerts an equal and opposite force *on* the water. This is illustrated in Fig. 2.1.4.

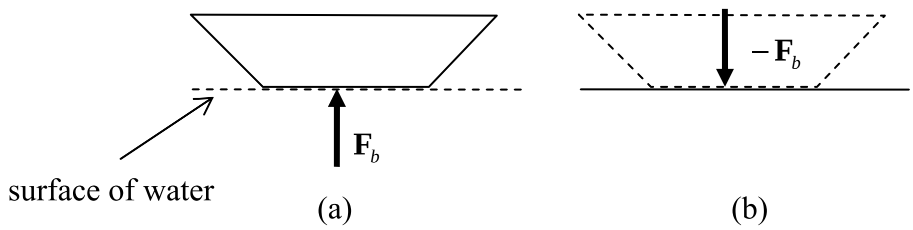

Figure 2.1.4: Newton's third law; (a) the water exerts a force on the boat, (b) the boat exerts an equal and opposite force on the water

Newton's laws are used in the analysis of the most basic problems and in the analysis of the most advanced, complex, problems. They appear in many guises and sometimes they appear hidden, but they are always there in a Solid Mechanics problem.

[3] **vector** quantities, that is, quantities which have both a magnitude and a direction associated with them, are represented by bold letters, like **F** here; scalars are represented by italics, like *m* here. The magnitude and direction of vectors are illustrated using arrows as in Fig. 2.1.2

2.2 The Statics of Particles

2.2.1 Equilibrium of a Particle

The statics of particles is the study of particles *at rest* under the action of forces. Such particles can be analysed using Newton's first law only. This situation is referred to as **equilibrium**, which is defined as follows:

Equilibrium of a Particle
A particle is in equilibrium when the resultant of all the forces acting on that particle is zero

In practical problems, one will want to introduce a coordinate system to describe the action of forces on a particle. It is important to note that a force exists independently of any coordinate system one might use to describe it. For example, consider the force **F** in Fig. 2.2.1. Using the vector polygon law, this force can be decomposed into combinations of any number of different individual forces; these individual forces are referred to as **components** of **F**. In particular, shown in Fig 2.2.1 are three cases in which **F** is decomposed into two rectangular (perpendicular) components, the components of **F** in "direction *x*" and in "direction *y*", $\mathbf{F}_x$ and $\mathbf{F}_y$.

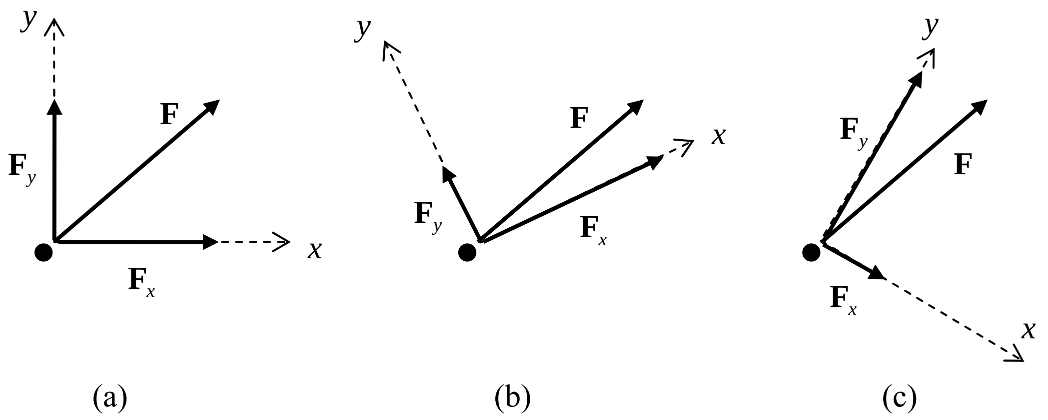

Figure 2.2.1: A force F decomposed into components $\mathbf{F}_x$ and $\mathbf{F}_y$ using three different coordinate systems

By resolving forces into rectangular components, one can obtain analytic solutions to problems, rather than relying on graphical solutions to problems, for example as done in Fig. 2.1.2. In order that the resultant force **F** on a body be zero, one must have that the resultant force in the *x* and *y* directions are zero individually[1], as illustrated in the following example.

[1] and in the *z* direction if one is considering a three dimensional problem

Example

Consider the particle in Fig. 2.2.2, subjected to forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$. The particle is in equilibrium and so by definition the resultant force is zero, $\mathbf{F} = \mathbf{0}$. The forces are decomposed into horizontal and vertical components $\mathbf{F}_{1x}, \mathbf{F}_{2x}, \mathbf{F}_{3x}$ and $\mathbf{F}_{1y}, \mathbf{F}_{2y}, \mathbf{F}_{3y}$. The horizontal forces may be added together to get a single horizontal force $\mathbf{F}_x$, which must equal zero. This force $\mathbf{F}_x$ should be evaluated using the vector polygon law but, since the individual forces $\mathbf{F}_{1x}, \mathbf{F}_{2x}, \mathbf{F}_{3x}$ all lie along the same line, one need only add together the *magnitudes* of these vectors, which involves simply an addition of *scalars*: $F_{1x} + F_{2x} + F_{3x} = 0$. Similarly, one has $F_{1y} + F_{2y} + F_{3y} = 0$. These equations could be used to evaluate, for example, the force $\mathbf{F}_1$, if only $\mathbf{F}_2$ and $\mathbf{F}_3$ were known.

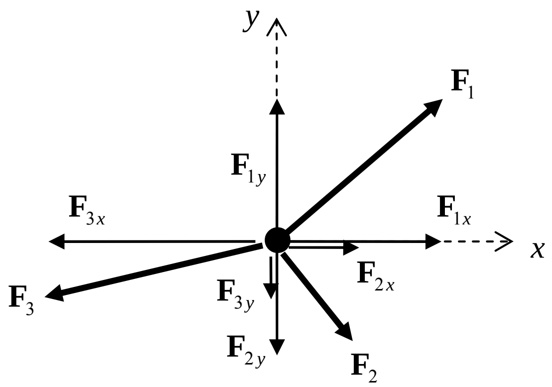

Figure 2.2.2: Calculating the resultant of three forces by decomposing them into horizontal and vertical components

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In general then, if a set of forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$ act on a particle, the particle is in equilibrium if and only if

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0 \qquad \textbf{Equations of Equilibrium (particle)} \qquad (2.2.1)$$

These are known as the **equations of equilibrium for a particle**. They are three equations and so can be used to solve problems involving three "unknowns", for example the three components of one of the forces. In two-dimensional problems (as in the next example), they are a set of two equations.

Example

Consider the system of two cables attached to a wall shown in Fig. 2.2.3. The cables meet at C, and this point is subjected to the two forces shown. To evaluate the forces of tension arising in the cables AC and BC, one can draw a free body diagram of the particle

C, i.e. the particle is isolated and all the forces acting *on* that particle are considered, Fig 2.2.3b.

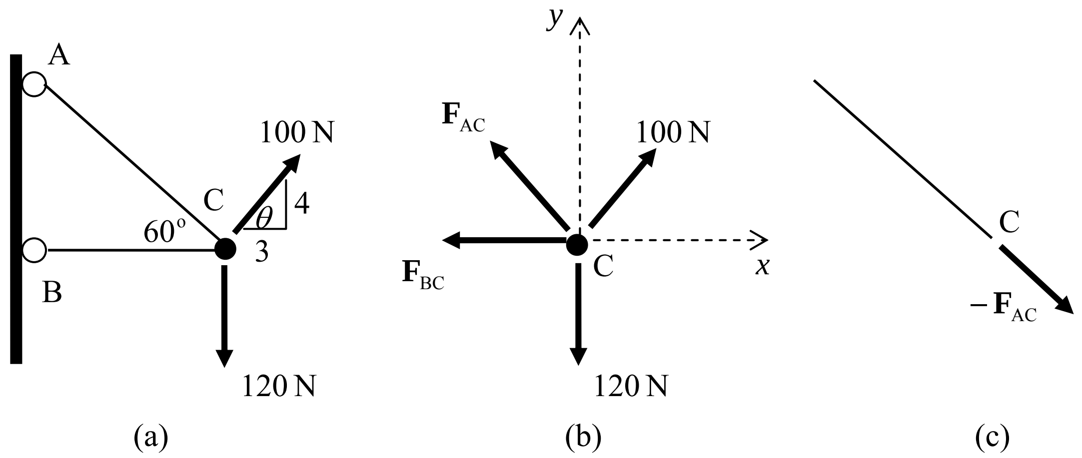


Figure 2.2.3: Calculating the tension in cables; (a) the cable system, (b) a free-body diagram of particle C, (c) cable AC in equilibrium

The equations of equilibrium for particle C are

$$\begin{aligned}\sum F_x &= -F_{\mathrm{BC}} - F_{\mathrm{AC}}\cos 60 + 100\cos\theta = 0,\\ \sum F_y &= F_{\mathrm{AC}}\cos 30 + 100\sin\theta - 120 = 0\end{aligned}$$

leading to $F_{\mathrm{AC}} = 46.2\,\mathrm{N}$, $F_{\mathrm{BC}} = 36.9\,\mathrm{N}$.

The cable exerts a tension/pulling force on particle C and so, from Newton's third law, C must exert an equal and opposite force on the cable, as illustrated in Fig. 2.2.3c.

■

The concept of the free body is essential to Solid Mechanics, for the most simple and most complex of problems. Again and again, problems will be solved by considering only a portion of the complete system, and analysing the forces acting *on* that portion only.

2.2.2 Rough and Smooth Surfaces

Fig 2.2.4a shows a particle in equilibrium, sitting on a rough surface and subjected to a force **F**. Such a surface is one where frictional forces are large enough to prevent tangential motion. The free body diagram of the particle is shown in Fig. 2.2.4b. The friction reaction force is $\mathbf{R}_f$ and the normal reaction force is $\mathbf{N}$ and these lead to the resultant reaction force **R** which, by Newton's first law, must balance **F**.

When a particle meets a smooth surface, there is no resistance to tangential movement. The particle is subjected to only a normal reaction force, and thus a particle in equilibrium can only sustain a purely normal force. This is illustrated in Fig. 2.2.4c.

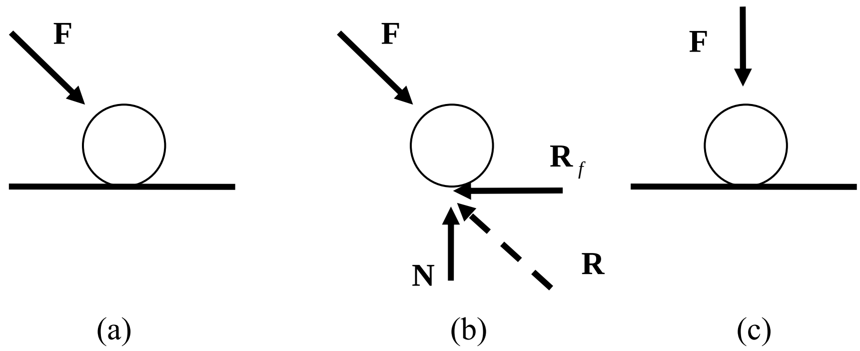
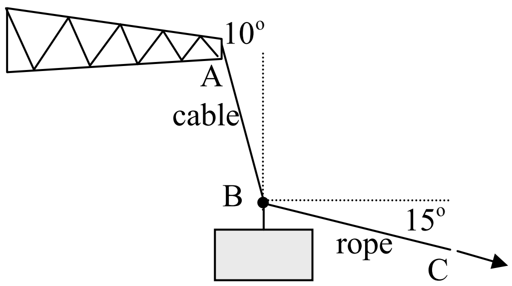


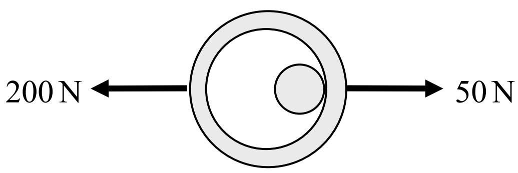
Figure 2.2.4: a particle sitting on a surface; (a) a rough surface, (b) a free-body diagram of the particle in (a), (c) a smooth surface

2.2.3 Problems

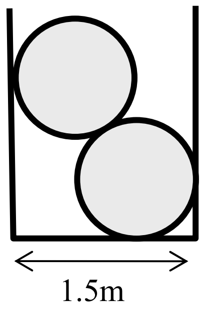
1. A 3000kg crate is being unloaded from a ship. A rope BC is pulled to position the crate correctly on the wharf. Use the Equations of Equilibrium to evaluate the tensions in the crane-cable AB and rope. [Hint: create a free body for particle B.]



2. A metal ring sits over a stationary post, as shown in the plan view below. Two forces act on the ring, in opposite directions. Draw a free body diagram of the ring including the reaction force of the post *on* the ring. Evaluate this reaction force. Draw a free body diagram of the post and show also the forces acting on it.



3. Two cylindrical barrels of radius 500mm are placed inside a container, a cross section of which is shown below. The mass of each barrel is 10kg. All surfaces are *smooth*. Draw free body diagrams of each barrel, including the reaction forces exerted by the container walls *on* the barrels, the weight of each barrel, which acts through the barrel centres, and the reaction forces of barrel on barrel. Apply the Equations of Equilibrium to each barrel. Evaluate all forces. What forces act on the container walls?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2.3 The Statics of Rigid Bodies

A material body can be considered to consist of a very large number of particles. A rigid body is one which does not deform, in other words the distance between the individual particles making up the rigid body remains unchanged under the action of external forces.

A new aspect of mechanics to be considered here is that a rigid body under the action of a force has a tendency to *rotate* about some axis. Thus, in order that a body be at rest, one not only needs to ensure that the resultant force is zero, but one must now also ensure that the forces acting on a body do not tend to make it rotate. This issue is addressed in what follows.

2.3.1 Moments, Couples and Equivalent Forces

When one swings a door on its hinges, it will move more easily if (i) one pushes hard, i.e. if the force is large, and (ii) if one pushes furthest from the hinges, near the edge of the door. It makes sense therefore to measure the rotational effect of a force on an object as follows:

The tendency of a force to make a rigid body rotate is measured by the **moment** of that force about an axis. The moment of a force **F** about an axis through a point o is defined as the product of the magnitude of **F** times the perpendicular distance *d* from the **line of action** of **F** and the axis o. This is illustrated in Fig. 2.3.1.

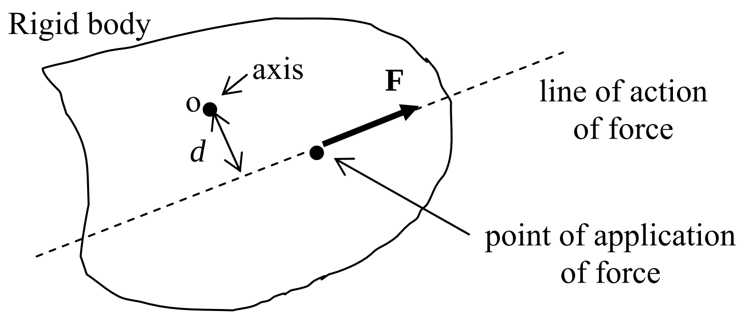


Figure 2.3.1: The moment of a force F about an axis o (the axis goes "into" the page)

The moment M_o of a force **F** can be written as

$$M_0 = Fd \tag{2.3.1}$$

Not only must the axis be specified (by the subscript *o*) when evaluating a moment, but the sense of that moment must be given; by convention, a tendency to rotate *counterclockwise* is taken to be a *positive* moment. Thus the moment in Fig. 2.3.1 is positive. The units of moment are the Newton metre (Nm)

Note that when the line of action of a force goes through the axis, the moment is zero.

It should be emphasized that there is not actually a physical axis, such as a rod, at the point *o* of Fig. 2.3.1; in this discussion, it is *imagined* that an axis is there.

Two forces of equal magnitude and acting along the same line of action have not only the same components F_x, F_y, but have equal moments about any axis. They are called **equivalent forces** since they have the same effect on a rigid body. This is illustrated in Fig. 2.3.2.

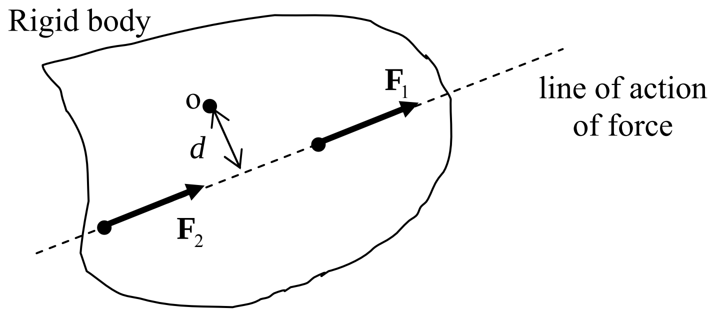


Figure 2.3.2: Two equivalent forces

Consider next the case of two forces of equal magnitude, parallel lines of action separated by distance *d*, and opposite sense. Any two such forces are said to form a **couple**. The only motion that a couple can impart is a rotation; unlike the forces of Fig. 2.3.2, the couple has no tendency to translate a rigid body. The moment of the couple of Fig. 2.3.3 about o is

$$M_o = Fd_2 - Fd_1 = Fd \qquad (2.3.2)$$

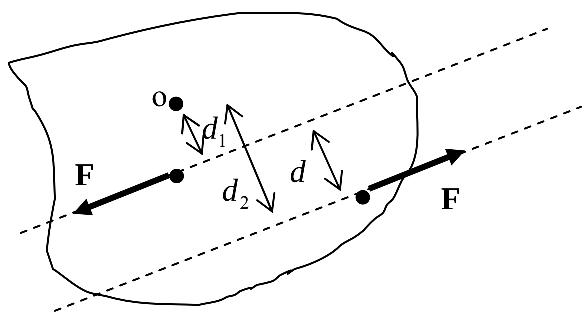


Figure 2.3.3: A couple

The sign convention which will be followed in most of what follows is that a couple is positive when it acts in a counterclockwise sense, as in Fig. 2.3.3.

It is straight forward to show the following three important properties of couples:

(a) the moment of Fig. 2.3.3 is also Fd about *any* axis in the rigid body, and so can be represented by *M*, without the subscript. In other words, this moment of the couple is independent of the choice of axis. {see ▲Problem 1}

(b) any two different couples having the same moment *M* are equivalent, in the sense that they tend to rotate the body in precisely the same way; it does not matter that the forces forming these couples might have different magnitudes, act in different directions and have different distances between them.

(c) any two couples may be replaced by a single couple of moment equal to the algebraic sum of the moments of the individual couples.

Example

Consider the two couples shown in Fig. 2.3.4a. These couples can conveniently be represented schematically by semi-circular arrows, as shown in Fig. 2.3.4b. They can also be denoted by the letter *M*, the magnitude of their moment, since the magnitude of the forces and their separation is unimportant, only their product. In this example, if the body is in static equilibrium, the couples must be equal and opposite, $M_2 = -M_1$, i.e. the sum of the moments is zero and the net effect is to impart zero rotation on the body.

Note that the curved arrow for M_2 has been drawn counterclockwise, even though it is negative. It could have been illustrated as in Fig. 2.3.4c, but the version of 2.3.4b is preferable as it is more consistent and reduces the likelihood of making errors when solving problems (see later).

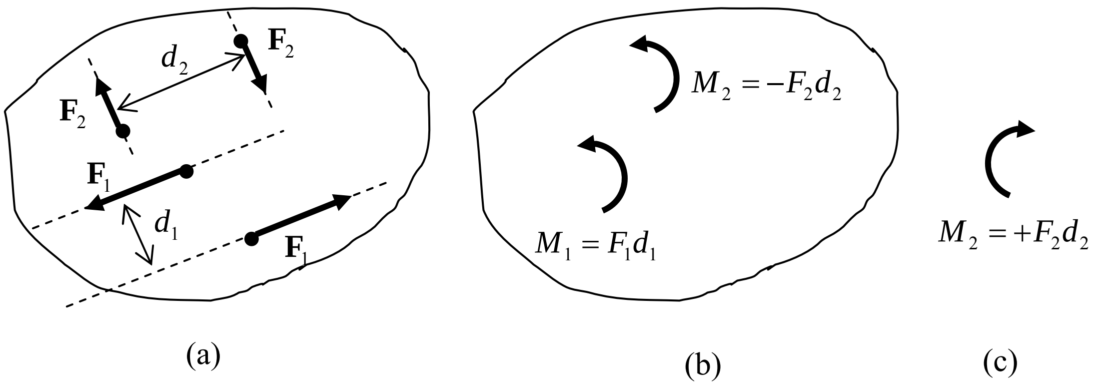


Figure 2.3.4: Two couples acting on a rigid body

■

A final point to be made regarding couples is the following: any force is equivalent to (i) a force acting at any (other) point and (ii) a couple. This is illustrated in Fig. 2.3.5.

Referring to Fig. 2.3.5, a force **F** acts at position A. This force tends to translate the rigid body along its line of action and also to rotate it about any chosen axis. The system of forces in Fig. 2.3.5b are equivalent to those in Fig. 2.3.5a: a set of equal and opposite forces have simply been added at position B. Now the force at A and one of the forces at B form a couple, of moment *M* say. As in the previous example, the couple can conveniently be represented by a curved arrow, and the letter *M*. For illustrative purposes, the curved arrow is usually grouped with the force **F** at B, as shown in Fig. 2.3.5c. However, note that the curved arrow representing the moment of a couple, which can be placed anywhere and have the same effect, *is not associated with any particular point in the rigid body*.

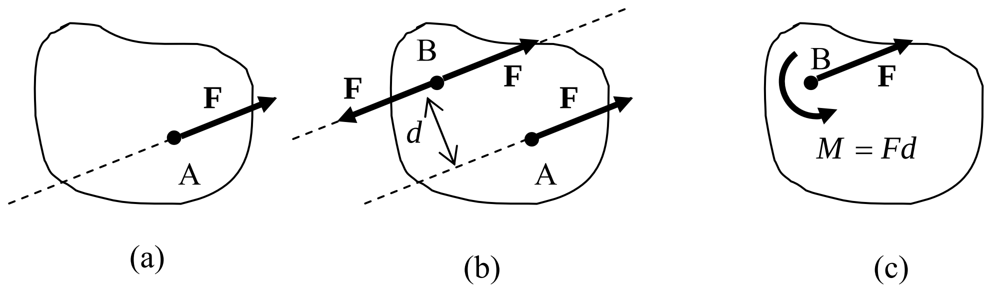


Figure 2.3.5: Equivalents force/moment systems; (a) a force F, (b) an equivalent system to (a), (c) an equivalent system involving a force and a couple *M*

Note that if the force at A was moved to a position other than B, the moment M of Fig. 2.3.5c would be different.

Example

Consider the spanner and bolt system shown in Fig. 2.3.6. A downward force of 200N is applied at the point shown. This force can be replaced by a force acting somewhere else, together with a moment. For the case of the force moved to the bolt-centre, the moment has the magnitude shown in Fig. 2.3.6b.

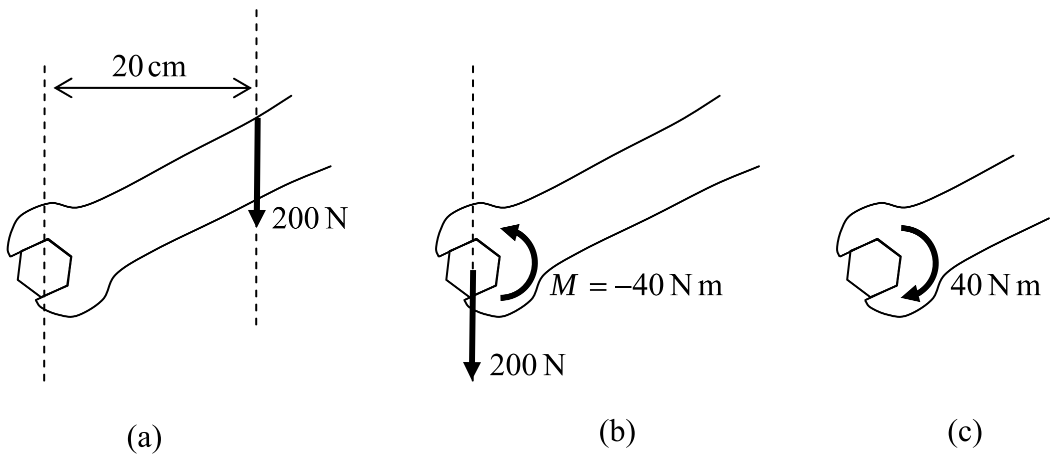


Figure 2.3.6: Equivalent force and force/moment acting on a spanner and bolt system

As mentioned, it is best to maintain consistency and draw the semi-circle representing the moment counterclockwise (positive) and given a value of -40 as in Fig. 2.3.6b; rather than as in Fig. 2.3.6c.

■

Example

Consider the plate subjected to the four external loads shown in Fig. 2.3.7a. An equivalent force-couple system **F**-M, with the force acting at the centre of the plate, can be calculated through

$$\sum F_x = 200\,\text{N}, \quad \sum F_y = 100\,\text{N}$$
$$\sum \text{M}_\text{o} = -(100)(100) - (50/\sqrt{2})(100) - (50/\sqrt{2})(100) + (200)(50) = -7071.07\,\text{Nmm}$$

and is shown in Fig. 2.3.7b. A **resultant force R** can also be derived, that is, an equivalent force positioned so that a couple is not necessary, as shown in Fig. 2.3.7.c.

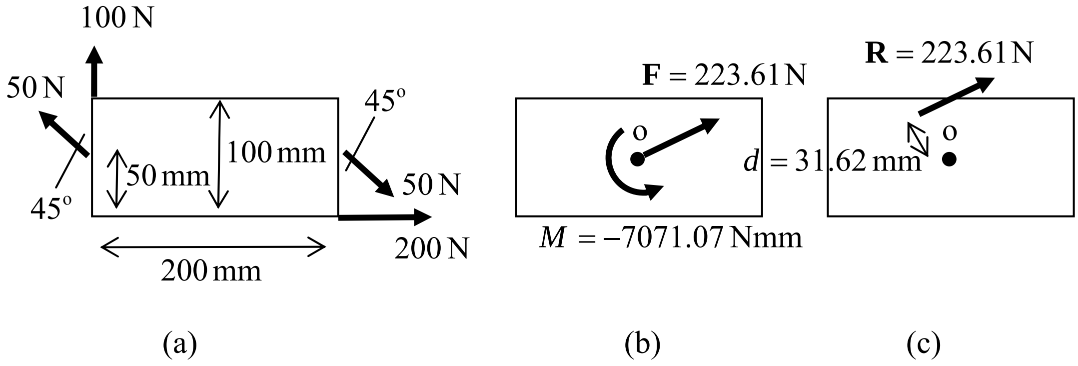


Figure 2.3.7: Forces acting on a plate; (a) individual forces, (b) an equivalent force-couple system at the plate-centre, (c) the resultant force

The force systems in the three figures are equivalent in the sense that they tend to impart (a) the same translation in the *x* direction, (b) the same translation in the *y* direction, and (c) the same rotation about *any* given point in the plate. For example, the moment about the upper left corner is

Fig 2.3.7a: $-(100)(0) - (50/\sqrt{2})(50) - (50/\sqrt{2})(150) + (200)(100)$
Fig 2.3.7b: $+(223.61)(89.44) - 7071$
Fig 2.3.7c: $+(223.61)(57.82)$

all leading to $M = 12928.93\,\text{Nmm}$ about that point.

■

2.3.2 Equilibrium of Rigid Bodies

The concept of equilibrium encountered earlier in the context of particles can now be generalized to the case of the rigid body:

Equilibrium of a Rigid Body A rigid body is in equilibrium when the external forces acting on it form a system of forces equivalent to zero

The necessary and sufficient conditions that a (two dimensional) rigid body is in equilibrium are then

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0 \quad \textbf{Equilibrium Equations (2D Rigid Body)} \quad (2.3.3)$$

that is, there is no resultant force and no resultant moment. Note that the $x-y$ axes and the axis of rotation o can be chosen completely arbitrarily: if the resultant force is zero, and the resultant moment about *one* axis is zero, then the resultant moment about *any* other axis in the body will be zero also.

2.3.3 Joints and Connections

Components in machinery, buildings etc., connect with each other and are supported in a number of different ways. In order to solve for the forces acting in such assemblies, one must be able to analyse the forces acting at such connections/supports.

One of the most commonly occurring supports can be idealised as a **roller support**, Fig. 2.3.8a. Here, the contacting surfaces are smooth and the roller offers only a normal reaction force (see §2.2.2). This reaction force is labelled $\mathbf{R}_y$, according to the conventional $x-y$ coordinate system shown. This is shown in the free-body diagram of the component.

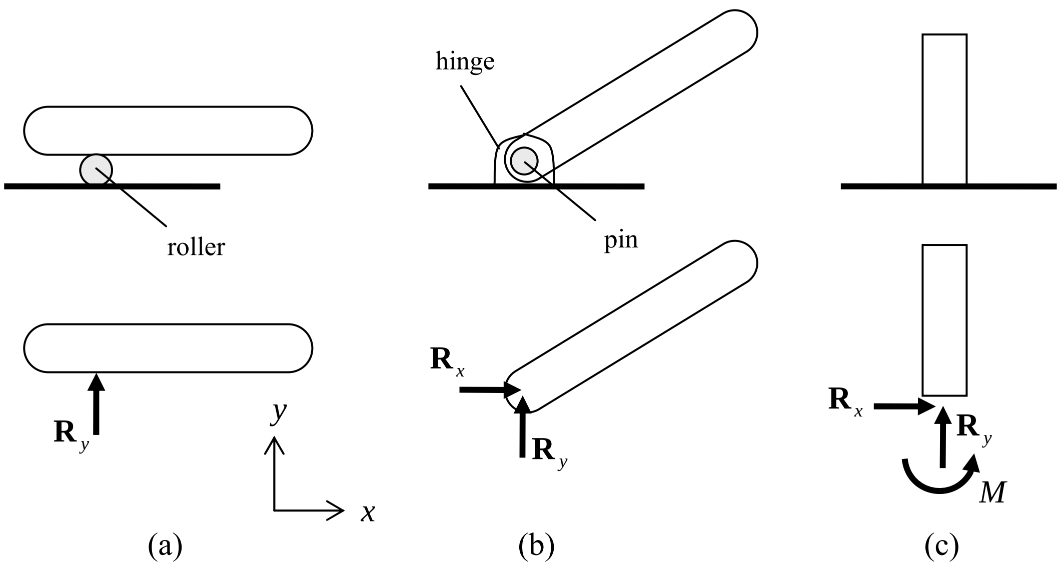


Figure 2.3.8: Supports and connections; (a) roller support, (b) pin joint, (c) clamped

Another commonly occurring connection is the **pin joint**, Fig. 2.3.8b. Here, the component is connected to a fixed hinge by a pin (going "into the page"). The component is thus constrained to move in one plane, and the joint does not provide resistance to this turning movement. The underlying support transmits a reaction force through the hinge pin to the component, which can have both normal ($\mathbf{R}_y$) and tangential ($\mathbf{R}_x$) components.

Finally, in Fig. 2.3.8c is shown a **fixed (clamped) joint**. Here the component is welded or glued and cannot move at the base. It is said to be **cantilevered**. The support in this case reacts with normal and tangential forces, but also with a couple of moment *M*, which resists any bending/turning at the base.

Example

For example, consider such a component loaded with a force **F** a distance *L* from the base, as shown in Fig. 2.3.9a. A free-body diagram of the component is shown in Fig. 2.3.9b. The known force **F** acts on the body and so do two unknown forces $\mathbf{R}_x$, $\mathbf{R}_y$, and a couple of moment *M*. The unknown forces and moment will be called **reactions** henceforth. If the component is static, the equilibrium equations 2.3.3 apply; one has, taking moments about the base of the component,

$$\sum F_x = F + R_x = 0, \quad \sum F_y = R_y = 0, \quad \sum M_{\mathrm{o}} = -FL + M = 0$$

and so

$$R_x = -F, \quad R_y = 0, \quad M = FL$$

The moment is positive and so acts in the direction shown in the Figure.

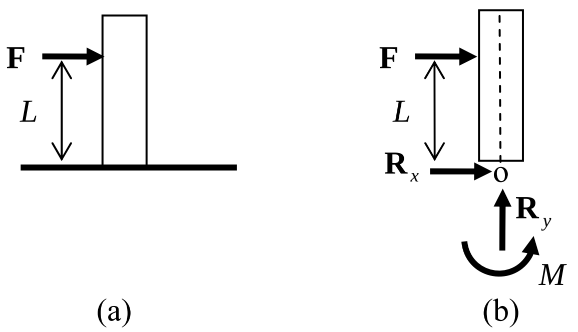


Figure 2.3.9: A loaded cantilevered component; (a) loaded component, (b) free body diagram of the component

The reaction moment of Fig. 2.3.9(b) can be experienced as follows: take a ruler and hold it firmly at one end, upright in your right hand. Simulate the applied force now by pushing against the ruler with a finger of your left hand. You will feel that, to maintain the ruler "vertical" at the base, you need to apply a twist with your right hand, in the direction of the moment shown in Fig. 2.3.9(b).

Note that, when solving this problem, moments were taken about the base. As mentioned already, one can take the moment about *any* point in the column. For example, taking the moment about the point where the force **F** is applied, one has

$$\sum M_{\mathrm{F}} = R_x L + M = 0$$

This of course leads to the same result as before, but the final calculation of the forces is now slightly more complicated; in general, it is easier if the axis is chosen to coincide with the point where the reaction forces act – this is because the reaction forces do not then appear in the moment equation: $\sum M_{\mathrm{o}} = -FL + M = 0$.

■

For ease of discussion, from now on, "couples" such as that encountered in Fig. 2.3.9 will simply be called "moments".

All the elements are now in place to tackle fairly complex static rigid body problems.

Example

Consider the plate subjected to the three external loads shown in Fig. 2.3.10a. The plate is supported by a roller at A and a pin-joint at B. The weight of the plate is assumed to be small relative to the applied loads and is neglected. A free body diagram of the plate is shown in Fig 2.3.10b. This shows all the forces acting *on* the plate. Reactions act at A and B: these forces represent the action of the base *on* the plate, preventing it from moving downward and horizontally. The equilibrium equations can be used to find the reactions:

$$\begin{aligned}
&\sum F_x = F_{xB} = 0 \quad \rightarrow \quad F_{xB} = 0 \\
&\sum F_y = +F_{yA} - 150 - 100 + 50 + F_{yB} = 0 \quad \rightarrow \quad F_{yA} + F_{yB} = 200\,\mathrm{N} \\
&\sum M_{\mathrm{A}} = -(150)(50) - (100)(120) + (50)(200) + F_{yB}(200) = 0 \ \rightarrow \quad F_{yB} = 47.5\,\mathrm{N},
\end{aligned}$$

$$\rightarrow \mathrm{F}_{yA} = 152.5\,\mathrm{N}$$

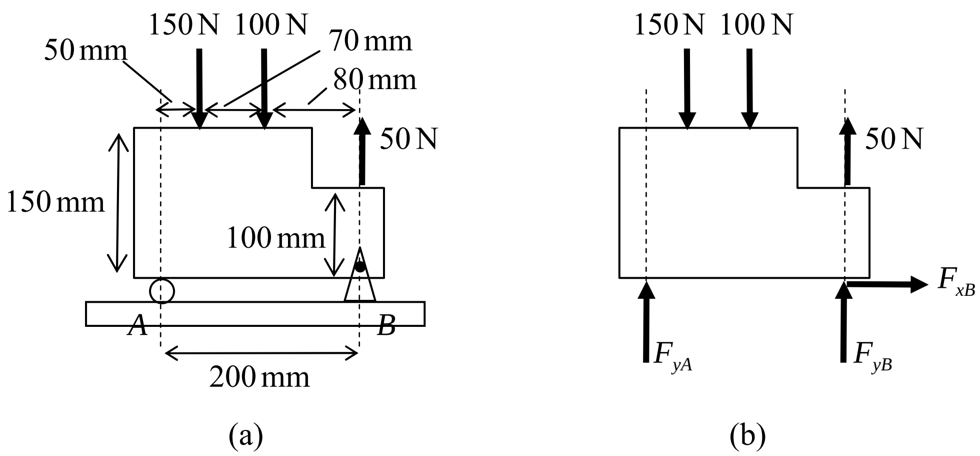


Figure 2.3.10: Equilibrium of a plate; (a) forces acting on the plate, (b) free-body diagram of the plate

The resultant moment was calculated by taking the moment about point A. As mentioned in relation to the previous example, one could have taken the moment about any other point in the plate. The "most convenient" point about which to take moments in this example would be point A or B, since in that case only one of the reaction forces will appear in the moment equilibrium equation.

■

In the above example there were three unknown reactions and three equilibrium equations with which to find them. If the roller was replaced with a pin, there would be four unknown reactions, and now there would not be enough equations with which to find the reactions. When this situation arises, the system is called **statically indeterminate**. To find the unknown reactions, one must relax the assumption of rigidity, and take into account the fact that all materials deform. By calculating deformations within the plate, the reactions can be evaluated. The deformation of materials is studied in the following chapters.

To end this Chapter, note the following:

(i) the equilibrium equations 2.3.3 result from Newton's laws, and are thus as valid for a body of water as they are for a body of hard steel; the external forces acting on a body of still water form a system of forces equivalent to zero.

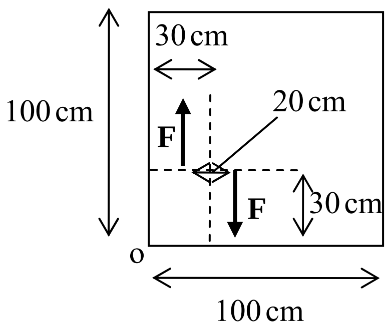
(ii) as mentioned already, Newton's laws apply not only to a complete body or structure, but to *any portion* of a body. The external forces acting *on* any free-body portion of static material form a system of forces equivalent to zero.

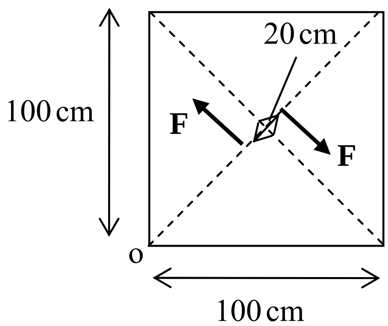
(iii) there is no such thing as a rigid body. Metals and other engineering materials can be considered to be "nearly rigid" as they do not deform by much under even fairly large loads. The analysis carried out in this Chapter is particularly relevant to these materials and in answering questions like: what forces act in the steel members of a suspension bridge under the load of self-weight and traffic? (which is just a more complicated version of the problem of Fig. 2.2.3 or Problem 3 below).

(iv) if the loads on the plate of Fig. 2.3.10a are too large, the plate will "break". The analysis carried out in this Chapter cannot answer where it will break or when it will break. The more sophisticated analysis carried out in the following Chapters is necessary to deal with this and many other questions of material response.

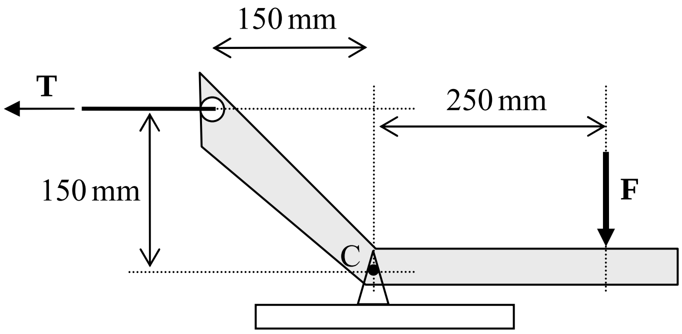
2.3.4 Problems

1. A plate is subjected to a couple Fd, with $d = 20\text{cm}$, as shown below left. Verify that the couple can be moved to the position shown below right, and the effect on the plate is the same, by showing that the moment about point o in both cases is $M = -20F$.

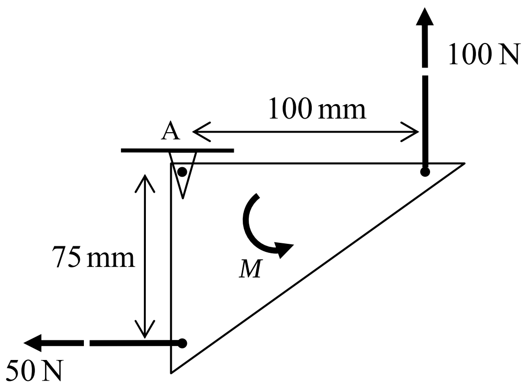




2. What force **F** must be applied to the following static component such that the tension in the cable, **T**, is 1kN? What are the reactions at the pin support C?



3. A machine part is hinged at A and subjected to two forces through cables as shown. What couple *M* needs to be applied to the machine part for equilibrium to be maintained? Where can this couple be applied?



3 Stress

Forces acting at the surfaces of components were considered in the previous chapter. The task now is to examine forces arising *inside* materials, **internal forces**. Internal forces are described using the idea of **stress**. There is a lot more to stress than the notion of "force over area", as will become clear in this chapter. First, the idea of surface (contact) stress distributions will be examined, together with their relationship to resultant forces and moments. Then internal stress and traction will be discussed. The means by which internal forces are described is through the **stress components**, for example σ_{zx}, σ_{yy}, and this "language" of sigmas and subscripts needs to be mastered in order to model sensibly the internal forces in real materials. **Stress analysis** involves representing the actual internal forces in a real physical component mathematically. Some of the limitations of this are discussed in §3.3.2.

Newton's laws are used to derive the **stress transformation equations**, and these are then used to derive expressions for the **principal stresses**, **stress invariants**, **principal directions** and **maximum shear stresses** acting at a material particle. The practical case of two dimensional **plane stress** is discussed.

3.1 Surface and Contact Stress

The concept of the force is fundamental to mechanics and many important problems can be cast in terms of forces only, for example the problems considered in Chapter 2. However, more sophisticated problems require that the action of forces be described in terms of *stress*, that is, force divided by area. For example, if one hangs an object from a rope, it is not the weight of the object which determines whether the rope will break, but the weight divided by the cross-sectional area of the rope, a fact noted by Galileo in 1638.

3.1.1 Stress Distributions

As an introduction to the idea of stress, consider the situation shown in Fig. 3.1.1a: a block of mass m and cross sectional area A sits on a bench. Following the methodology of Chapter 2, an analysis of a free-body of the block shows that a force equal to the weight mg acts upward on the block, Fig. 3.1.1b. Allowing for more detail now, this force will actually be distributed over the surface of the block, as indicated in Fig. 3.1.1c. Defining the stress to be force divided by area, the stress acting on the block is

$$\sigma = \frac{mg}{A} \qquad (3.1.1)$$

The unit of stress is the Pascal (Pa): 1Pa is equivalent to a force of 1 Newton acting over an area of 1 metre squared. Typical units used in engineering applications are the kilopascal, kPa (10^3 Pa), the megapascal, MPa (10^6 Pa) and the gigapascal, GPa (10^9 Pa).

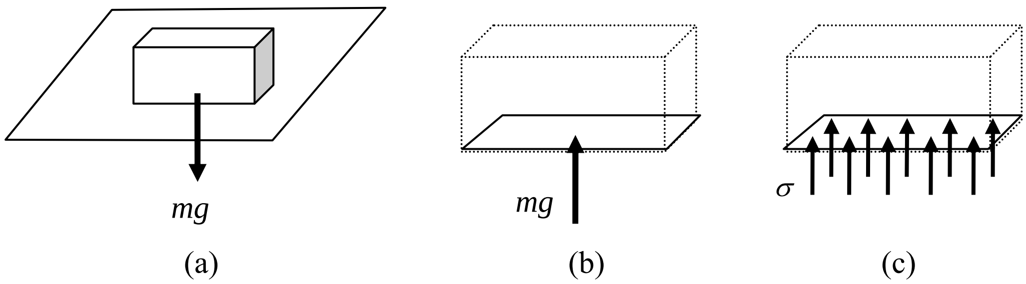


Figure 3.1.1: a block resting on a bench; (a) weight of the block, (b) reaction of the bench on the block, (c) stress distribution acting on the block

The stress distribution of Fig. 3.1.1c acts on the block. By Newton's third law, an equal and opposite stress distribution is exerted by the block on the bench; one says that the weight force of the block is *transmitted* to the underlying bench.

The stress distribution of Fig. 3.1.1 is **uniform**, i.e. constant everywhere over the surface. In more complex and interesting situations in which materials contact, one is more likely to obtain a *non-uniform* distribution of stress. For example, consider the case of a metal ball being pushed into a similarly stiff object by a force F, as illustrated in Fig. 3.1.2.[1] Again, an equal force F acts on the underside of the ball, Fig. 3.1.2b. As with the block, the force will actually be distributed over a **contact region**. It will be shown in Part II that the ball (and the large object) will deform and a circular contact region will arise where the ball and object meet[2], and that the stress is largest at the centre of the contact surface, dying away to zero at the edges of contact, Fig. 3.1.2c ($\sigma_1 > \sigma_2$ in Fig. 3.1.2c). In this case, with stress σ not constant, one can only write, Fig. 3.1.2d,

$$F = \int_A dF = \int_A \sigma dA \tag{3.1.2}$$

The stress varies from point to point over the surface but the sum (or integral) of the stresses (times areas) equals the total force applied to the ball.

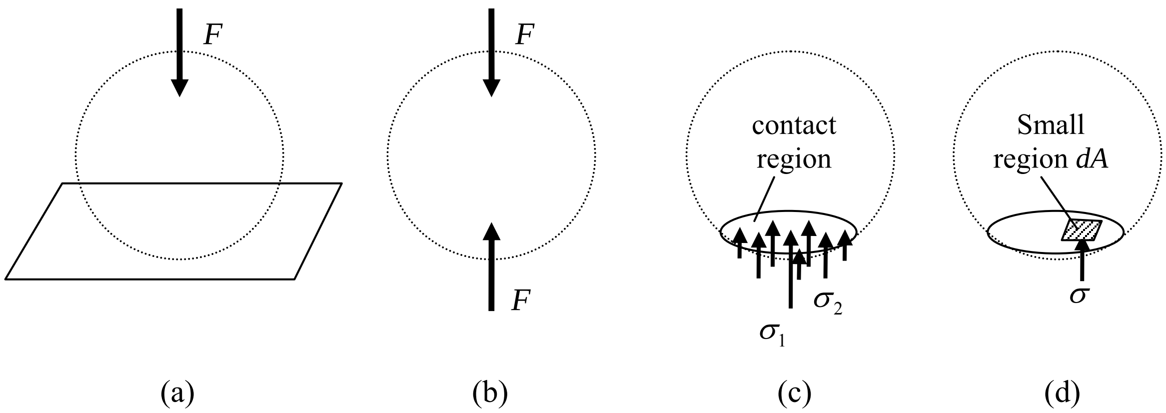


Figure 3.1.2: a ball being forced into a large object, (a) force applied to ball, (b) reaction of object on ball, (c) a non-uniform stress distribution over the contacting surface, (d) the stress acting on a small (infinitesimal) area

A given stress distribution gives rise to a resultant force, which is obtained by integration, Eqn. 3.1.2. It will also give rise to a resultant moment. This is examined in the following example.

Example

Consider the surface shown in Fig. 3.1.3, of length 2m and depth 2m (into the page). The stress over the surface is given by $\sigma = x$ kPa, with x measured in m from the left-hand side of the surface.

The force acting on an element of length dx at position x is (see Fig. 3.1.3b)

$$dF = \sigma\, dA = \left(x\,\mathrm{kPa}\right)\times\left(dx\,\mathrm{m}\times 2\,\mathrm{m}\right)$$

[1] the weight of the ball is neglected here
[2] the radius of which depends on the force applied and the materials in contact

The resultant force is then, from Eqn. 3.1.2

$$F = \int_A dF = 2\int_0^2 x dx \left(\mathrm{kPa\,m^2}\right) = 4\,\mathrm{kN}$$

The moment of the stress distribution is given by

$$M_0 = \int_A dM = \int_A \sigma \times l\, dA \tag{3.1.3}$$

where l is the length of the moment-arm from the chosen axis.

Taking the axis to be at $x = 0$, the moment-arm is $l = x$, Fig. 3.1.3b, and

$$M_{x=0} = \int_A dM = 2\int_0^2 x \times x\, dx \left(\mathrm{kPa\,m^3}\right) = \frac{16}{3}\,\mathrm{kN\,m}$$

Taking moments about the right-hand end, $x = 2$, one has

$$M_{x=2} = \int_A dM = -2\int_0^2 x \times (2 - x) dx \left(\mathrm{kPa\,m^3}\right) = -\frac{8}{3}\,\mathrm{kN\,m}$$

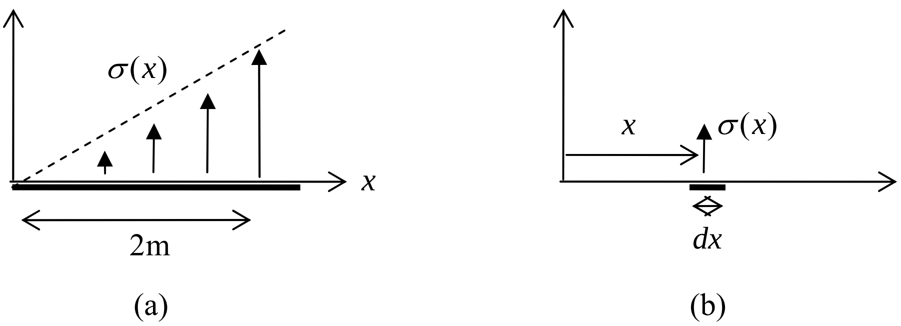


Figure 3.1.3: a non-uniform stress acting over a surface; (a) the stress distribution, (b) stress acting on an element of size d*x*

■

3.1.2 Equivalent Forces and Moments

Sometimes it is useful to replace a stress distribution σ with an **equivalent force** F, i.e. a force equal to the resultant force of the distribution and one which also give the same moment about any axis as the distribution. Formulae for equivalent forces are derived in what follows for triangular and arbitrary linear stress distributions.

Triangular Stress Distribution

Consider the triangular stress distribution shown in Fig. 3.1.4. The stress at the end is σ_0, the length of the distribution is L and the thickness "into the page" is t. The equivalent force is, from Eqn. 3.1.2,

$$F = t\sigma_0 \int_0^L \frac{x}{L} dx = \frac{1}{2}\sigma_0 Lt \qquad (3.1.4)$$

which is just the average stress times area. The point of action of this force should be such that the moment of the force is equivalent to the moment of the stress distribution. Taking moments about the left hand end, for the distribution one has, from 3.1.3,

$$M_o = t\int_0^L x\sigma(x)dx = \frac{1}{3}\sigma_0 L^2 t$$

Placing the force at position $x = x_c$, Fig. 3.1.4, the moment of the force is $M_o = (\sigma_0 Lt/2)x_c$. Equating these expressions leads to the position at which the equivalent force acts:

$$x_c = \frac{2}{3}L. \qquad (3.1.5)$$

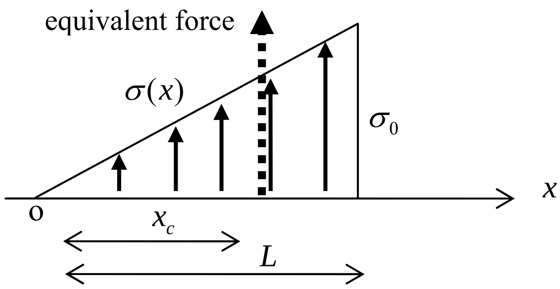


Figure 3.1.4: triangular stress distribution and equivalent force

Note that the moment about *any* axis is now the same for both the stress distribution and the equivalent force.

■

Arbitrary Linear Stress Distribution

Consider the linear stress distribution shown in Fig. 3.1.5. The stress at the ends are σ_1 and σ_2 and this time the equivalent force is

$$F = t\int_0^L [\sigma_1 + (\sigma_2 - \sigma_1)(x/L)]dx = Lt(\sigma_1 + \sigma_2)/2 \qquad (3.1.6)$$

Taking moments about the left hand end, for the distribution one has

$$M_o = t\int_0^L x\sigma(x)dx = L^2 t(\sigma_1 + 2\sigma_2)/6$$

The moment of the force is $M_o = Lt(\sigma_1 + \sigma_2)x_c / 2$. Equating these expressions leads to

$$x_c = \frac{L(\sigma_1 + 2\sigma_2)}{3(\sigma_1 + \sigma_2)} \tag{3.1.7}$$

Eqn. 3.1.5 follows from 3.1.7 by setting $\sigma_1 = 0$.

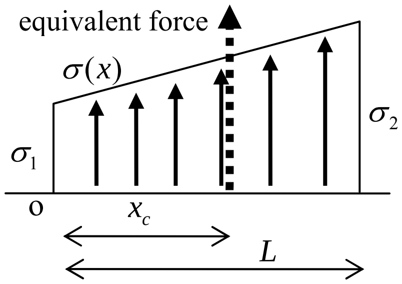


Figure 3.1.5: a non-uniform stress distribution and equivalent force

■

The Centroid

Generalising the above cases, the line of action of the equivalent force for any arbitrary stress distribution $\sigma(x)$ is

$$\boxed{x_c = \frac{t\int x\sigma(x)dx}{t\int \sigma(x)dx} = \frac{\int xdF}{F}} \quad \textbf{Centroid} \tag{3.1.8}$$

This location is known as the **centroid** of the distribution.

Note that most of the discussion above is for two-dimensional cases, i.e. the stress is assumed constant "into the page". Three dimensional problems can be tackled in the same way, only now one must integrate two-dimensionally over a surface rather than one-dimensionally over a line.

Also, the forces considered thus far are *normal* forces, where the force acts perpendicular to a surface, and they give rise to **normal stresses**. Normal stresses are also called **pressures** when they are compressive as in Figs. 3.1.1-2.

3.1.3 Shear Stress

Consider now the case of **shear forces**, that is, forces which act tangentially to surfaces.

A normal force *F* acts on the block of Fig. 3.1.6a. The block does not move and, to maintain equilibrium, the force is resisted by a friction force $F = \mu mg$, where μ is the coefficient of friction. A free body diagram of the block is shown in Fig. 3.1.6b. Assuming a uniform distribution of stress, the stress and resultant force arising on the surfaces of the block and underlying object are as shown. The stresses are in this case called **shear stresses**.

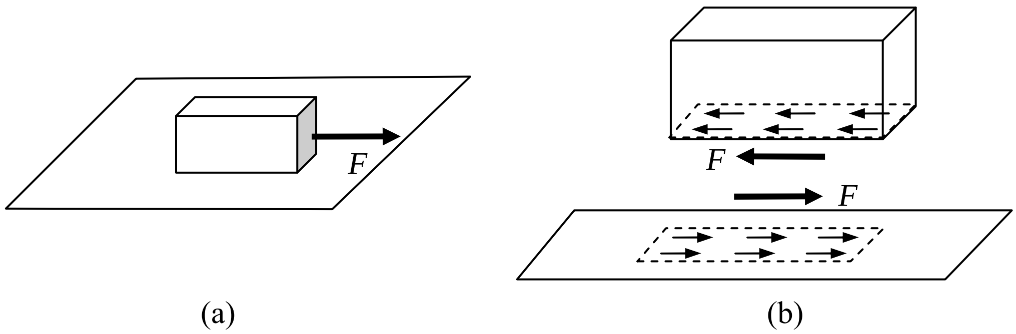


Figure 3.1.6: shear stress; (a) a force acting on a block, (b) shear stresses arising on the contacting surfaces

3.1.4 Combined Normal and Shear Stress

Forces acting inclined to a surface are most conveniently described by decomposing the force into components normal and tangential to the surface. Then one has both normal stress σ_N and shear stress σ_S, as in Fig. 3.1.7.

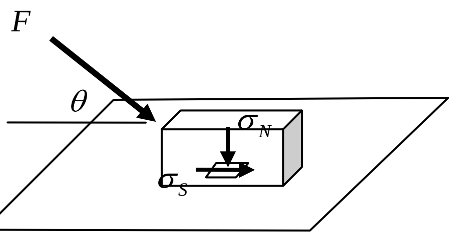
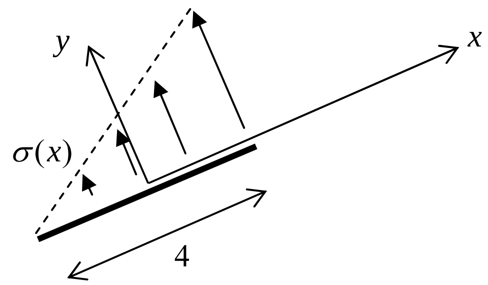


Figure 3.1.7: a force F giving rise to normal and shear stress over the contacting surfaces

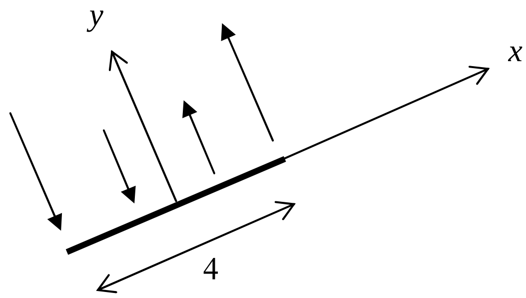
The stresses considered in this section are examples of **surface stresses** or **contact stresses**. They arise when materials meet at a common surface. Other examples would be sea-water pressurising a material in deep water and the stress exerted by a train wheel on a train track.

3.1.5 Problems

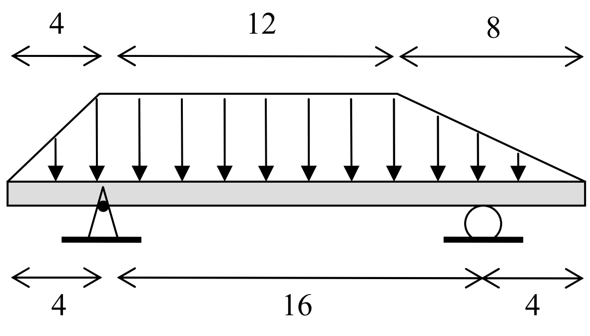
1. Consider the surface shown below, of length 4cm and unit depth (1cm into the page). The stress over the surface is given by $\sigma = 2 + x$ kPa, with x measured in cm from the surface *centre*.
   (a) Evaluate the resultant force acting on the surface (in Newtons).
   (b) What is the moment about an axis (into the page) through the left-hand end of the surface?
   (c) What is the moment about an axis (into the page) through the centre of the surface?



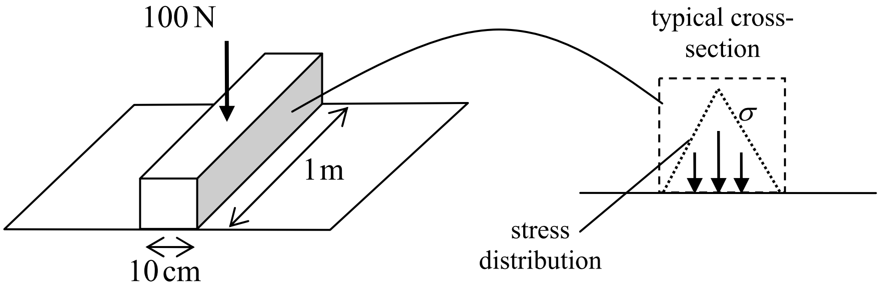
2. Consider the surface shown below, of length 4mm and unit depth (1mm into the page). The stress over the surface is given by $\sigma = x$ MPa, with x measured from the surface centre. What is the total force acting on the surface, and the moment acting about the centre of the surface?



3. Find the reaction forces (per unit length) at the pin and roller for the following beam, which is subjected to a varying pressure distribution, the maximum pressure being $\sigma(x) = 20\,\text{kPa}$ (all lengths are in cm – give answer in N/m)
   [Hint: first replace the stress distribution with three equivalent forces]



4. A block of material of width 10cm and length 1m is pushed into an underlying substrate by a normal force of 100 N. It is found that a uniform triangular normal stress distribution arises at the contacting surfaces, that is, the stress is maximum at the centre and dies off linearly to zero at the block edges, as sketched below right. What is the maximum pressure acting on the surface?



3.2 Body Forces

Surface forces act on surfaces. As discussed in the previous section, these are the forces which arise when bodies are in contact and which give rise to stress distributions. Surface forces also arise *inside* materials, acting on internal surfaces, Fig. 3.2.1a, as will be discussed in the following section.

To complete the description of forces acting on real materials, one needs to deal with forces which arise even when bodies are not in contact; one can think of these forces as *acting at a distance*, for example the force of gravity. To describe these forces, one can define the **body force**, which acts on volume elements of material. Fig. 3.2.1b shows a sketch of a volume element subjected to a magnetic body force and a gravitational body force $\mathbf{F}_g$.

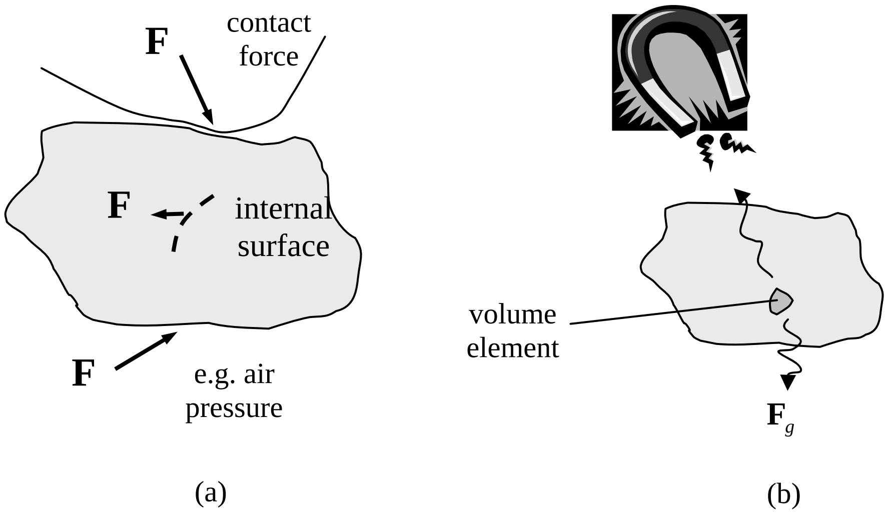


Figure 3.2.1: forces acting on a body; (a) surface forces acting on surfaces, (b) body forces acting on a material volume element

3.2.1 Weight

The most important body force is the force due to gravity, i.e. the weight force. In Chapter 2 there were examples involving the weight of components. In those cases it was simply stated that the weight could be taken to be a single force acting at the component centre (for example, Problem 3 in §2.2.3). This is true when the component is symmetrical, for example, in the shape of a circle or a square. However, it is not true in general for a component of arbitrary shape.

In what follows, the important case of a flat object of arbitrary shape will be examined.

The weight of a small volume element ΔV of material of density ρ is $dF = \rho g \Delta V$ and the total weight is

$$F = \int_V \rho g dV \tag{3.2.1}$$

Consider the general two-dimensional case, Fig. 3.2.2, where material elements of area ΔA_i (and constant thickness *t*) are subjected to forces $\Delta F_i = t\rho g \Delta A_i$.

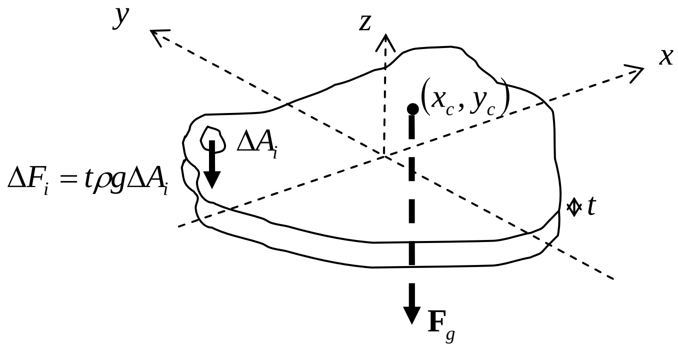


Figure 3.2.2: Resultant Weight on a body

The resultant, i.e. equivalent, weight force due to all elements, for a component with uniform density, is

$$F = \int dF = t\rho g \int dA = \rho g t A,$$

where *A* is the cross-sectional area.

The resultant moments about the *x* and *y* axes, which can be positioned anywhere in the body, are $M_x = t\rho g \int y dA$ and $M_y = t\rho g \int x dA$ respectively; the moment ΔM_x is shown in Fig. 3.2.3. The equivalent weight force is thus positioned at (x_c, y_c), Fig. 3.2.2, where

$$\boxed{x_c = \frac{\int x dA}{A}, \quad y_c = \frac{\int y dA}{A}} \qquad \textbf{Centroid of Area} \tag{3.2.2}$$

The position (x_c, y_c) is called the **centroid of the area**. The quantities $\int x dA$, $\int y dA$, are called the **first moments of area** about, respectively, the *y* and *x* axes.

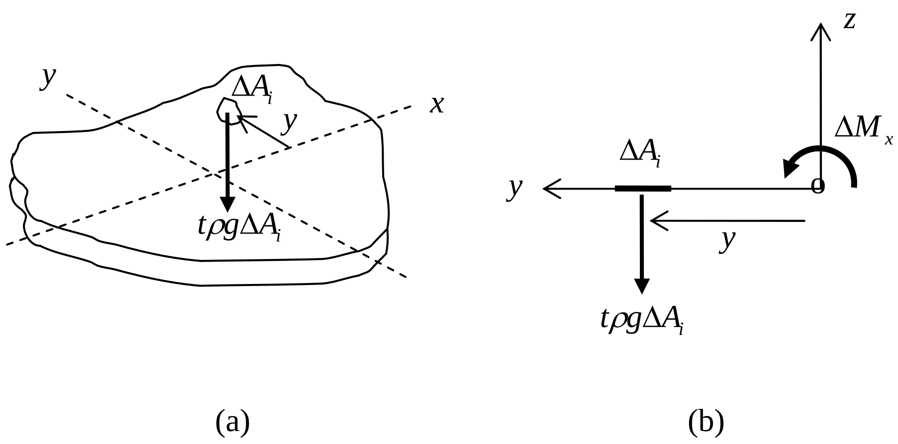
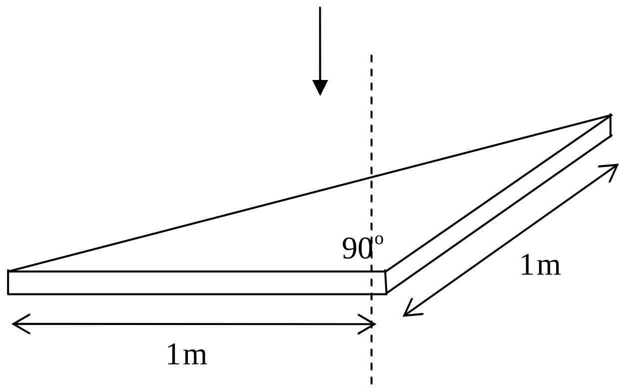


Figure 3.2.3: The moment M_x; (a) full view, (b) plane view

3.2.2 Problems

1. Where does the resultant force due to gravity act in the triangular component shown below? (Gravity acts downward in the direction of the arrow shown, perpendicular to the component's surface)



3.3 Internal Stress

The idea of stress considered in §3.1 is not difficult to conceptualise since objects interacting with other objects are encountered all around us. A more difficult concept is the idea of forces and stresses acting *inside* a material, "within the interior where neither eye nor experiment can reach" as Euler put it. It took many great minds working for centuries on this question to arrive at the concept of stress we use today, an idea finally brought to us by Augustin Cauchy, who presented a paper on the subject to the Academy of Sciences in Paris, in 1822.

Augustin Cauchy

3.3.1 Cauchy's Concept of Stress

Uniform Internal Stress

Consider first a long slender block of material subject to equilibrating forces F at its ends, Fig. 3.3.1a. If the complete block is in equilibrium, then any sub-division of the block must be in equilibrium also. By imagining the block to be cut in two, and considering free-body diagrams of each half, as in Fig. 3.3.1b, one can see that forces F must be acting *within* the block so that each half is in equilibrium. Thus *external loads create internal forces*; internal forces represent the action of one part of a material on another part of the same material across an internal surface. If the material out of which the block is made is uniform over this cut, one can take it that a uniform stress $\sigma = F / A$ acts over this interior surface, Fig. 3.3.1b.

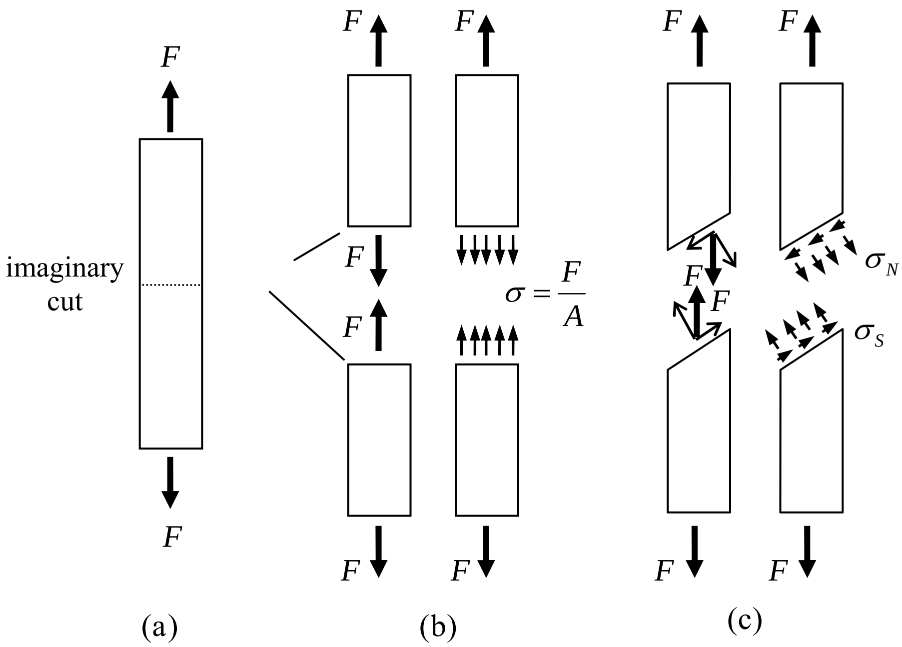


Figure 3.3.1: a slender block of material; (a) under the action of external forces F, (b) internal normal stress σ, (c) internal normal and shear stress

Note that, if the internal forces were not acting over the internal surfaces, the two half-blocks of Fig. 3.3.1b would fly apart; one can thus regard the internal forces as those required to maintain material in an un-cut state.

If the internal surface is at an incline, as in Fig. 3.3.1c, then the internal force required for equilibrium will not act normal to the surface. There will be components of the force normal and tangential to the surface, and thus both normal (σ_N) and shear (σ_S) stresses must arise. Thus, even though the material is subjected to a purely normal load, internal shear stresses develop.

From Fig. 3.3.2a, the normal and shear stresses arising on an interior surface inclined at angle θ to the horizontal are {▲Problem 1}

$$\sigma_N = \frac{F}{A}\cos^2\theta, \qquad \sigma_S = \frac{F}{A}\sin\theta\cos\theta \tag{3.3.1}$$

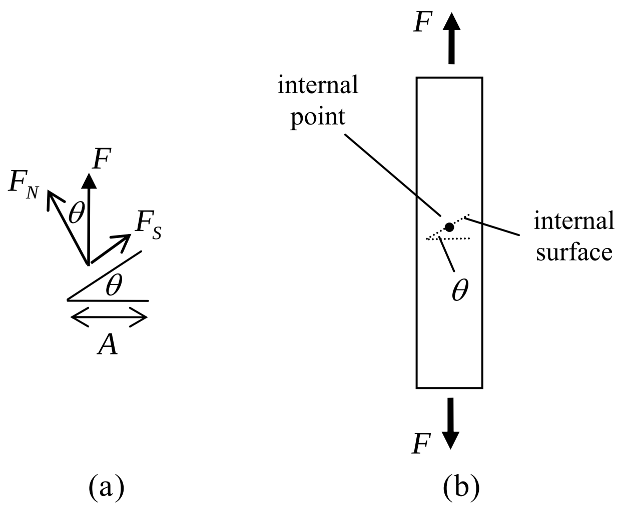


Figure 3.3.2: stress on inclined surface; (a) decomposing the force into normal and shear forces, (b) stress at an internal point

Although stress is associated with surfaces, one can speak of the stress "at a point". For example, consider some point interior to the block, Fig 3.3.2b. The stress there evidently depends on which surface through that point is under consideration. From Eqn. 3.3.1a, the normal stress at the point is a maximum F/A when $\theta = 0$ and a minimum of zero when $\theta = 90^\circ$. The maximum normal stress arising at a point within a material is of special significance, for example it is this stress value which often determines whether a material will fail ("break") there. It has a special name: the **maximum principal stress**. From Eqn. 3.3.1b, the **maximum shear stress** at the point is $\pm F/2A$ and arises on surfaces inclined at $\pm 45^\circ$.

Non-Uniform Internal Stress

Consider a more complex geometry under a more complex loading, as in Fig. 3.3.3. Again, using equilibrium arguments, there will be some stress distribution acting over any given internal surface. To evaluate these stresses is not an easy matter, and much of Part

II is devoted to doing just that. Suffice to say here that they will invariably be *non-uniform* over a surface, that is, the stress at some particle will differ from the stress at a neighbouring particle.

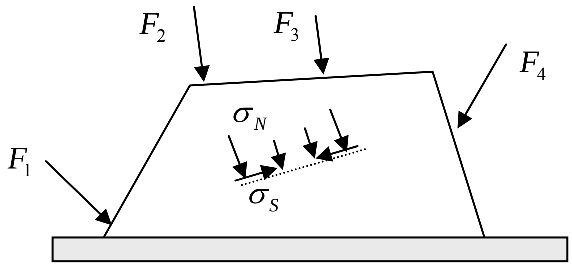


Figure 3.3.3: a component subjected to a complex loading, giving rise to a non-uniform stress distribution over an internal surface

Traction and the Physical Meaning of Internal Stress

All materials have a complex molecular microstructure and each molecule exerts a force on each of its neighbours. The complex interaction of countless molecular forces maintains a body in equilibrium in its unstressed state. When the body is disturbed and deformed into a new equilibrium position, net forces act, Fig. 3.3.4a. An imaginary plane can be drawn through the material, Fig. 3.3.4b. Unlike some of his predecessors, who attempted the extremely difficult task of accounting for all the molecular forces, Cauchy discounted the molecular structure of matter and simply replaced the molecular forces acting on the plane by a single force **F**, Fig 3.3.4c. This is the force exerted by the molecules above the plane *on* the material below the plane and can be attractive or repulsive. Different planes can be taken through the *same* portion of material and, in general, a *different* force will act on the plane, Fig 3.3.4d.

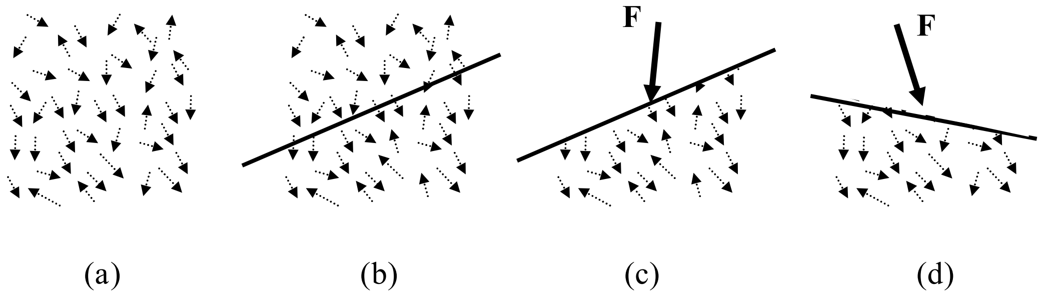


Figure 3.3.4: a multitude of molecular forces represented by a single force; (a) molecular forces, a plane drawn through the material, replacing the molecular forces with an equivalent force F, a different equivalent force F acts on a different plane through the same material

The definition of stress will now be made more precise. First, define the **traction** at some particular point in a material as follows: take a plane of surface area *S* through the point, on which acts a force *F*. Next shrink the plane – as it shrinks in size both *S* and *F* get smaller, and the direction in which the force acts may change, but eventually the ratio F/S will remain constant and the force will act in a particular direction, Fig. 3.3.5. The limiting value of this ratio of force over surface area is defined as the **traction vector** (or **stress vector**) **t**:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \tag{3.3.2}$$

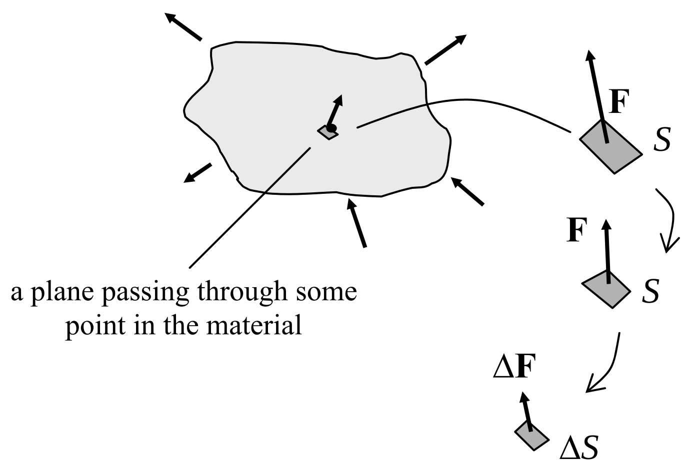
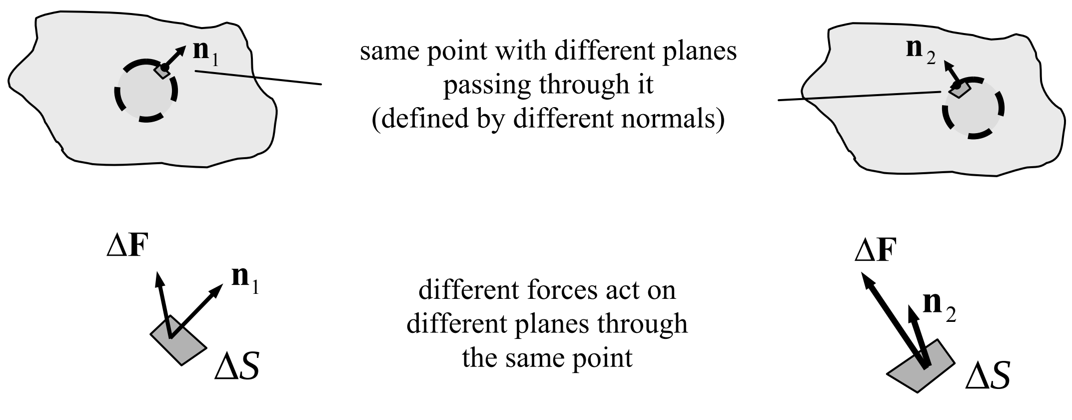


Figure 3.3.5: the traction vector - the limiting value of force over area, as the surface area of the element on which the force acts is shrunk

An infinite number of traction vectors act at any single point, since an infinite number of different planes pass through a point. Thus the notation $\lim_{\Delta S \to 0} \Delta F / \Delta S$ is ambiguous. For this reason the plane on which the traction vector acts must be specified; this can be done by specifying the normal **n** to the surface on which the traction acts, Fig 3.3.6. The traction is thus a special vector – associated with it is not only the direction in which it acts but also a second direction, the normal to the plane upon which it acts.



$$\mathbf{t}^{(\mathbf{n}_1)} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} \qquad \mathbf{t}^{(\mathbf{n}_2)} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S}$$

Figure 3.3.6: two different traction vectors acting at the same point

Stress Components

The traction vector can be decomposed into components which act normal and parallel to the surface upon which it acts. These components are called the **stress components**, or simply **stresses**, and are denoted by the symbol σ; subscripts are added to signify the surface on which the stresses act and the directions in which the stresses act.

Consider a particular traction vector acting on a surface element. Introduce a Cartesian coordinate system with base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ so that one of the base vectors is a normal to the surface, and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 3.3.7, the $\mathbf{k}$ direction is taken to be normal to the plane, and $\mathbf{t}^{(\mathbf{k})} = t_x \mathbf{i} + t_y \mathbf{j} + t_z \mathbf{k}$.

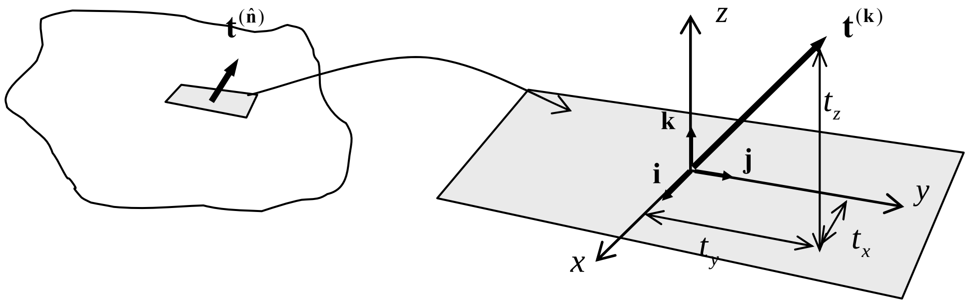


Figure 3.3.7: the components of the traction vector

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the *direction of the normal* to the plane and the second denotes the *direction of the component*. Thus, re-drawing Fig. 3.3.7 as Fig. 3.3.8: $\mathbf{t}^{(\mathbf{k})} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$. The first two stresses, the components acting tangential to the surface, are shear stresses, whereas σ_{zz}, acting normal to the plane, is a normal stress[1].

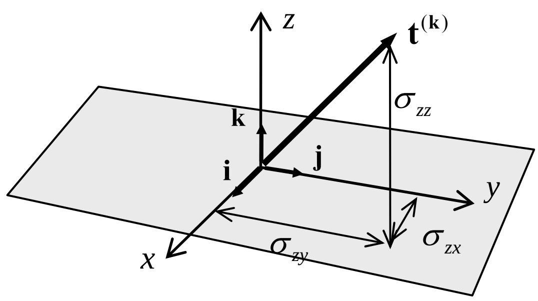


Figure 3.3.8: stress components – the components of the traction vector

The traction vector shown in Figs. 3.3.7, 3.3.8, represents the force (per unit area) exerted by the material above the surface *on* the material below the surface. By Newton's third

[1] this convention for the subscripts is not universally followed. Many authors, particularly in the mathematical community, use the exact opposite convention, the first subscript to denote the direction and the second to denote the normal. It turns out that *both conventions are equivalent*, since, as will be shown later, the stress is symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$

law, an equal and opposite traction must be exerted by the material below the surface on the material above the surface, as shown in Fig. 3.3.9 (thick dotted line). If $\mathbf{t}^{(\mathbf{k})}$ has stress components $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$, then so should $\mathbf{t}^{(-\mathbf{k})}$: $\mathbf{t}^{(-\mathbf{k})} = \sigma_{zx}(-\mathbf{i}) + \sigma_{zy}(-\mathbf{j}) + \sigma_{zz}(-\mathbf{k}) = -\mathbf{t}^{(\mathbf{k})}$.

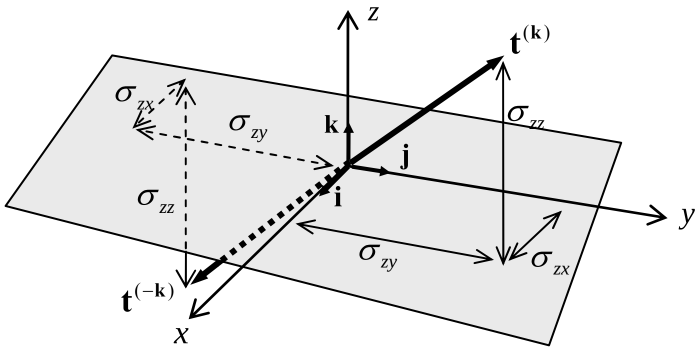


Figure 3.3.9: equal and opposite traction vectors – each with the same stress components

Sign Convention for Stress Components

The following convention is used:

> The stress is *positive* when the direction of the normal *and* the direction of the stress component are both positive *or* both negative
> The stress is *negative* when one of the directions is positive and the other is negative

According to this convention, the three stresses in Figs. 3.3.7-9 are all positive.

Looking at the two-dimensional case for ease of visualisation, the (positive and negative) normal stresses and shear stresses on either side of a surface are as shown in Fig. 3.3.10. Normal stresses which "pull" (tension) are positive; normal stresses which "push (compression) are negative. Note that the shear stresses always go in opposite directions.

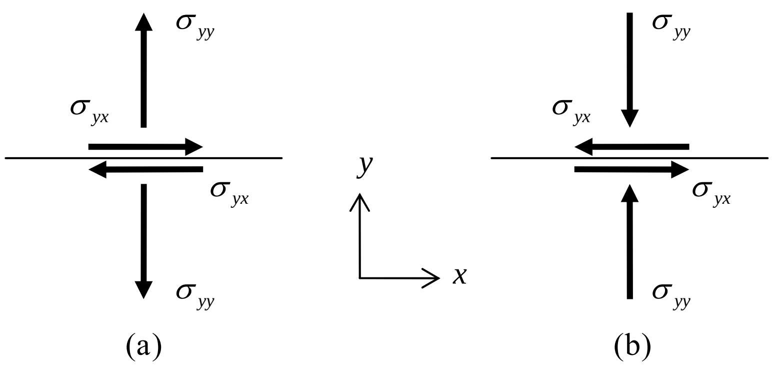
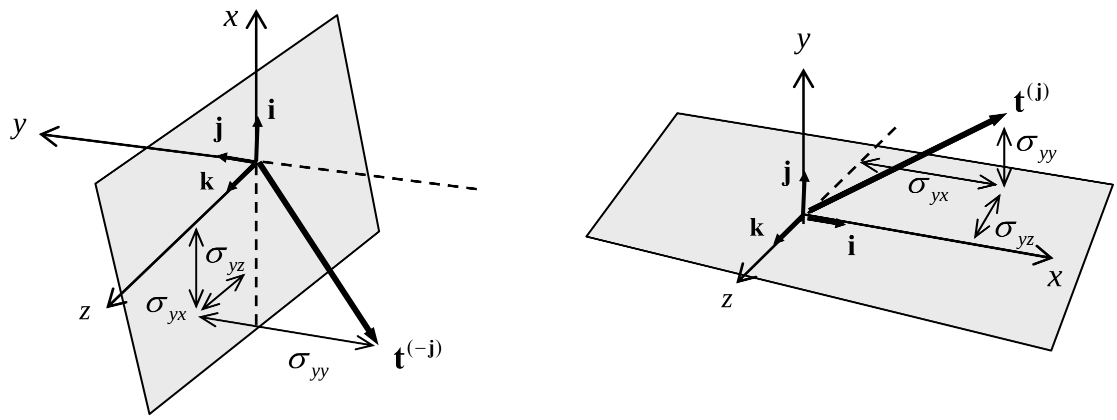


Figure 3.3.10: stresses acting on either side of a material surface: (a) positive stresses, (b) negative stresses

Examples of negative stresses are shown in Fig. 3.3.11 {▲Problem 4}.



$$\mathbf{t}^{(-\mathbf{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$$

(a)

$$\mathbf{t}^{(\mathbf{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$$

(b)

Figure 3.3.11: examples of negative stress components

3.3.2 Real Problems and Saint-Venant's Principle

Some examples have been given earlier of external forces acting on materials. In reality, an external force will be applied to a real material component in a complex way. For example, suppose that a block of material, welded to a large object at one end, is pulled at its other end by a rope attached to a metal hoop, which is itself attached to the block by a number of bolts, Fig. 3.3.12a. The block can be idealised as in Fig 3.3.12b; here, the precise details of the region in which the external force is applied are neglected.

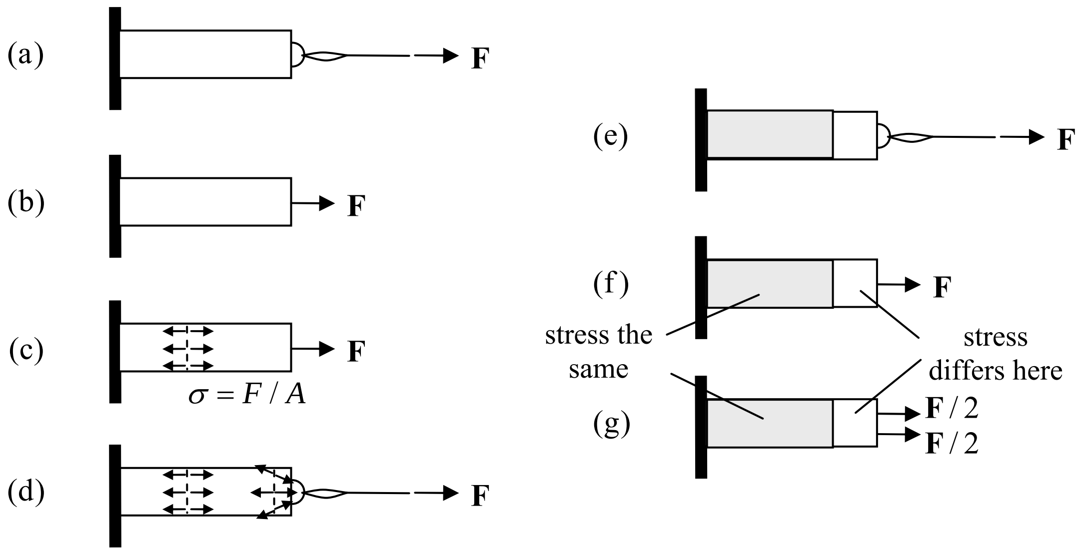


Figure 3.3.12: a block subjected to an external force: (a) real case, (b) ideal model, (c) stress in ideal model, (d) stress in actual material, (e) the stress in the real material, away from the right hand end, is modelled well by either (f) or (g)

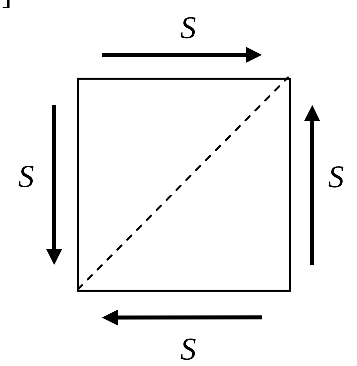
According to the earlier discussion, the stress in the ideal model is as in Fig. 3.3.12c. One will find that, in the *real* material, the stress is indeed (approximately) as predicted, but only at an appreciable distance from the right hand end. Near where the rope is attached, the force will differ considerably, as sketched in Fig.3.3.12d.

Thus the ideal models of the type discussed in this section, and in much of this book, are useful only in predicting the stress field in real components in regions away from points of application of loads. This does not present too much of a problem, since the stresses internal to a structure in such regions are often of most interest. If one wants to know what happens near the bolted connection, then one will have to create a complex model incorporating all the details and the problem will be more difficult to solve.

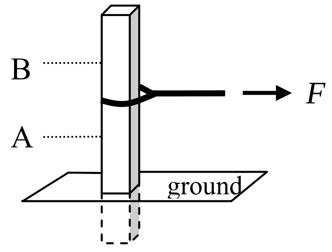
It is an *experimental* fact that if two different force systems are applied to a material, but they are *equivalent force systems*, as in Fig. 3.3.12(f,g), then the stress fields in regions away from where the loads are applied will be the same. This is known as **Saint-Venant's Principle**. Typically, one needs to move a distance away from where the loads are applied roughly equal to the distance over which the loads are applied.

3.3.3 Problems

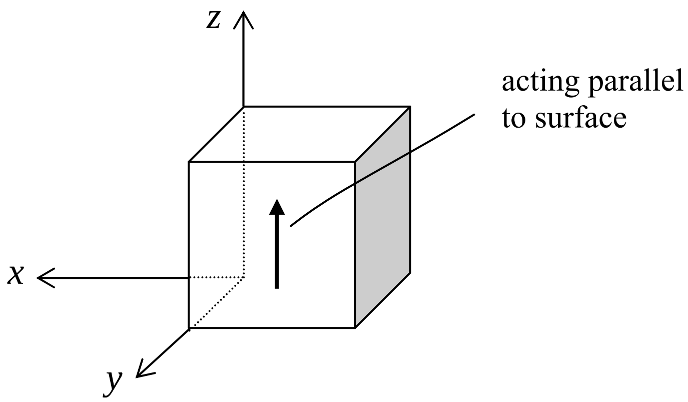
1. Derive Eqns. 3.3.1.
2. The four sides of a square block are subjected to equal *forces S*, as illustrated. The length of each side is l and the block has unit depth (into the page). What normal and shear *stresses* act along the (dotted) diagonal? [Hint: draw a free body diagram of the upper left hand triangle.]



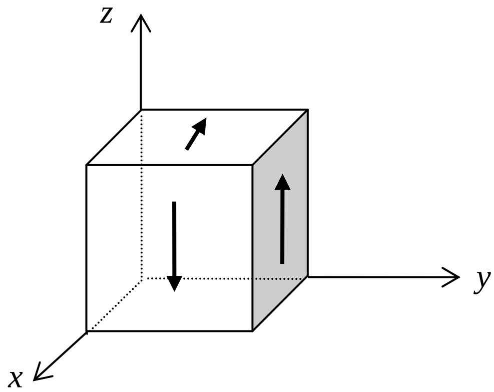
3. A shaft is concreted firmly into the ground. A thick steel rope is looped around the shaft and a force is applied normal to the shaft, as shown. The shaft is in static equilibrium. Draw a free body diagram of the shaft (from the top down to ground level) showing the forces/moments acting on the shaft (including the reaction forces at the ground-level; ignore the weight of the shaft). Draw a free body diagram of the section of shaft from the top down to the cross section at A. Draw a free body diagram of the section of shaft from the top to the cross section at B. Roughly sketch the stresses acting over the (horizontal) internal surfaces of the shaft at A and B.



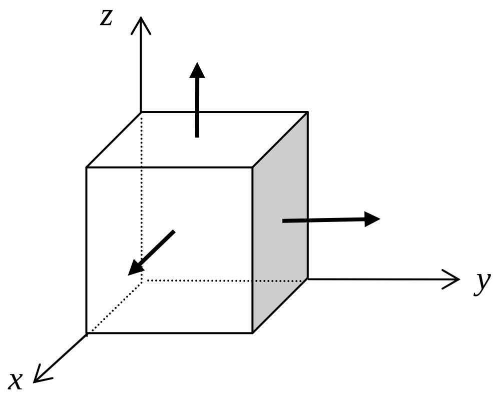
4. In Fig. 3.3.11, which of the stress components is/are negative?
5. Label the following stress component acting on an internal material surface. Is it a positive or negative stress?



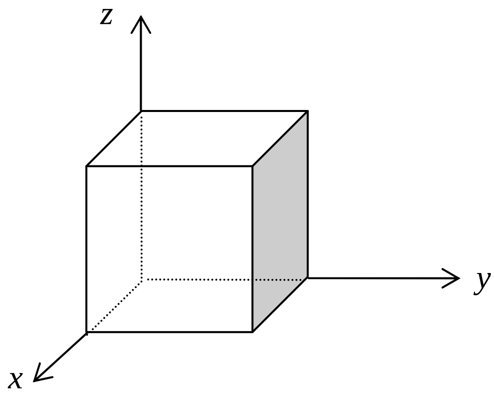
6. Label the following shear stresses. Are they positive or negative?



7. Label the following normal stresses. Are they positive or negative?



8. By the definition of the traction vector **t** which acts on the $x-z$ plane, $\mathbf{t}^{(\mathrm{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$. Sketch these three stress components on the figure below.


z
y
x

3.4 Equilibrium of Stress

Consider two perpendicular planes passing through a point *p*. The stress components acting on these planes are as shown in Fig. 3.4.1a. These stresses are usually shown together acting on a small material element of finite size, Fig. 3.4.1b. It has been seen that the stress may vary from point to point in a material but, if the element is very small, the stresses on one side can be taken to be (more or less) equal to the stresses acting on the other side. By convention, in analyses of the type which will follow, all stress components shown are *positive*.

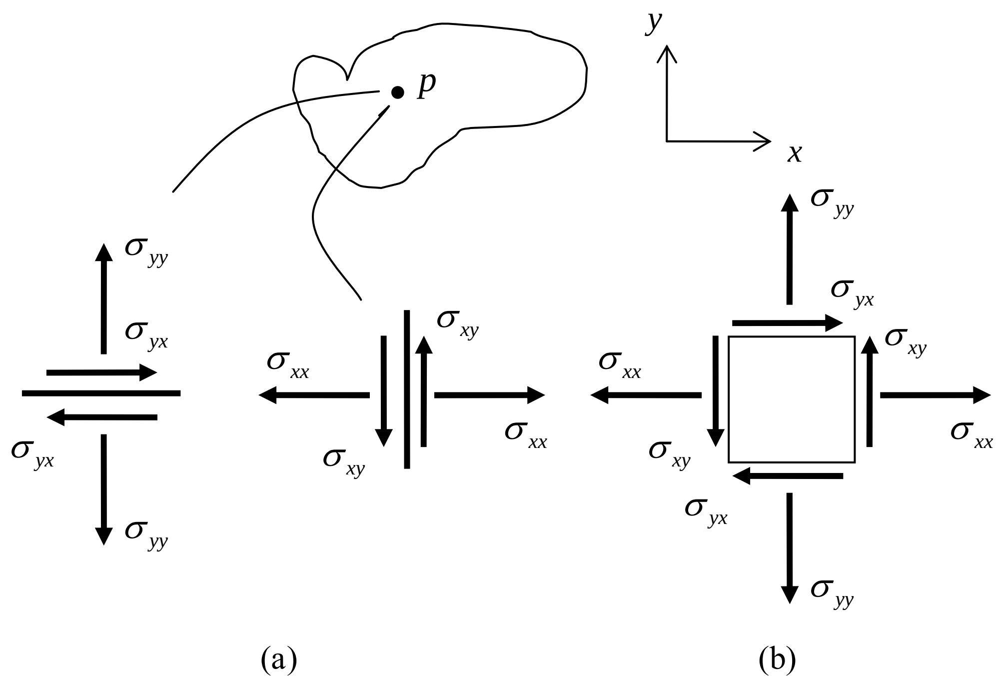


Figure 3.4.1: stress components acting on two perpendicular planes through a point; (a) two perpendicular surfaces at a point, (b) small material element at the point

The four stresses can conveniently be written in the matrix form:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \qquad (3.4.1)$$

It will be shown below that the stress components acting on *any* other plane through *p* can be evaluated from a knowledge of only these stress components.

3.4.1 Symmetry of the Shear Stress

Consider the material element shown in Fig. 3.4.1b, reproduced in Fig. 3.4.2a below. The element has dimensions is $\Delta x \times \Delta y$ and is subjected to uniform stresses over its sides. The resultant forces of the stresses acting on each side of the element act through the side-centres, and are shown in Fig. 3.4.2b. The stresses shown are positive, but note how *positive stresses can lead to negative forces*, depending on the definition of the $x-y$ axes used. The resultant force on the complete element is seen to be zero.

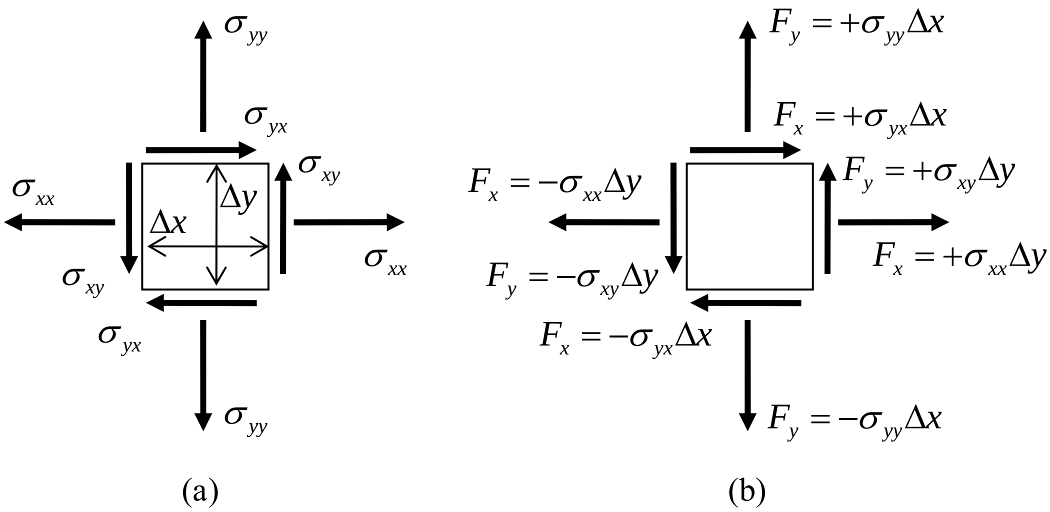


Figure 3.4.2: stress components acting on a material element; (a) stresses, (b) resultant forces on each side

By taking moments about any point in the block, one finds that {▲Problem 1}

$$\sigma_{xy} = \sigma_{yx} \tag{3.4.2}$$

Thus the shear stresses acting on the element are all equal, and for this reason the σ_{yx} stresses are usually labelled σ_{xy}, Fig. 3.4.3a, or simply labelled τ, Fig. 3.4.3b.

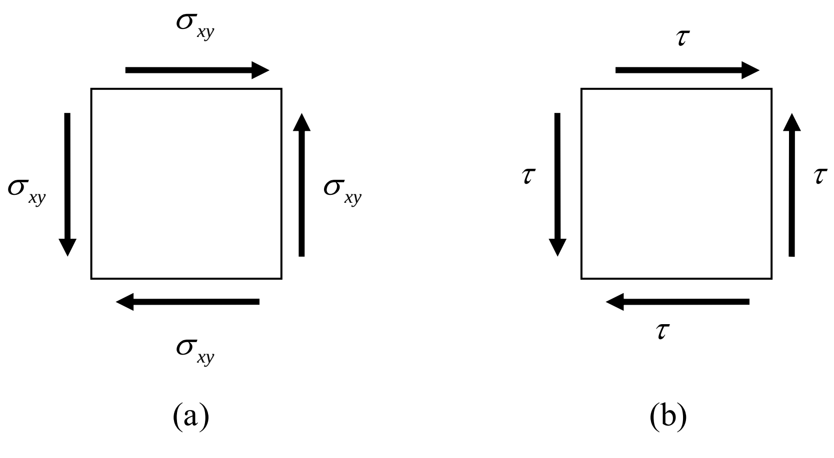


Figure 3.4.3: shear stress acting on a material element

3.4.2 Three Dimensional Stress

The three-dimensional counterpart to the two-dimensional element of Fig. 3.4.2 is shown in Fig. 3.4.4. Again, all stresses shown are positive.

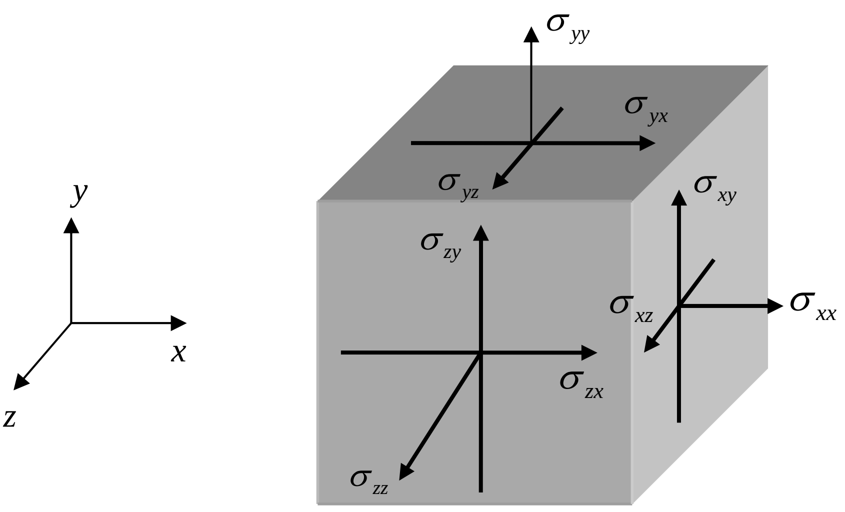


Figure 3.4.4: a three dimensional material element

Moment equilibrium in this case requires that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy} \tag{3.4.3}$$

The nine stress components, six of which are independent, can now be written in the matrix form

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{3.4.4}$$

A vector **F** has one direction associated with it and is characterised by *three* components (F_x, F_y, F_z). The stress is a quantity which has two directions associated with it (the direction of a force and the normal to the plane on which the force acts) and is characterised by the *nine* components of Eqn. 3.4.4. Such a mathematical object is called a **tensor**. Just as the three components of a vector change with a change of coordinate axes (for example, as in Fig. 2.2.1), so the nine components of the **stress tensor** change with a change of axes. This is discussed in the next section for the two-dimensional case. (The concept of a tensor will be examined more closely in Books II and especially IV.)

3.4.3 Stress Transformation Equations

Consider the case where the nine stress components acting on three perpendicular planes through a material particle are known. These components are σ_{xx}, σ_{xy}, etc. when using x, y, z axes, and can be represented by the cube shown in Fig. 3.4.5a. Rotate now the planes about the three axes – these new planes can be represented by the rotated cube shown in Fig. 3.4.5b; the axes normal to the planes are now labelled x', y', z' and the corresponding stress components with respect to these new axes are $\sigma'_{xx}, \sigma'_{xy}$, etc.

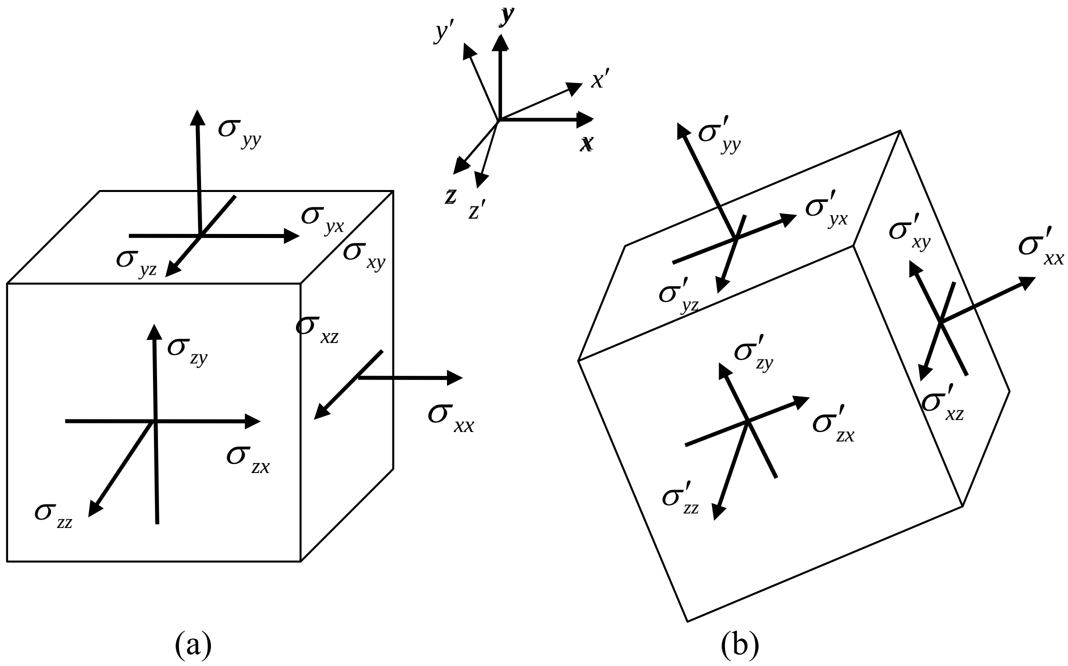


Figure 3.4.5: a three dimensional material element; (a) original element, (b) rotated element

There is a relationship between the stress components σ_{xx}, σ_{xy}, etc. and the stress components $\sigma'_{xx}, \sigma'_{xy}$, etc. The relationship can be derived using Newton's Laws. The equations describing the relationship in the fully three-dimensional case are very lengthy – they will be discussed in Books II and IV. Here, the relationship for the two-dimensional case will be derived – this 2D relationship will prove very useful in analysing many practical situations.

Two-dimensional Stress Transformation Equations

Assume that the stress components of Fig. 3.4.6a are known. It is required to find the stresses arising on other planes through *p*. Consider the perpendicular planes shown in Fig. 3.4.65b, obtained by rotating the original element through a positive (counterclockwise) angle θ. The new surfaces are defined by the axes $x' - y'$.

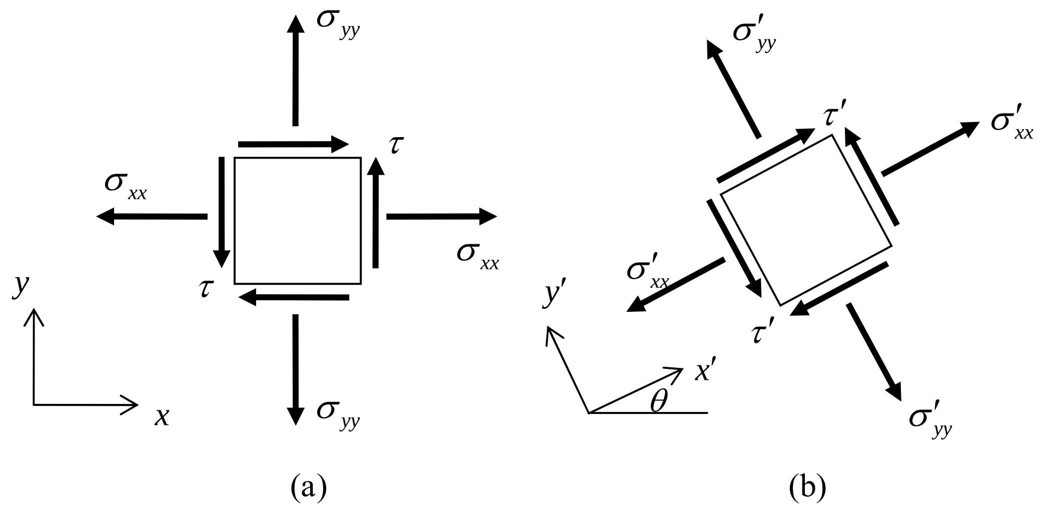


Figure 3.4.6: stress components acting on two different sets of perpendicular surfaces, i.e. in two different coordinate systems; (a) original system, (b) rotated system

To evaluate these new stress components, consider a triangular element of material at the point, Fig. 3.4.7. Carrying out force equilibrium in the direction x', one has (with unit depth into the page)

$$\sum F_{x'}: \quad \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta = 0 \qquad (3.4.5)$$

Since $|OB| = |AB|\cos\theta,\ |OA| = |AB|\sin\theta$, and dividing through by $|AB|$,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \qquad (3.4.6)$$

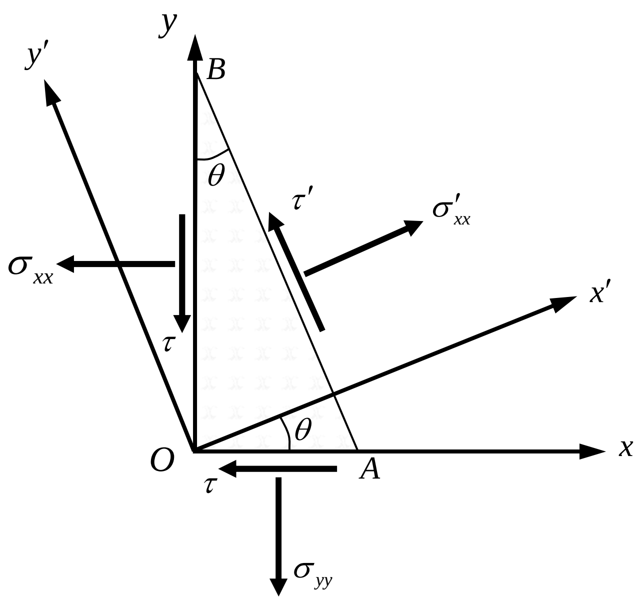


Figure 3.4.7: a free body diagram of a triangular element of material

The forces can also be resolved in the y' direction and one obtains the relation

$$\tau' = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau\cos 2\theta \qquad (3.4.7)$$

Finally, consideration of the element in Fig. 3.4.8 yields two further relations, one of which is the same as Eqn. 3.4.6.

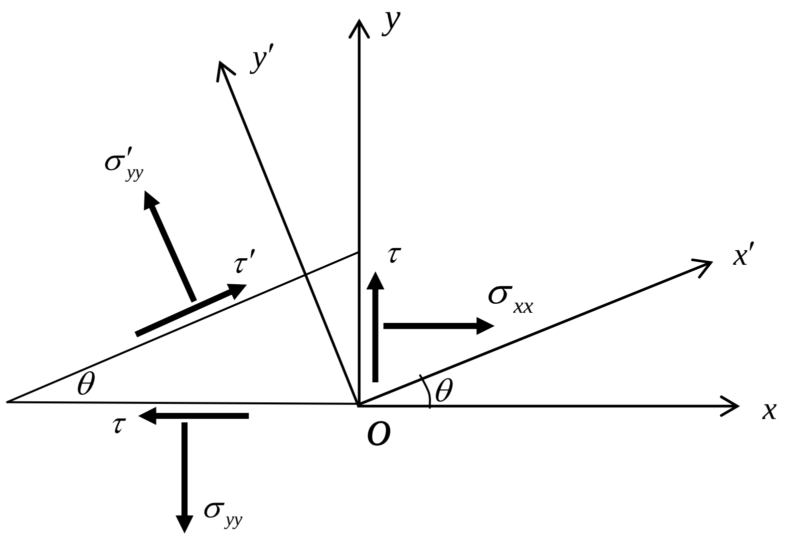


Figure 3.4.8: a free body diagram of a triangular element of material

In summary, one obtains the **stress transformation equations**:

$$\boxed{\begin{aligned} \sigma'_{xx} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\ \sigma'_{yy} &= \sin^2\theta\sigma_{xx} + \cos^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\ \sigma'_{xy} &= \sin\theta\cos\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy} \end{aligned}}$$

2D Stress Transformation Equations (3.4.8)

These equations have many uses, as will be seen in the next section.

In matrix form,

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (3.4.9)$$

Body Force, Acceleration and Non-Uniform Stress

Here, it will be shown that the Stress Transformation Equations are valid also when (i) there are body forces, (ii) the body is accelerating and (iii) the stress and other quantities are not uniform.

Suppose that a body force $\mathbf{F}_b = (\mathbf{F}_b)_x \mathbf{i} + (\mathbf{F}_b)_y \mathbf{j}$ acts on the material and that the material is accelerating with an acceleration $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$. The components of body force and acceleration are shown in Fig. 3.4.9 (a reproduction of Fig. 3.4.7). The body force will vary depending on the size of the material under consideration, e.g. the force of gravity $\mathbf{F}_b = m\mathbf{g}$ will be larger for larger materials; therefore consider a quantity which is independent of the amount of material: the body force per unit mass, $\mathbf{F}_b / m$. Then, Eqn 3.4.5 now reads

$$\begin{aligned} \sum F_{x'} : \quad & \sigma'_{xx} |AB| - \sigma_{xx} |OB| \cos\theta - \sigma_{yy} |OA| \sin\theta - \tau |OB| \sin\theta - \tau |OA| \cos\theta \\ & + (\mathbf{F}_b / m)_x \, m \cos\theta + (\mathbf{F}_b / m) m \sin\theta + m a_x \cos\theta + m a_y \sin\theta = 0 \end{aligned} \tag{3.4.10}$$

where *m* is the mass of the triangular portion of material. The volume of the triangle is $|AB|^2 / \sin 2\theta$ so that, this time, when 3.4.10 is divided through by $|AB|$, one has

$$\begin{aligned} \sigma'_{xx} &= \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + \tau \sin 2\theta \\ & \quad - |AB| \rho \left\{ (\mathbf{F}_b / m)_x / 2\sin\theta + (\mathbf{F}_b / m) / 2\cos\theta + a_x / 2\sin\theta + a_y / 2\cos\theta \right\} \end{aligned} \tag{3.4.11}$$

where ρ is the density. Now, as the element is shrunk in size down to the vertex *O*, $|AB| \to 0$, and Eqn. 3.4.6 is recovered. Thus the Stress Transformation Equations are valid provided the material under consideration is very small; in the limit, they are valid "at the point" *O*.

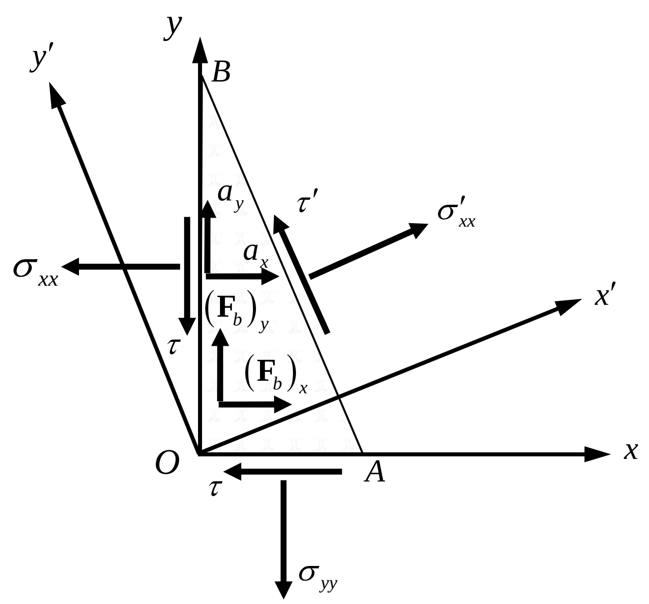


Figure 3.4.9: a free body diagram of a triangular element of material, including a body force and acceleration

Finally, consider the case where the stress is not uniform over the faces of the triangular portion of material. Intuitively, it can be seen that, if one again shrinks the portion of material down in size to the vertex *O*, the Stress Transformation Equations will again be valid, with the quantities $\sigma'_{xx}, \sigma_{xx}, \sigma_{yy}$ etc. being the values "at" the vertex. To be more precise, consider the σ_{xx} stress acting over the face $|OB|$ in Fig. 3.4.10. No matter how the stress varies in the material, if the distance $|OB|$ is small, the stress can be approximated by a linear stress distribution, Fig. 3.4.10b. This linear distribution can itself be decomposed into two components, a uniform stress of magnitude σ^o_{xx} (the value of σ_{xx} at the vertex) and a triangular distribution with maximum value $\Delta\sigma_{xx}$. The resultant force on the face is then $|OB|\left(\sigma^o_{xx} + \Delta\sigma_{xx}/2\right)$. This time, as the element is shrunk in size, $\Delta\sigma_{xx} \to 0$ and Eqn. 3.4.6 is again recovered. The same argument can be used to show that the Stress Transformation Equations are valid for any varying stress, body force or acceleration.

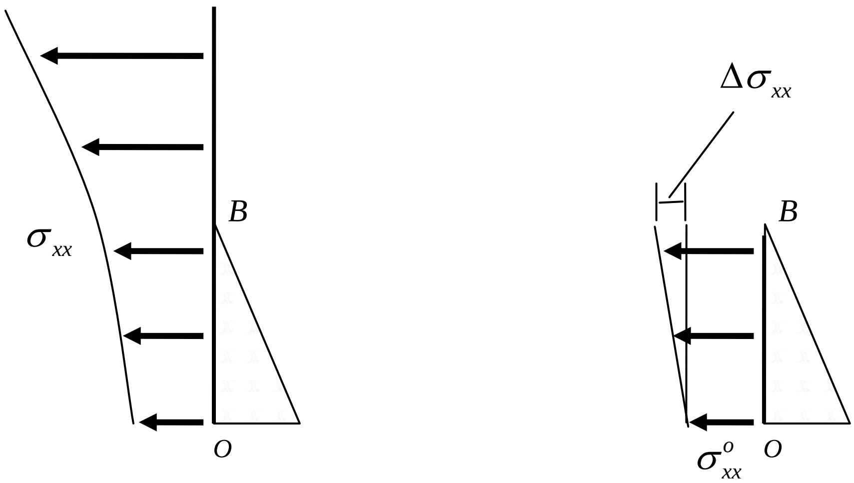


Figure 3.4.10: stress varying over a face; (a) stress is linear over OB if OB is small, (b) linear distribution of stress as a uniform stress and a triangular stress

Three Dimensions Re-visited

As the planes were rotated in the two-dimensional analysis, no consideration was given to the stresses acting in the "third dimension". Considering again a three dimensional block, Fig. 3.4.11, there is only one traction vector acting *on* the $x-y$ plane at the material particle, **t**. This traction vector can be described in terms of the x, y, z axes as $\mathbf{t} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$, Fig 3.4.11a. Alternatively, it can be described in terms of the x', y', z' axes as $\mathbf{t} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}' + \sigma'_{zz}\mathbf{k}'$, Fig 3.4.11b.

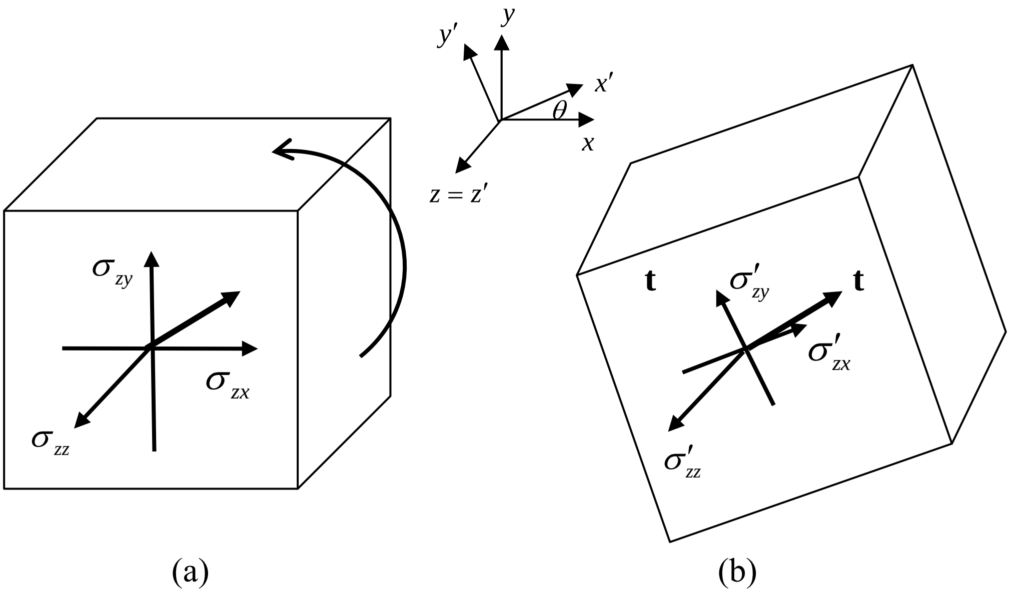


Figure 3.4.11: a three dimensional material element; (a) original element, (b) rotated element (rotation about the *z* axis)

With the rotation only happening *in* the $x-y$ plane, about the *z* axis, one has $\sigma_{zz} = \sigma'_{zz}$, $\mathbf{k} = \mathbf{k}'$. One can thus examine the two dimensional $x-y$ plane shown in Fig. 3.4.12, with

$$\sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}'. \tag{3.4.12}$$

Using some trigonometry, one can see that

$$\begin{aligned} \sigma'_{zx} &= +\sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta \\ \sigma'_{zy} &= -\sigma_{zx}\sin\theta + \sigma_{zy}\cos\theta \end{aligned}. \tag{3.4.13}$$

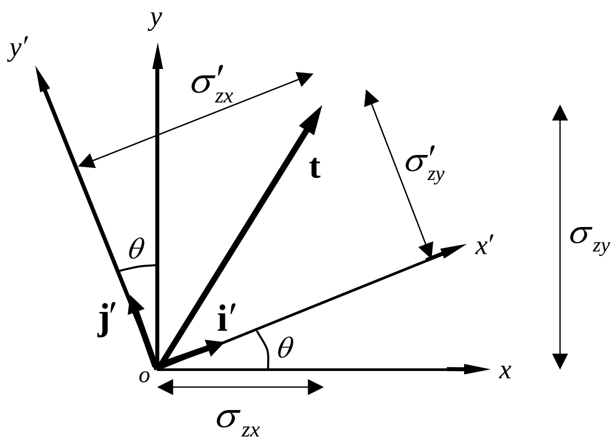


Figure 3.4.12: the traction vector represented using two different coordinate systems

3.4.4 Problems

1. Derive Eqns. 3.4.2 by taking moments about the lower left corner of the block in Fig. 3.4.2.

2. Suppose that the stresses acting on two perpendicular planes through a point are
$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$
Use the stress transformation formulae to evaluate the stresses acting on two new perpendicular planes through the point, obtained from the first set by a positive rotation of 30°. Use the conventional notation $x' - y'$ to represent the coordinate axes parallel to these new planes.

3.5 Plane Stress

This section is concerned with a special two-dimensional state of stress called **plane stress**. It is important for two reasons: (1) it arises in real components (particularly in thin components loaded in certain ways), and (2) it is a two dimensional state of stress, and thus serves as an excellent introduction to more complicated three dimensional stress states.

3.5.1 Plane Stress

The state of plane stress is defined as follows:

Plane Stress:
If the stress state at a material particle is such that the only non-zero stress components act in one plane only, the particle is said to be in plane stress.

The axes are usually chosen such that the $x-y$ plane is the plane in which the stresses act, Fig. 3.5.1.

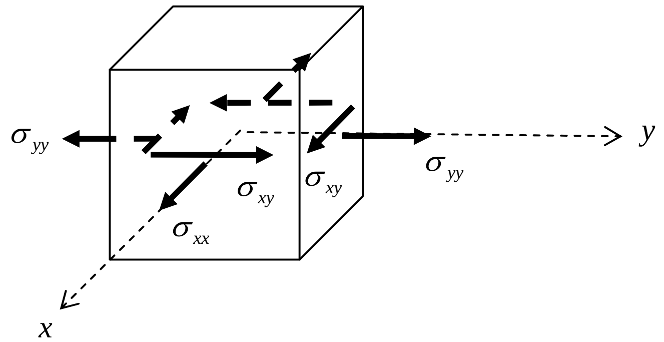


Figure 3.5.1: non-zero stress components acting in the $x-y$ plane

The stress can be expressed in the matrix form 3.4.1.

Example

The thick block of uniform material shown in Fig. 3.5.2, loaded by a constant stress σ_o in the x direction, will have $\sigma_{xx} = \sigma_0$ and all other components zero everywhere. It is therefore in a state of plane stress.

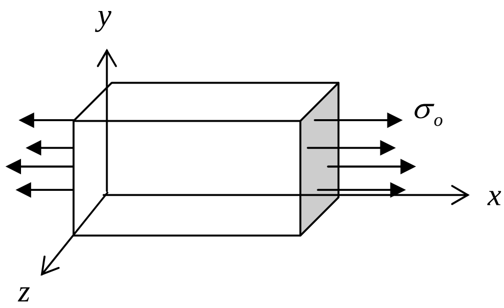


Figure 3.5.2: a thick block of material in plane stress

3.5.2 Analysis of Plane Stress

Next are discussed the **stress invariants**, **principal stresses** and **maximum shear stresses** for the two-dimensional plane state of stress, and tools for evaluating them. These quantities are useful because they tell us the complete state of stress at a point in simple terms. Further, these quantities are directly related to the strength and response of materials. For example, the way in which a material plastically (permanently) deforms is often related to the maximum shear stress, the directions in which flaws/cracks grow in materials is often related to the principal stresses, and the energy stored in materials is often a function of the stress invariants.

Stress Invariants

A stress invariant is some function of the stress components which is independent of the coordinate system being used; in other words, they have the same value no matter where the $x-y$ axes are drawn through a point. In a two dimensional space there are two stress invariants, labelled I_1 and I_2. These are

$$\boxed{\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} \\ I_2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned}} \qquad \textbf{Stress Invariants} \qquad (3.5.1)$$

These quantities can be proved to be invariant directly from the stress transformation equations, Eqns. 3.4.8 {▲Problem 1}. Physically, invariance of I_1 and I_2 means that they are the same for any chosen perpendicular planes through a material particle.

Combinations of the stress invariants are also invariant, for example the important quantity

$$\tfrac{1}{2}I_1 \pm \sqrt{\tfrac{1}{4}I_1^2 - I_2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\tfrac{1}{4}\left(\sigma_{xx}-\sigma_{yy}\right)^2 + \sigma_{xy}^2} \qquad (3.5.2)$$

Principal Stresses

Consider a material particle for which the stress, with respect to some $x - y$ coordinate system, is

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.5.3}$$

The stress acting on different planes through the point can be evaluated using the Stress Transformation Equations, Eqns. 3.4.8, and the results are plotted in Fig. 3.5.3. The original planes are re-visited after rotating 180°.

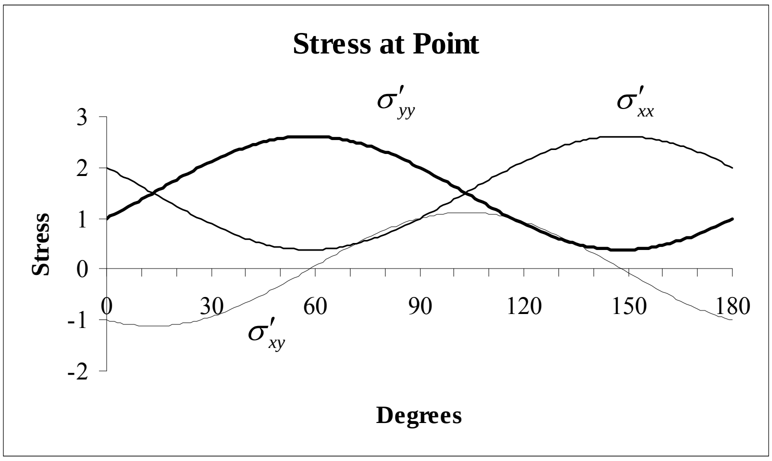


Figure 3.5.3: stresses on different planes through a point

It can be seen that there are two perpendicular planes for which the shear stress is zero, for θ ≈ 58° and $\theta \approx (58+90)^\circ$. In fact it can be proved that for every point in a material there are two (and only two) perpendicular planes on which the shear stress is zero (see below). These planes are called the **principal planes**. It will also be noted from the figure that the normal stresses acting on the planes of zero shear stress are either a maximum or minimum. Again, this can be proved (see below). These normal stresses are called principal stresses. The principal stresses are labelled σ_1 and σ_2, Fig. 3.5.4.

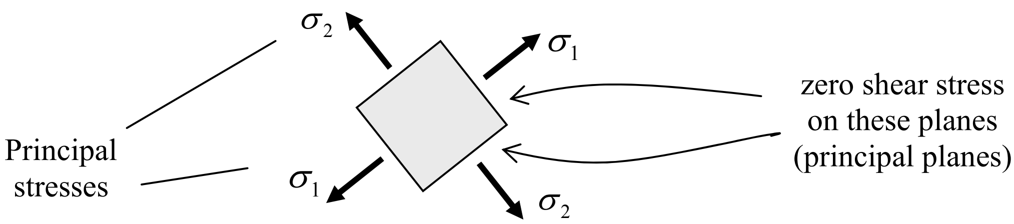


Figure 3.5.4: principal stresses

The principal stresses can be obtained by setting $\sigma'_{xy} = 0$ in the Stress Transformation Equations, Eqns. 3.4.8, which leads to the value of θ for which the planes have zero shear stress:

$$\boxed{\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}} \qquad \textbf{Location of Principal Planes} \qquad (3.5.4)$$

For the example stress state, Eqn. 3.5.3, this leads to

$$\theta = \frac{1}{2}\arctan(-2)$$

and so the perpendicular planes are at $\theta = -31.72°\ (148.28°)$ and $\theta = 58.3°$.

Explicit expressions for the principal stresses can be obtained by substituting the value of θ from Eqn. 3.5.4 into the Stress Transformation Equations, leading to (see the Appendix to this section, §3.5.8)

$$\boxed{\begin{aligned}\sigma_1 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma_2 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2}\end{aligned}} \qquad \textbf{Principal Stresses} \qquad (3.5.5)$$

For the example stress state Eqn.3.5.3, one has

$$\sigma_1 = \frac{3+\sqrt{5}}{2} \approx 2.62, \qquad \sigma_2 = \frac{3-\sqrt{5}}{2} \approx 0.38$$

Note here that one uses the symbol σ_1 to represent the maximum principal stress and σ_2 to represent the minimum principal stress. By maximum, it is meant the algebraically largest stress so that, for example, $+1 > -3$.

From Eqns. 3.5.2, 3.5.5, the principal stresses are invariant; they are intrinsic features of the stress state at a point and do not depend on the coordinate system used to describe the stress state.

The question now arises: why are the principal stresses so important? One part of the answer is that the maximum principal stress is the largest normal stress acting on any plane through a material particle. This can be proved by differentiating the stress transformation formulae with respect to θ,

$$\frac{d\sigma'_{xx}}{d\theta} = -\sin 2\theta(\sigma_{xx} - \sigma_{yy}) + 2\cos 2\theta\sigma_{xy}$$
$$\frac{d\sigma'_{yy}}{d\theta} = +\sin 2\theta(\sigma_{xx} - \sigma_{yy}) - 2\cos 2\theta\sigma_{xy} \qquad (3.5.6)$$
$$\frac{d\sigma'_{xy}}{d\theta} = -\cos 2\theta(\sigma_{xx} - \sigma_{yy}) - 2\sin 2\theta\sigma_{xy}$$

The maximum/minimum values can now be obtained by setting these expressions to zero. One finds that the normal stresses are a maximum/minimum at the very value of θ in Eqn. 3.5.4 – the value of θ for which the shear stresses are zero – the principal planes.

Very often the only thing one knows about the stress state at a point are the principal stresses. In that case one can derive a very useful formula as follows: align the coordinate axes in the principal directions, so

$$\sigma_{xx} = \sigma_1, \quad \sigma_{yy} = \sigma_2, \quad \sigma_{xy} = 0 \qquad (3.5.7)$$

Using the transformation formulae with the relations $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ then leads to

$$\boxed{\begin{aligned} \sigma'_{xx} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \sigma'_{yy} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \sigma'_{xy} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta \end{aligned}} \qquad (3.5.8)$$

Here, θ is measured *from* the principal directions, as illustrated in Fig. 3.5.5.

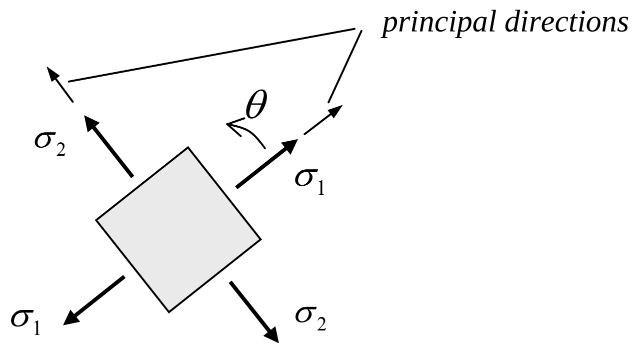


Figure 3.5.5: principal stresses and principal directions

The Third Principal Stress

Although plane stress is essentially a two-dimensional stress-state, it is important to keep in mind that any real material is three-dimensional. The stresses acting *on* the $x-y$ plane are the normal stress σ_{zz} and the shear stresses σ_{zx} and σ_{zy}, Fig. 3.5.6. These are all zero (in plane stress). It was discussed above how the principal stresses occur on planes of zero shear stress. Thus the σ_{zz} stress is also a principal stress. Technically speaking, there are always three principal stresses in three dimensions, and (at least) one of these will be zero in plane stress. This fact will be used below in the context of maximum shear stress.

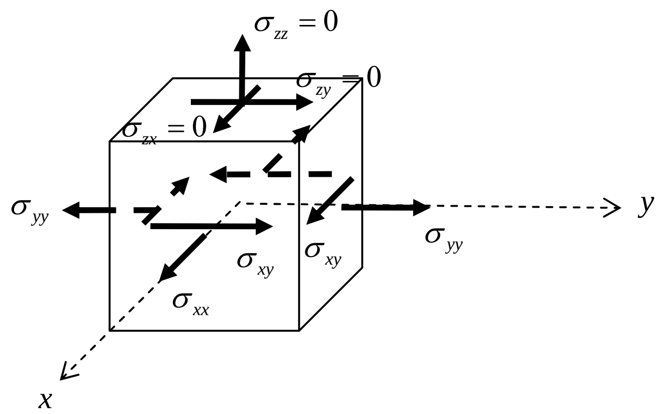


Figure 3.5.6: stresses acting on the $x-y$ plane

Maximum Shear Stress

Eqns. 3.5.8 can be used to derive an expression for the maximum shear stress. Differentiating the expression for shear stress with respect to θ, setting to zero and solving, shows that the maximum/minimum occurs at $\theta = \pm 45$, in which case

$$\sigma_{xy}\Big|_{\theta=+45} = -\frac{1}{2}(\sigma_1 - \sigma_2), \qquad \sigma_{xy}\Big|_{\theta=-45} = +\frac{1}{2}(\sigma_1 - \sigma_2)$$

or

$$\boxed{\max(\sigma_{xy}) = \frac{1}{2}|\sigma_1 - \sigma_2|} \qquad \textbf{Maximum Shear Stress} \qquad (3.5.9)$$

Thus the shear stress reaches a maximum on planes which are oriented at $\pm 45°$ to the principal planes, and the value of the shear stress acting on these planes is as given above. Note that the formula Eqn. 3.5.9 does not let one know in which *direction* the shear stresses are acting but this is not usually an important issue. Many materials respond in certain ways when the maximum shear stress reaches a critical value, and the actual direction of shear stress is unimportant. The direction of the maximum principal stress is, on the other hand, important – a material will in general respond differently according to whether the normal stress is compressive or tensile.

The normal stress acting on the planes of maximum shear stress can be obtained by substituting $\theta = \pm 45$ back into the formulae for normal stress in Eqn. 3.5.8, and one sees that

$$\sigma'_{xx} = \sigma'_{yy} = (\sigma_1 + \sigma_2)/2 \tag{3.5.10}$$

The results of this section are summarised in Fig. 3.5.7.

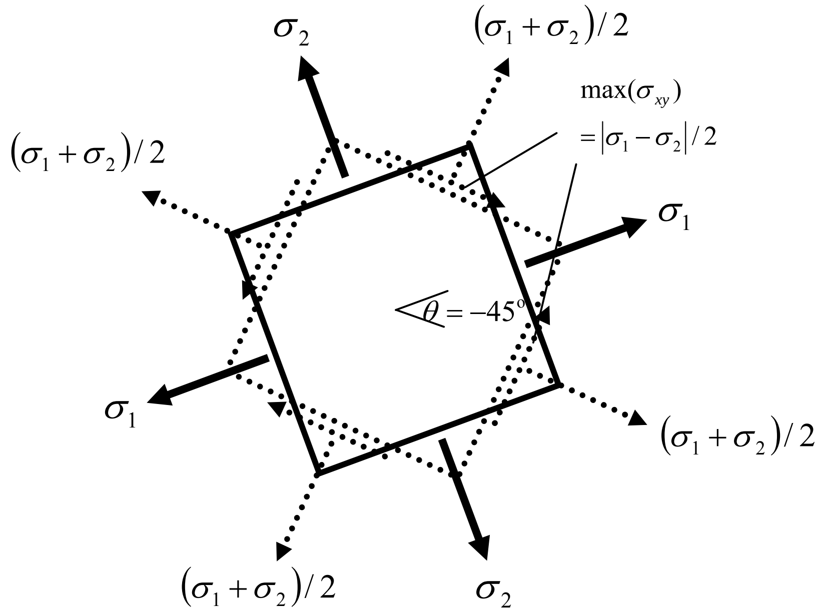
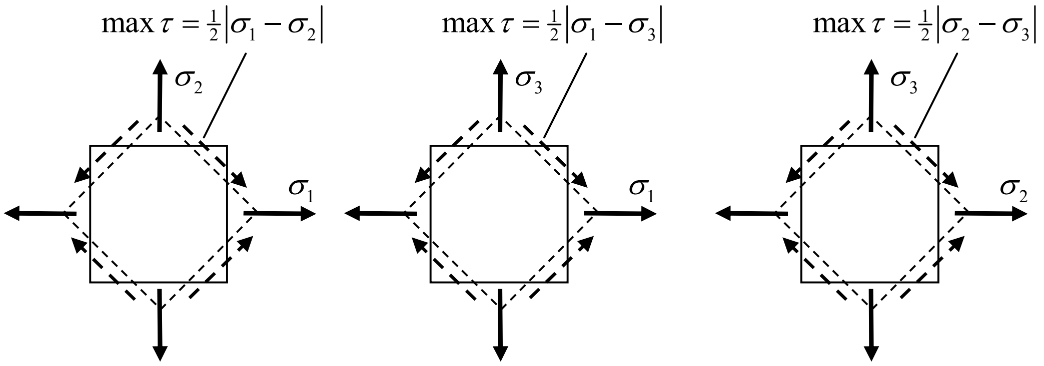


Figure 3.5.7: principal stresses and maximum shear stresses acting on the $x - y$ plane

The maximum shear stress in the $x - y$ plane was calculated above, Eqn. 3.5.9. This is not necessarily the maximum shear stress acting at the material particle. In general, it can be shown that the maximum shear stress is the maximum of the following three terms (see the Appendix to this section, §3.5.8):

$$\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1 - \sigma_3|, \quad \frac{1}{2}|\sigma_2 - \sigma_3|$$

The first term is the maximum shear stress in the $1-2$ plane, i.e. the plane containing the σ_1 and σ_2 stresses (and given by Eqn. 3.5.9). The second term is the maximum shear stress in the $1-3$ plane and the third term is the maximum shear stress in the $2-3$ plane. These are sketched in Fig. 3.5.8 below.



(a) 1-2 plane (a) 1-3 plane (a) 2-3 plane

Figure 3.5.8: principal stresses and maximum shear stresses

In the case of plane stress, $\sigma_3 = \sigma_{zz} = 0$, and the maximum shear stress will be

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1|, \quad \frac{1}{2}|\sigma_2|\right\} \tag{3.5.11}$$

3.5.3 Stress Boundary Conditions

When solving problems, information is usually available on what is happening at the boundaries of materials. This information is called the **boundary conditions**. Information is usually not available on what is happening in the interior of the material – information there is obtained by solving the equations of mechanics.

A number of different conditions can be known at a boundary, for example it might be known that a certain part of the boundary is fixed so that the displacements there are zero. This is known as a **displacement boundary condition**. On the other hand the stresses over a certain part of the material boundary might be known. These are known as **stress boundary conditions** – this case will be examined here.

General Stress Boundary Conditions

It has been seen already that, when one material contacts a second material, a force, or distribution of stress arises. This force F will have arbitrary direction, Fig. 3.5.9a, and can be decomposed into the sum of a normal stress distribution σ_N and a shear distribution σ_S, Fig. 3.5.9b. One can introduce a coordinate system to describe the applied stresses, for example the $x-y$ axes shown in Fig. 3.5.9c (the axes are most conveniently defined to be normal and tangential to the boundary).

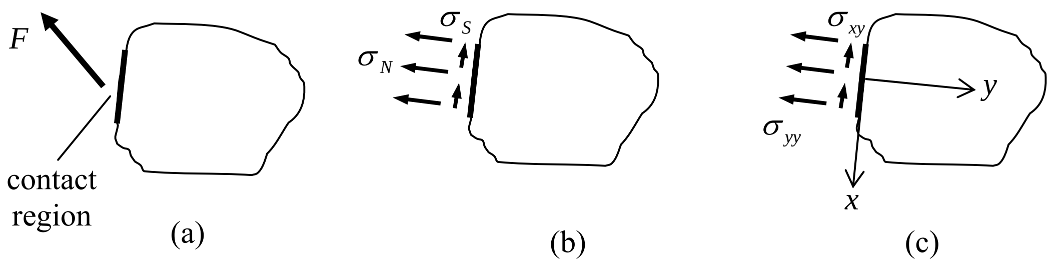


Figure 3.5.9: Stress boundary conditions; (a) force acting on material due to contact with a second material, (b) the resulting normal and shear stress distributions, (c) applied stresses as stress components in a given coordinate system

Figure 3.5.10 shows the same component as Fig. 3.5.9. Shown in detail is a small material element at the boundary. From equilibrium of the element, stresses σ_{xy}, σ_{yy}, equal to the applied stresses, must be acting inside the material, Fig. 3.5.10a. Note that the **tangential stresses**, which are the σ_{xx} stresses in this example, can take on any value and the element will still be in equilibrium with the applied stresses, Fig. 3.5.10b.

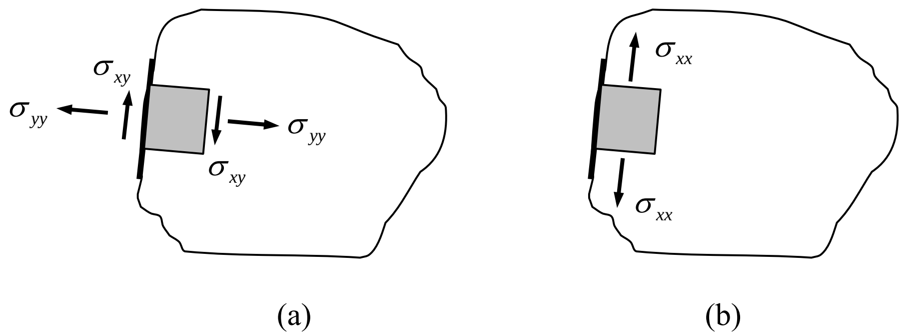


Figure 3.5.10: Stresses acting on a material element at the boundary, (a) normal and shear stresses, (b) tangential stresses

Thus, if the applied stresses are *known*, then so also are the normal and shear stresses acting at the boundary of the material.

Stress Boundary Conditions at a Free Surface

A free surface is a surface that has "nothing" on one side and so there is nothing to provide reaction forces. Thus there must also be no normal or shear stress on the other side (the inside).

This leads to the following, Fig. 3.5.11:

Stress boundary conditions at a free surface:
the normal and shear stress at a free surface are zero

This simple fact is used again and again to solve practical problems.

Again, the stresses acting normal to any other plane at the surface do not have to be zero – they can be balanced as, for example, the tangential stresses σ_T and the stress $\overline{\sigma}$ in Fig. 3.5.11.

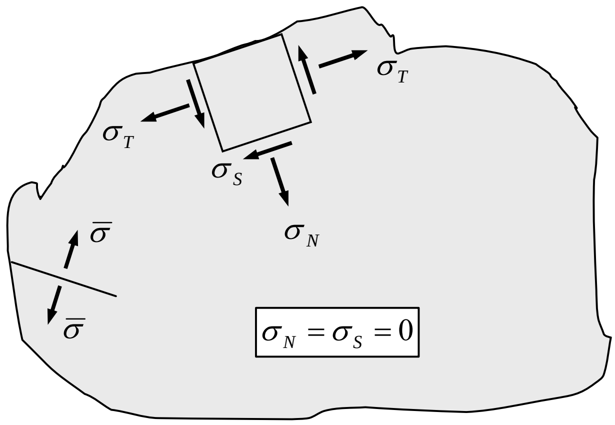


Figure 3.5.11: A free surface - the normal and shear stresses there are zero

Atmospheric Pressure

There *is* something acting on the outside "free" surfaces of materials – the atmospheric pressure. This is a type of stress which is **hydrostatic**, that is, it acts normal at all points, as shown in Fig. 3.5.12. Also, it does not vary much. This pressure is present when one characterises a material, that is, when its material properties are determined from tests and so on, for example, its Young's Modulus (see Chapter 5). The atmospheric pressure is therefore a datum – stresses are really measured relative to this value, and so the atmospheric pressure is ignored.

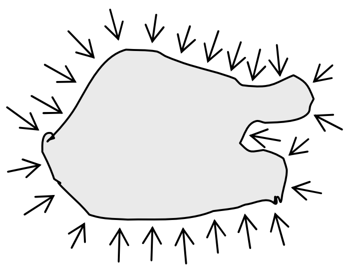


Figure 3.5.12: a material subjected to atmospheric pressure

3.5.4 Thin Components

Consider a thin component as shown in Fig. 3.5.13. With the coordinate axes aligned as shown, and with the large face free of loading, one has $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$. Strictly speaking, these stresses are zero *only* at the free surfaces of the material but, because it is thin, these stresses should not vary much from zero within. Taking the "z" stresses to be identically zero throughout the material, the component is in a state of plane stress[1]. On the other hand, were the sheet not so thin, the stress components that were zero at the free-surfaces might well deviate significantly from zero deep within the material.

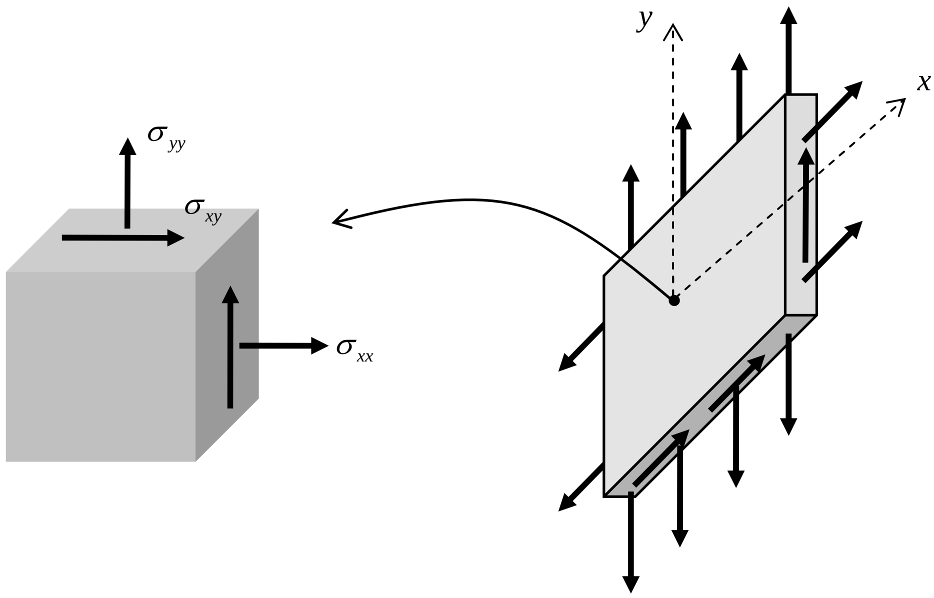


Figure 3.5.13: a thin material loaded in-plane, leading to a state of plane stress

When analysing plane stress states, only one cross section of the material need be considered. This is illustrated in Fig. 3.5.14.

[1] it will be shown in Book II that, when the applied stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ vary only *linearly* over the thickness of the component, the stresses $\sigma_{zz}, \sigma_{zx}, \sigma_{zy}$ are exactly zero throughout the component, otherwise they are only approximately zero

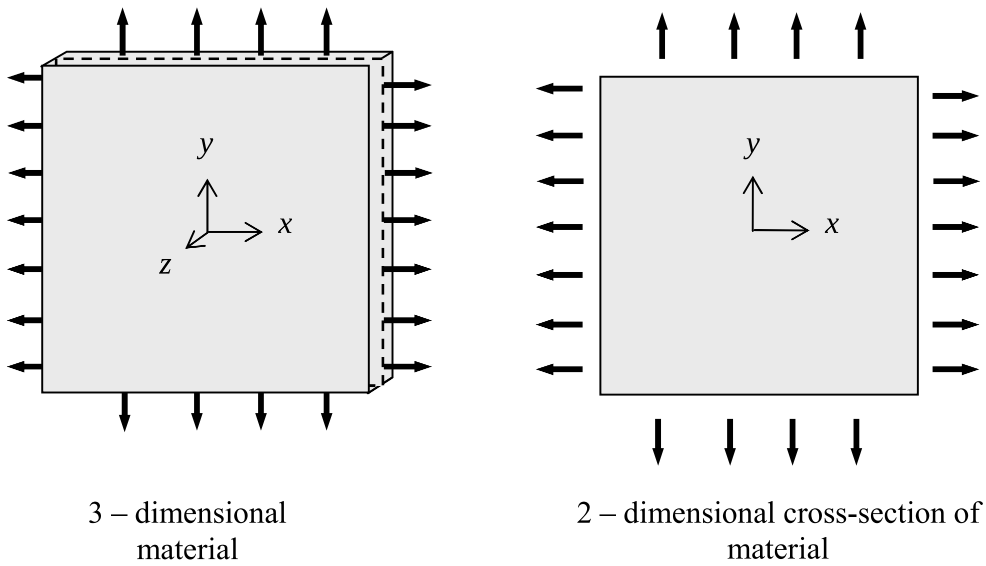


Figure 3.5.14: one two-dimensional cross-section of material

Note that, although the stress normal to the plane, σ_{zz}, is zero, the three dimensional sheet of material *is* deforming in this direction – it will obviously be getting thinner under the tensile loading shown in Fig. 3.5.14.

Note that plane stress arises in *all* thin materials (loaded in –plane), no matter what they are made of.

3.5.5 Mohr's Circle

Otto Mohr devised a way of describing the state of stress at a point using a single diagram, called the **Mohr's circle**.

To construct the Mohr circle, first introduce the **stress coordinates** (σ, τ), Fig. 3.5.15; the abscissae (horizontal) are the normal stresses σ and the ordinates (vertical) are the shear stresses τ. On the horizontal axis, locate the principal stresses σ_1, σ_2, with $\sigma_1 > \sigma_2$. Next, draw a circle, centred at the average principal stress $(\sigma, \tau) = ((\sigma_1 + \sigma_2)/2, 0)$, having radius $(\sigma_1 - \sigma_2)/2$.

The normal and shear stresses acting on a single plane are represented by a single point on the Mohr circle. The normal and shear stresses acting on two perpendicular planes are represented by two points, one at *each end of a diameter* on the Mohr circle. Two such diameters are shown in the figure. The first is horizontal. Here, the stresses acting on two perpendicular planes are $(\sigma, \tau) = (\sigma_1, 0)$ and $(\sigma, \tau) = (\sigma_2, 0)$ and so this diameter represents the principal planes/stresses.

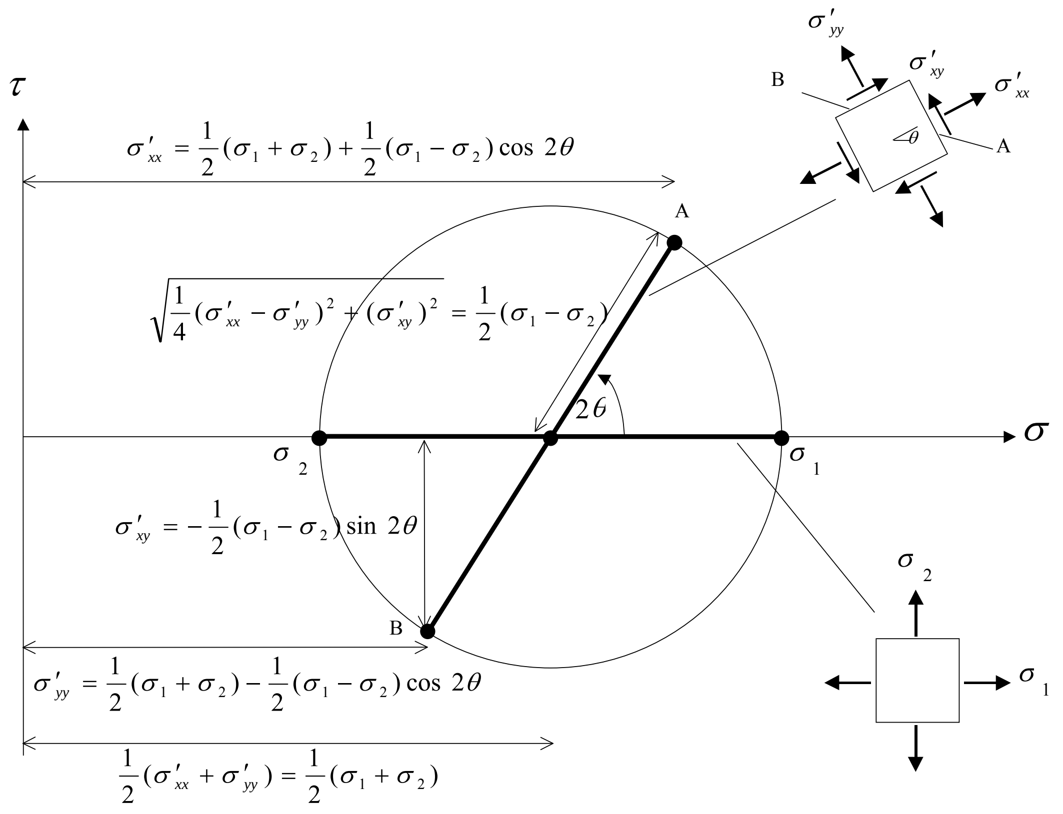


Figure 3.5.15: Mohr's Circle

The stresses on planes rotated by an amount θ from the principal planes are given by Eqn. 3.5.8. Using elementary trigonometry, these stresses are represented by the points A and B in Fig. 3.5.15. Note that a rotation of θ in the physical plane corresponds to a rotation of 2θ in the Mohr diagram.

Note also that the conventional labeling of shear stress has to be altered when using the Mohr diagram. On the Mohr circle, a shear stress is positive if it yields a clockwise moment about the centre of the element, and is "negative" when it yields a negative moment. For example, at point A the shear stress is "positive" ($\tau > 0$), which means the direction of shear on face A of the element is actually opposite to that shown. This agrees with the formula $\sigma'_{xy} = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$, which is less than zero for $\sigma_1 > \sigma_2$ and $\theta \le 90^\circ$. At point B the shear stress is "negative" ($\tau < 0$), which again agrees with formula.

3.5.6 Stress Boundary Conditions (continued)

Consider now in more detail a surface between two different materials, Fig. 3.5.16. One says that the normal and shear stresses are **continuous** across the surface, as illustrated.

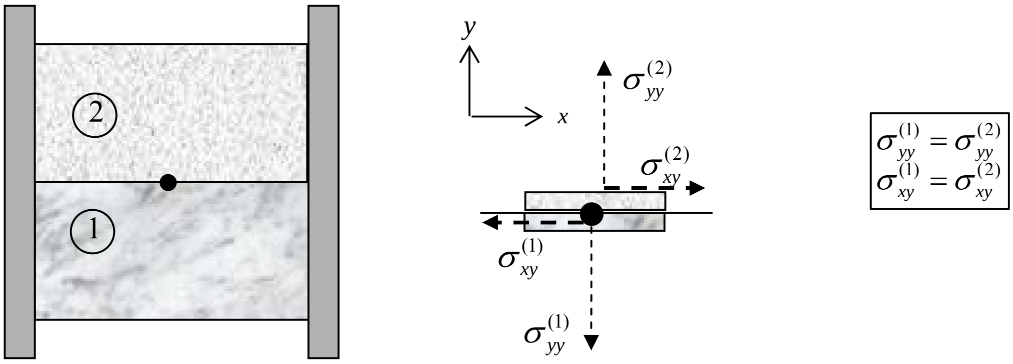
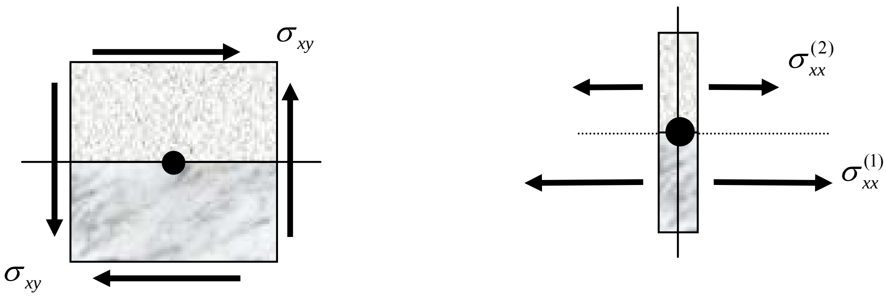


Figure 3.5.16: normal and shear stress continuous across an interface between two different materials, material '1' and material '2'

Note also that, since the shear stress σ_{xy} is the same on both sides of the surface, the shear stresses acting on both sides of a perpendicular plane passing *through* the interface between the materials, by the symmetry of stress, must also be the same, Fig. 3.5.17a.



(a) (b)

Figure 3.5.17: stresses at an interface; (a) shear stresses continuous across the interface, (b) tangential stresses not necessarily continuous

However, again, the tangential stresses, those acting parallel to the interface, do *not* have to be equal. For example, shown in Fig. 3.5.17b are the tangential stresses acting in the upper material, $\sigma_{xx}^{(2)}$ - they balance no matter what the magnitude of the stresses $\sigma_{xx}^{(1)}$.

Description of Boundary Conditions

The following example brings together the notions of stress boundary conditions, stress components, equilibrium and equivalent forces.

Example

Consider the plate shown in Fig. 3.5.18. It is of width $2a$, height *b* and depth *t*. It is subjected to a tensile stress *r*, pressure *p* and shear stresses *s*. The applied stresses are uniform through the thickness of the plate. It is welded to a rigid base.

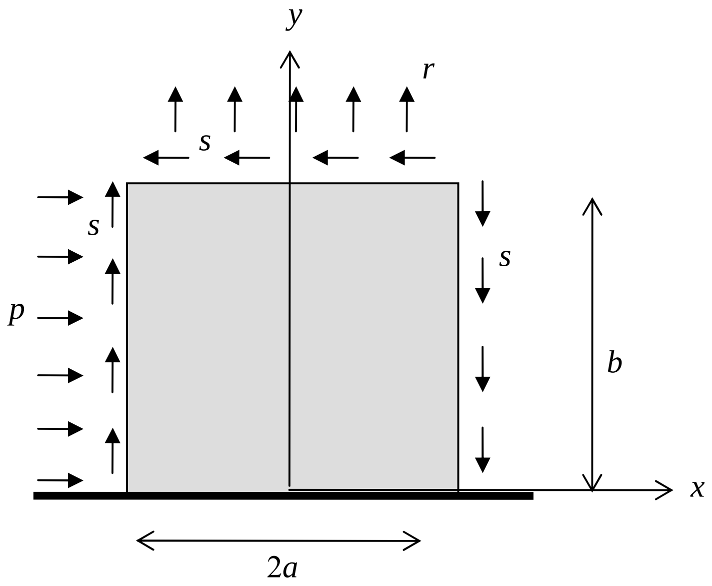


Figure 3.5.18: a plate subjected to stress distributions

Using the $x-y$ axes shown, the stress boundary conditions can be expressed as:

$$\text{Left-hand surface:} \quad \begin{cases} \sigma_{xx}(-a,y) = -p \\ \sigma_{xy}(-a,y) = -s \end{cases}, \quad 0 < y < b$$

$$\text{Top surface:} \quad \begin{cases} \sigma_{yy}(x,b) = +r \\ \sigma_{xy}(x,b) = -s \end{cases}, \quad -a < x < +a$$

$$\text{Right-hand surface:} \quad \begin{cases} \sigma_{xx}(+a,y) = 0 \\ \sigma_{xy}(+a,y) = -s \end{cases}, \quad 0 < y < b$$

Note carefully the description of the normal and shear stresses over each side and the signs of the stress components.

The stresses at the lower edge are unknown (there is a displacement boundary condition there: zero displacement). They will in general not be uniform. Using the given $x-y$ axes, these unknown reaction stresses, exerted by the base on the plate, are (see Fig 3.5.19)

Lower surface: $\begin{cases} \sigma_{yy}(x,0) \\ \sigma_{xy}(x,0) \end{cases}, \qquad -a < x < +a$

Note the directions of the arrows in Fig. 3.5.19, they have been drawn in the direction of positive $\sigma_{yy}(x,0)$, $\sigma_{xy}(x,0)$.

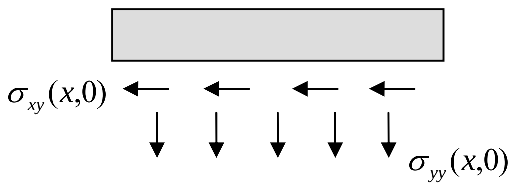


Figure 3.5.19: unknown reaction stresses acting on the lower edge

For force equilibrium of the complete plate, consider the free-body diagram 3.5.20; shown are the resultant forces of the stress distributions. Force equilibrium requires that

$$\sum F_x = bpt - 2ast - t\int_{-a}^{+a} \sigma_{xy}(x,0)dx = 0$$

$$\sum F_y = 2art - t\int_{-a}^{+a} \sigma_{yy}(x,0)dx = 0$$

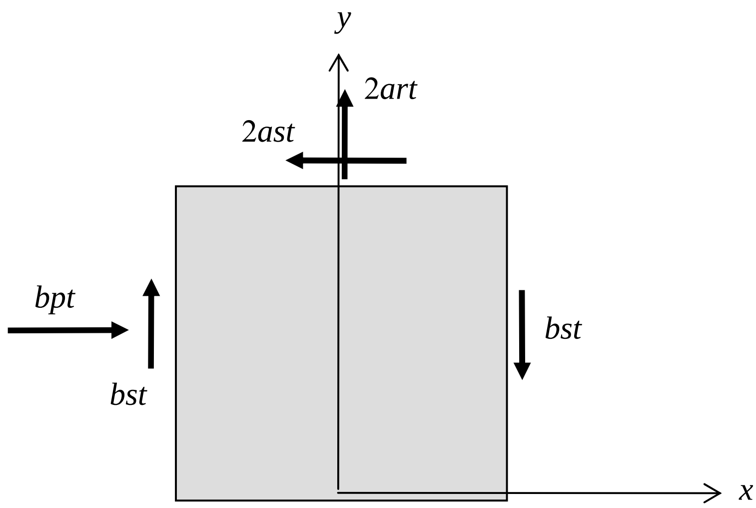


Figure 3.5.20: a free-body diagram of the plate in Fig. 3.5.18 showing the known resultant forces (forces on the lower boundary are not shown)

For moment equilibrium, consider the moments about, for example, the lower left-hand corner. One has

$$\sum M_0 = -bpt(b/2) + 2ast(b) + 2art(a) - bst(2a) - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x)dx = 0$$

If one had taken moments about the top-left corner, the equation would read

$$\sum M_0 = +bpt(b/2) + 2art(a) - bst(2a)$$
$$- t\int_{-a}^{+a} \sigma_{xy}(x,0) \times b\,dx - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x)dx = 0$$

■

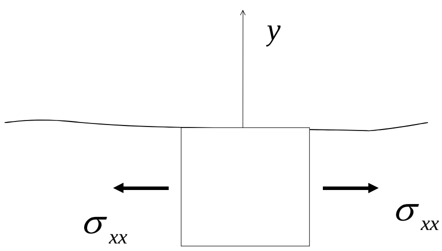
3.5.7 Problems

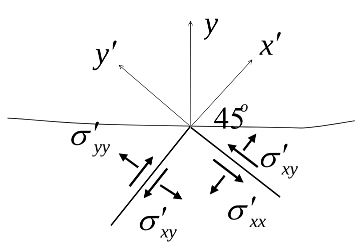
1. Prove that the function $\sigma_x + \sigma_y$, i.e. the sum of the normal stresses acting at a point, is a stress invariant. [Hint: add together the first two of Eqns. 3.4.8.]

2. Consider a material in plane stress conditions. An element at a free surface of this material is shown below left. Taking the coordinate axes to be orthogonal to the surface as shown (so that the tangential stress is σ_{xx}), one has

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{bmatrix}$$

   (a) what are the two in-plane principal stresses at the point? Which is the maximum and which is the minimum?
   (b) examine planes inclined at 45° to the free surface, as shown below right. What are the stresses acting on these planes and what have they got to do with maximum shear stress?





3. The stresses at a point in a state of plane stress are given by

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & 2 \end{bmatrix}$$

   (a) Draw a little box to represent the point and draw some arrows to indicate the magnitude and direction of the stresses acting at the point.
   (b) What relationship exists between Oxy and a second coordinate set $Ox'y'$, such that the shear stresses are zero in $Ox'y'$?
   (c) Find the two in-plane principal stresses.

(d) Draw another box whose sides are aligned to the principal directions and draw some arrows to indicate the magnitude and direction of the principal stresses acting at the point.
(e) Check that the sum of the normal stresses at the point is an invariant.

4. A material particle is subjected to a state of stress given by

$$[\sigma_{ij}] = \begin{bmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the principal stresses (all three), maximum shear stresses (see Eqn. 3.5.11), and the direction of the planes on which these stresses act.

5. Consider the following state of stress (with respect to an x, y, z coordinate system):

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Use the stress transformation equations to derive the stresses acting on planes obtained from the original planes by a counterclockwise rotation of 45° about z axis.
(b) What is the maximum normal stress acting at the point?
(c) What is the maximum shear stress? On what plane(s) does it act? (See Eqn. 3.5.11.)

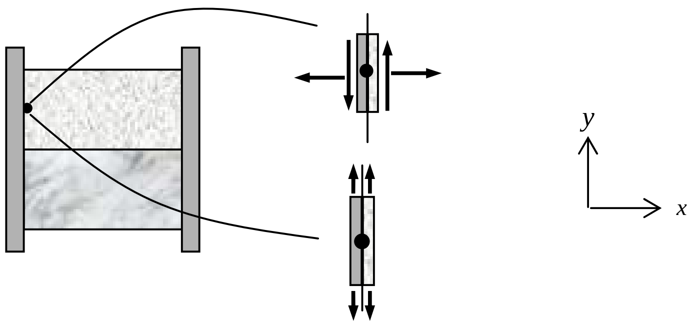
6. Consider the two dimensional stress state

$$[\sigma_{ij}] = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

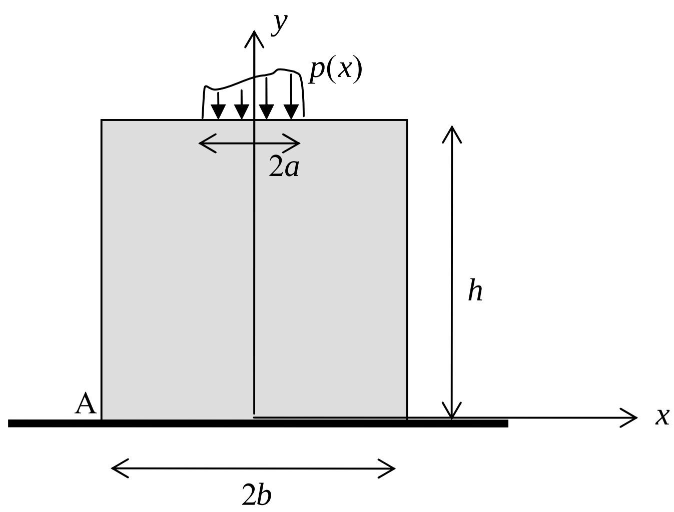
Show that this is an **isotropic state of stress**, that is, the stress components are the same on *all* planes through a material particle.

7. (a) Is a trampoline (the material you jump on) in a state of plane stress? When someone is actually jumping on it?
(b) Is a picture hanging on a wall in a state of plane stress?
(c) Is a glass window in a state of plane stress? On a very windy day?
(d) A piece of rabbit skin is stretched in a testing machine – is it in a state of plane stress?

8. Consider the point shown below, at the boundary between a wall and a dissimilar material. Label the stress components displayed using the coordinate system shown. Which stress components are continuous across the wall/material boundary? (Add a superscript 'w' for the stresses in the wall.)



9. A thin metal plate of width $2b$, height h and depth t is loaded by a pressure distribution $p(x)$ along $-a < x < +a$ and welded at its base to the ground, as shown in the figure below. Write down expressions for the stress boundary conditions (two on each of the three edges). Write down expressions for the force equilibrium of the plate and moment equilibrium of the plate about the corner A.



3.5.8 Appendix to §3.5

A Note on the Formulae for Principal Stresses

To derive Eqns. 3.5.5, first rewrite the transformation equations in terms of 2θ using $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to get

$$
\begin{aligned}
\sigma'_{xx} &= \tfrac{1}{2}(1+\cos 2\theta)\sigma_{xx} + \tfrac{1}{2}(1-\cos 2\theta)\sigma_{yy} + \sin 2\theta\sigma_{xy} \\
\sigma'_{yy} &= \tfrac{1}{2}(1-\cos 2\theta)\sigma_{xx} + \tfrac{1}{2}(1+\cos 2\theta)\sigma_{yy} - \sin 2\theta\sigma_{xy} \\
\sigma'_{xy} &= \tfrac{1}{2}\sin 2\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy}
\end{aligned}
$$

Next, from Eqn. 3.5.4,

$$\sin 2\theta = \frac{2\sigma_{xy}}{\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2}}, \qquad \cos 2\theta = \frac{\sigma_{xx} - \sigma_{yy}}{\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2}}$$

Substituting into the rewritten transformation formulae then leads to

$$\begin{aligned} \sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma'_{xy} &= 0 \end{aligned}$$

Here $\sigma'_{xx} > \sigma'_{yy}$ so that the maximum principal stress is $\sigma_1 = \sigma'_{xx}$ and the minimum principal stress is $\sigma'_2 = \sigma'_{yy}$. Here it is implicitly assumed that $\tan 2\theta > 0$, i.e. that $0 < 2\theta < 90$ or $180 < 2\theta < 270$. On the other hand one could assume that $\tan 2\theta < 0$, i.e. that $90 < 2\theta < 180$ or $270 < 2\theta < 360$, in which case one arrives at the formulae

$$\begin{aligned} \sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \end{aligned}$$

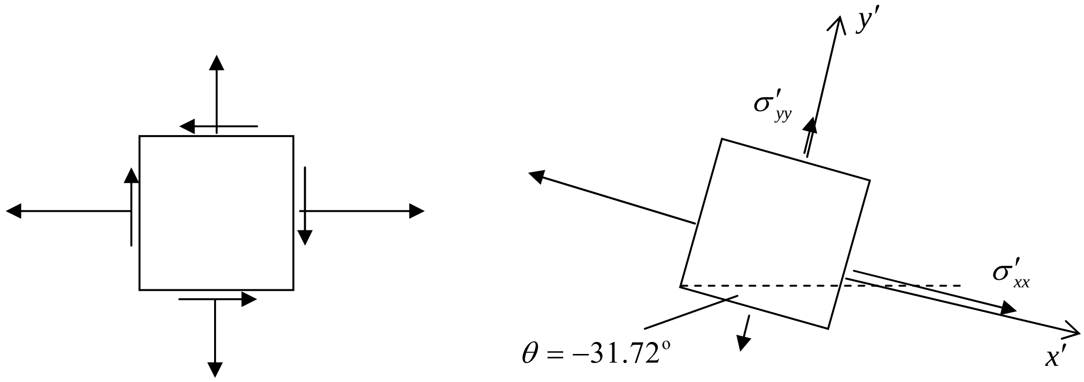
The results can be summarised as Eqn. 3.5.5,

$$\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma_2 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \end{aligned}$$

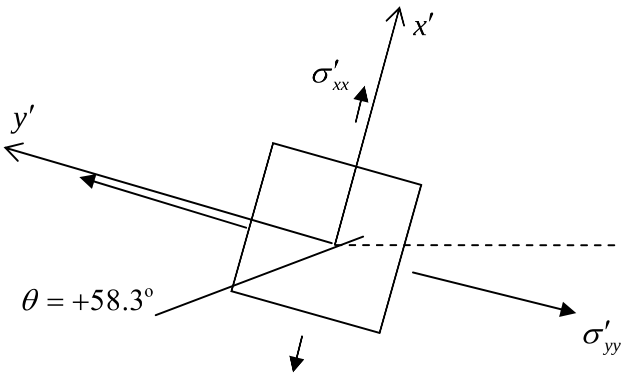
These formulae do not tell one on which of the two principal planes the maximum principal stress acts. This might not be an important issue, but if this information is required one needs to go directly to the stress transformation equations. In the example stress state, Eqn. 3.5.3, one has

$$\begin{aligned} \sigma'_{xx} &= \cos^2\theta(2) + \sin^2\theta(1) + \sin 2\theta(-1) \\ \sigma'_{yy} &= \sin^2\theta(2) + \cos^2\theta(1) - \sin 2\theta(-1) \end{aligned}$$

For $\theta = -31.72^\circ\ (148.28^\circ)$, $\sigma'_{xx} = 2.62$ and $\sigma'_{yy} = 0.38$. So one has the situation shown below.



If one takes the other angle, $\theta = 58.3°$, one has $\sigma'_{xx} = 0.38$ and $\sigma'_{yy} = 2.62$, and the situation below



A Note on the Maximum Shear Stress

Shown below left is a box element with sides perpendicular to the $1,2,z$ axes, i.e. aligned with the principal directions. The stresses in the new x', y' axis system shown are given by Eqns. 3.5.8, with θ measured from the principal directions:

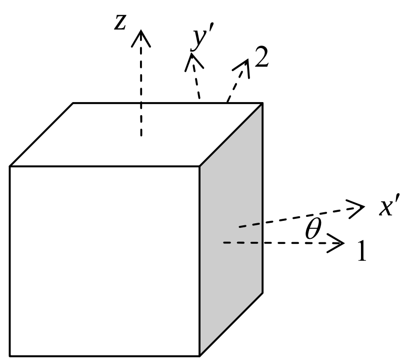
$$\begin{aligned}
\sigma'_{xx} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\
\sigma'_{yy} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\
\sigma'_{xy} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta
\end{aligned}$$

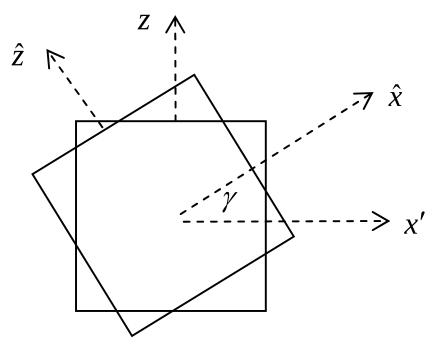
Now as well as rotating around in the $1-2$ plane through an angle θ, rotate also in the x', z plane through an angle γ (see below right). This rotation leads to the new stresses

$$\hat{\sigma}_{xx} = \cos^2\gamma\sigma'_{xx} + \sin^2\gamma\sigma_{zz} + \sin 2\gamma\sigma_{x'z}$$
$$\hat{\sigma}_{zz} = \sin^2\gamma\sigma'_{xx} + \cos^2\gamma\sigma_{zz} - \sin 2\gamma\sigma_{x'z}$$
$$\hat{\sigma}_{xz} = \sin\gamma\cos\gamma(\sigma_{zz} - \sigma'_{xx}) + \cos 2\gamma\sigma_{x'z}$$

In plane stress, $\sigma_{zz} = \sigma_{x'z} = 0$, so one has the stresses

$$\hat{\sigma}_{xx} = \cos^2\gamma\sigma'_{xx}, \quad \hat{\sigma}_{yy} = \sin^2\gamma\sigma'_{xx}, \quad \hat{\sigma}_{xy} = -\tfrac{1}{2}\sin 2\gamma\sigma'_{xx}$$





The shear stress can be written out in full:

$$\hat{\sigma}_{xy}(\gamma,\theta) = -\tfrac{1}{2}\sin 2\gamma\left[\tfrac{1}{2}(\sigma_1+\sigma_2) + \tfrac{1}{2}(\sigma_1-\sigma_2)\cos 2\theta\right].$$

This is a function of two variables; its minimum value can be found by setting the partial derivatives with respect to these variables to zero. Differentiating,

$$\partial\hat{\sigma}_{xy}/\partial\gamma = -\cos 2\gamma\left[\tfrac{1}{2}(\sigma_1+\sigma_2) + \tfrac{1}{2}(\sigma_1-\sigma_2)\cos 2\theta\right]$$
$$\partial\hat{\sigma}_{xy}/\partial\theta = -\tfrac{1}{2}\sin 2\gamma\left[-(\sigma_1-\sigma_2)\sin 2\theta\right]$$

Setting to zero gives the solutions $\sin 2\theta = 0$, $\cos 2\gamma = 0$, i.e. $\theta = 0$, $\gamma = 45^{\circ}$. Thus the maximum shear stress occurs at 45° to the $1-2$ plane, and in the $1-z$, i.e. $1-3$ plane (as in Fig. 3.5.8b). The value of the maximum shear stress here is then $|\hat{\sigma}_{xy}| = |\tfrac{1}{2}\sigma_1|$, which is the expression in Eqn. 3.5.11.

4 Strain

The concept of strain is introduced in this Chapter. The approximation to the **true strain** of the **engineering strain** is discussed. The practical case of two dimensional **plane strain** is discussed, along with the **strain transformation formulae**, **principal strains**, **principal strain directions** and the **maximum shear strain**.

4.1 Strain

If an object is placed on a table and then the table is moved, each material particle moves in space. The particles undergo a **displacement**. The particles have moved in space as a **rigid body**. The material remains unstressed. On the other hand, when a material is acted upon by a set of forces, it *changes size and/or shape*, it **deforms**. This deformation is described using the concept of **strain**. The study of motion, without reference to the forces which cause such motion, is called **kinematics**.

4.1.1 One Dimensional Strain

The Engineering Strain

Consider a slender rod, fixed at one end and stretched, as illustrated in Fig. 4.1.1; the original position of the rod is shown dotted.

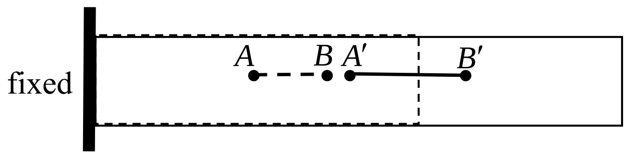


Figure 4.1.1: the strain at a point *A* in a stretched slender rod; *AB* is a line element in the unstretched rod, $A'B'$ is the same line element in the stretched rod

There are a number of different ways in which this stretching/deformation can be described. Here, what is perhaps the simplest measure, the **engineering strain**, will be used. To determine the strain at point *A*, Fig. 4.1.1, consider a small line element *AB* emanating from *A* in the unstretched rod. The points *A* and *B* move to A' and B' when the rod has been stretched. The (engineering) strain ε at *A* is then[1]

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} \tag{4.1.1}$$

The strain at other points in the rod can be evaluated in the same way.

If a line element is stretched to twice its original length, the strain is 1. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.5. The strain is often expressed as a percentage; a 100% strain is a strain of 1, a 200% strain is a strain of 2, etc. Most engineering materials, such as metals and concrete, undergo very small strains in practical applications, in the range 10^{-6} to 10^{-2}; rubbery materials can easily undergo large strains of 100%.

Consider now two adjacent line elements *AE* and *EB* (not necessarily of equal length), which move to $A'E'$ and $E'B'$, Fig. 4.1.2. If the rod is stretching **uniformly**, that is, if all

[1] this is the strain at point *A*. The strain at *B* is evidently the same – one can consider the line element *AB* to emanate fom point *B* (it does not matter whether the line element emanates out from the point to the "left" or to the "right")

line elements are stretching in the same proportion along the length of the rod, then $|A'E'|/|AE| = |E'B'|/|EB|$, and $\varepsilon^{(A)} = \varepsilon^{(E)}$; the strain is the same at all points along the rod.

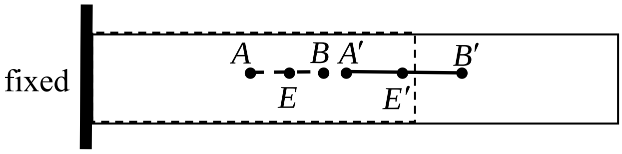


Figure 4.1.2: the strain at a point *A* and the strain at point *E* in a stretched rod

In this case, one could equally choose the line element *AB* or the element *AE* in the calculation of the strain at *A*, since

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} = \frac{|A'E'| - |AE|}{|AE|}$$

In other words it does not matter what the length of the line element chosen for the calculation of the strain at *A* is. In fact, if the length of the rod before stretching is L_0 and after stretching it is L, Fig. 4.1.3, the strain everywhere is (this is equivalent to choosing a "line element" extending the full length of the rod)

$$\varepsilon = \frac{L - L_0}{L_0} \qquad (4.1.2)$$

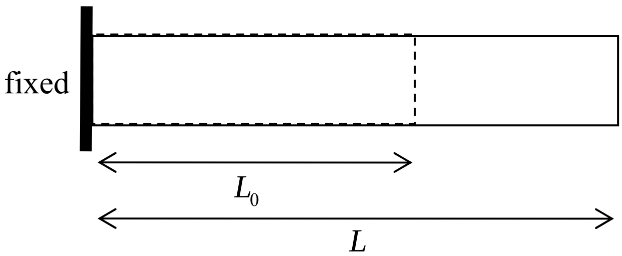


Figure 4.1.3: a stretched slender rod

On the other hand, when the strain is *not* uniform, for example $|A'E'|/|AE| \neq |E'B'|/|EB|$, then the length of the line element does matter. In this case, to be precise, the line element *AB* in the definition of strain in Eqn. 4.1.1 should be "infinitely small"; the smaller the line element, the more accurate will be the evaluation of the strain. The strains considered in this book will be mainly uniform; non-uniform strain will be dealt with in detail in Book II.

Displacement, Strain and Rigid Body Motions

To highlight the difference between displacement and strain, and their relationship, consider again the stretched rod of Fig 4.1.1. Fig 4.1.4 shows the same rod: the two points *A* and *B* undergo displacements $u^{(A)} = |AA'|, u^{(B)} = |BB'|$. The strain at *A*, Eqn 4.1.1, can be re-expressed in terms of these displacements:

$$\varepsilon^{(A)} = \frac{u^{(B)} - u^{(A)}}{|AB|} \tag{4.1.3}$$

In words, the strain is a measure of the *change* in displacement as one moves along the rod.

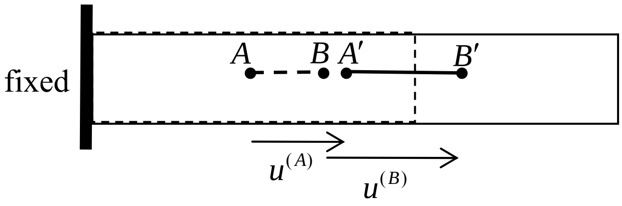


Figure 4.1.4: displacements in a stretched rod

Consider a line element emanating from the left-hand fixed end of the rod. The displacement at the fixed end is zero. However, the strain at the fixed end is *not* zero, since the line element there will change in length. This is a case where the displacement is zero but the strain is not zero.

Consider next the case where the rod is not fixed and simply moves/translates in space, without any stretching, Fig. 4.1.5. This is a case where the displacements are all non-zero (and in this case everywhere the same) but the strain is everywhere zero. This is in fact a feature of a good measure of strain: it should be zero for any rigid body motion; the strain should only measure the deformation.

Figure 4.1.5: a rigid body translation of a rod

Note that if one knows the strain at all points in the rod, one cannot be sure of the rod's exact position in space – again, this is because strain does not include information about possible rigid body motion. To know the precise position of the rod, one must also have some information about the displacements.

The True Strain

As mentioned, there are many ways in which deformation can be measured. Many different strains measures are in use apart from the engineering strain, for example the Green-Lagrange strain and the Euler-Almansi strain: referring again to Fig. 4.1.1, these are

$$\text{Green-Lagrange } \varepsilon^{(A)} = \frac{|A'B'|^2 - |AB|^2}{2|AB|^2}, \qquad \text{Euler-Alamnsi } \varepsilon^{(A)} = \frac{|A'B'|^2 - |AB|^2}{2|A'B'|^2} \tag{4.1.4}$$

Many of these strain measures are used in more advanced theories of material behaviour, particularly when the deformations are very large. Apart from the engineering strain, just one other measure will be discussed in any detail here: the **true strain** (or **logarithmic strain**), since it is often used in describing material testing (see Chapter 5).

The true strain may be defined as follows: define a small increment in strain to be the change in length divided by the *current* length: $d\varepsilon_t = dL / L$. As the rod of Fig. 4.11 stretches (uniformly), this current length continually changes, and the total strain thus defined is the accumulation of these increments:

$$\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_0}\right). \tag{4.1.5}$$

If a line element is stretched to twice its original length, the (true) strain is 0.69. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.69. The fact that a stretching and a contraction of the material by the same factor results in strains which differ only in sign is one of the reasons for the usefulness of the true strain measure.

Another reason for its usefulness is the fact that the true strain is additive. For example, if a line element stretches in two steps from lengths L_1 to L_2 to L_3, the total true strain is

$$\varepsilon_t = \ln\left(\frac{L_3}{L_2}\right) + \ln\left(\frac{L_2}{L_1}\right) = \ln\left(\frac{L_3}{L_1}\right),$$

which is the same as if the stretching had occurred in one step. This is not true of the engineering strain.

The true strain and engineering strain are related through (see Eqn. 4.1.2, 4.1.5)

$$\varepsilon_t = \ln(1+\varepsilon) \tag{4.1.6}$$

One important consequence of this relationship is that the smaller the deformation, the less the difference between the two strains. This can be seen in Table 4.1 below, which shows the values of the engineering and true strains for a line element of initial length 1mm, at different stretched lengths. (In fact, using a Taylor series expansion, $\varepsilon_t = \ln(1+\varepsilon) \approx \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots$, for small ε.) Almost all strain measures in use are similar in this way: they are defined such that they are more or less equal when the deformation is small. Put another way, when the deformations are small, it does not really matter which strain measure is used, since they are all essentially the same – in that case it is sensible to use the simplest measure.

L_0 (mm)	L (mm)	ε	ε_t
1	2	1	0.693
1	1.5	0.5	0.405
1	1.4	0.4	0.336
1	1.3	0.3	0.262
1	1.2	0.2	0.182
1	1.1	0.1	0.095
1	1.01	0.01	0.00995
1	1.001	0.001	0.000995

Table 4.1: true strain and engineering strain at different stretches

It should be emphasised that one strain measure, e.g. engineering or true, is not more "correct" or better than the other; the usefulness of a strain measure will depend on the application.

4.1.2 Two Dimensional Strain

The two dimensional case is similar to the one dimensional case, in that material deformation can be described by imagining the material to be a collection of small line elements. As the material is deformed, the line elements stretch, or get shorter, only now they can also rotate in space relative to each other. This movement of line elements is encompassed in the idea of strain: the "strain at a point" is all the stretching, contracting and rotating of *all* line elements emanating from that point, with all the line elements together making up the continuous material, as illustrated in Fig. 4.1.6.

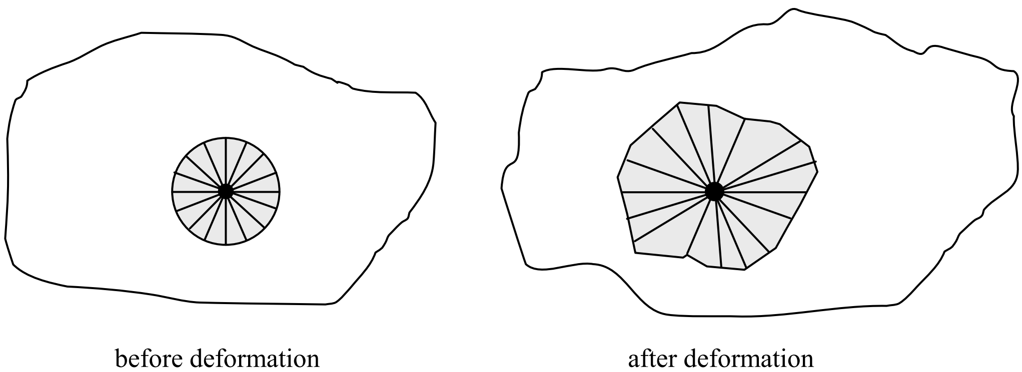


Figure 4.1.6: a deforming material element; original state of line elements and their final position after straining

It turns out that the strain at a point is completely characterised by the movement of *any two mutually perpendicular line-segments*. If it is known how these perpendicular line-segments are stretching, contracting and rotating, it will be possible to determine how any other line element at the point is behaving, by using a **strain transformation rule** (see later). This is analogous to the way the stress at a point is characterised by the stress acting on perpendicular planes through a point, and the stress components on other planes can be obtained using the stress transformation formulae.

So, for the two-dimensional case, consider two perpendicular line-elements emanating from a point. When the material that contains the point is deformed, two things (can) happen:

(1) the line segments will *change length* and
(2) the *angle* between the line-segments *changes*.

The change in length of line-elements is called **normal strain** and the change in angle between initially perpendicular line-segments is called **shear strain**.

As mentioned earlier, a number of different definitions of strain are in use; here, the following, most commonly used, definition will be employed, which will be called the **exact strain**:

Normal strain in direction x: (denoted by ε_{xx})
change in length (per unit length) of a line element originally lying in the $x-$direction

Normal strain in direction y: (denoted by ε_{yy})
change in length (per unit length) of a line element originally lying in the $y-$direction

Shear strain: (denoted by ε_{xy})
(half) the change in the original right angle between the two perpendicular line elements

Referring to Fig. 4.1.7, the (exact) strains are

$$\varepsilon_{xx} = \frac{A'B' - AB}{AB}, \quad \varepsilon_{yy} = \frac{A'C' - AC}{AC}, \quad \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda). \tag{4.1.7}$$

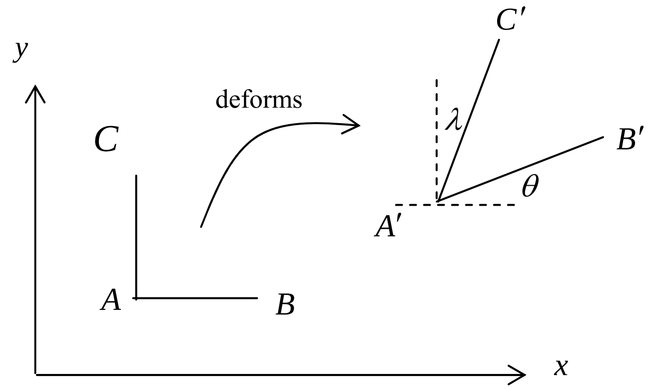


Figure 4.1.7: strain at a point *A*

These 2D strains can be represented in the matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.1.8}$$

As with the stress, the strain matrix is symmetric, with, by definition, $\varepsilon_{xy} = \varepsilon_{yz}$.

Note that the point A in Fig. 4.1.7 has also undergone a displacement $\mathbf{u}(A)$. This displacement has two components, u_x and u_y, as shown in Fig. 4.1.8 (and similarly for the points B and C).

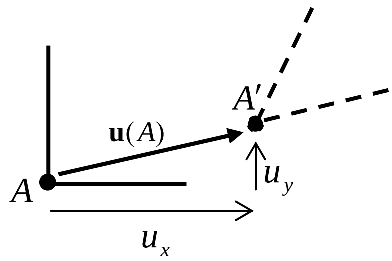


Figure 4.1.8: displacement of a point *A*

The line elements not only change length and the angle between them changes – they can also move in space as rigid-bodies. Thus, for example, the normal and shear strain in the three examples shown in Fig. 4.1.9 are the same, even though the displacements occurring in each case are different – *strain is independent of rigid body motions*.

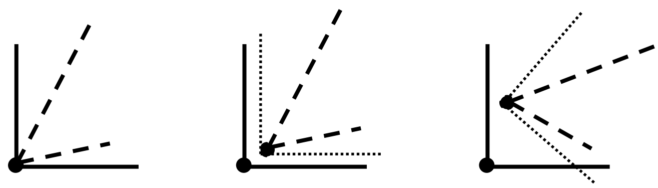


Figure 4.1.9: rigid body motions

The Engineering Strain

Suppose now that the deformation is very small, so that, in Fig. 4.1.10, $A'B' \approx A'B^*$ – here $A'B^*$ is the projection of $A'B'$ in the x – direction. In that case,

$$\varepsilon_{xx} \approx \frac{A'B^* - AB}{AB}. \qquad (4.1.9)$$

Similarly, one can make the approximations

$$\varepsilon_{yy} \approx \frac{A'C^* - AC}{AC}, \quad \varepsilon_{xy} \approx \frac{1}{2}\left(\frac{B^*B'}{AB} + \frac{C^*C'}{AC}\right), \qquad (4.1.10)$$

the expression for shear strain following from the fact that, for a *small angle*, the angle (measured in radians) is approximately equal to the tan of the angle.

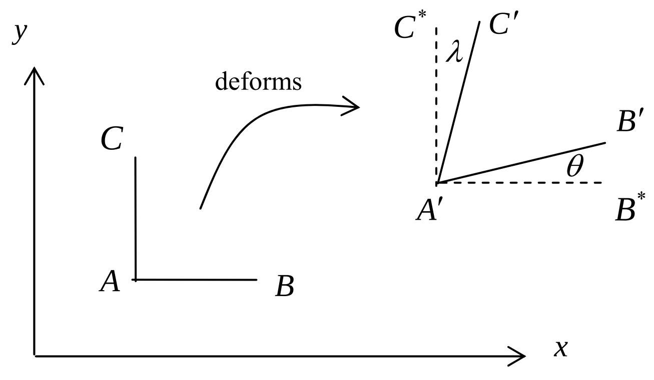


Figure 4.1.10: small deformation

This approximation for the normal strains is called the **engineering strain** or **small strain** or **infinitesimal strain** and is valid when the *deformations are small*. The advantage of the small strain approximation is that the mathematics is simplified greatly.

Example

Two perpendicular lines are etched onto the fuselage of an aircraft. During testing in a wind tunnel, the perpendicular lines deform as in Fig. 4.1.10. The coordinates of the line end-points (referring to Fig. 4.1.10) are:

$$
\begin{array}{ll}
C:(0.0000,1.0000) & C':(0.0025,1.0030) \\
A:(0.0000,0.0000) & A':(0.0000,0.0000) \\
B:(1.0000,0.0000) & B':(1.0045,0.0020)
\end{array}
$$

The exact strains are, from Eqn. 4.1.9, (to 8 decimal places)

$$
\varepsilon_{xx} = \frac{\sqrt{\left|A'B^*\right|^2 + \left|B^*B'\right|^2}}{|AB|} - 1 = 0.00450199
$$

$$
\varepsilon_{yy} = \frac{\sqrt{\left|A'C^*\right|^2 + \left|C^*C'\right|^2}}{|AC|} - 1 = 0.00300312
$$

$$
\varepsilon_{xy} = \frac{1}{2}\left(\arctan\left(\frac{\left|B^*B'\right|}{\left|A'B^*\right|} \right) + \arctan\left(\frac{\left|C^*C'\right|}{\left|A'C^*\right|} \right) \right) = 0.00224178
$$

The engineering strains are, from Eqns. 4.1.10-11,

$$
\varepsilon_{xx} = \frac{\left|A'B^*\right|}{|AB|} - 1 = 0.0045, \quad \varepsilon_{yy} = \frac{\left|A'C^*\right|}{|AC|} - 1 = 0.003, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\left|B^*B'\right|}{|AB|} + \frac{\left|C^*C'\right|}{|AC|} \right) = 0.00225
$$

As can be seen, for the small deformations which occurred, the errors in making the small-strain approximation are extremely small, less than 0.11% for all three strains.

■

Small strain is useful in characterising the small deformations that take place in, for example, (1) engineering materials such as concrete, metals, stiff plastics and so on, (2) linear viscoelastic materials such as many polymeric materials (see Chapter 9), (3) some porous media such as soils and clays at moderate loads, (4) almost any material if the loading is not too high.

Small strain is inadequate for describing large deformations that occur, for example, in many rubbery materials, soft tissues, engineering materials at large loads, etc. In these cases the more precise definition 4.1.7 (or a variant of it), as developed and used in Book III, is required. That said, the engineering strain and the concepts associated with it are an excellent introduction to the more involved large deformation strain measures.

In one dimension, there is no distinction between the exact strain and the engineering strain – they are the same. Differences arise between the two in the two-dimensional case when the material shears (as in the example above), or rotates as a rigid body (as will be discussed further below).

Engineering Shear Strain and Tensorial Shear Strain

The definition of shear strain introduced above is the **tensorial shear strain** ε_{xy}. The **engineering shear strain**[2] γ_{xy} is defined as twice this angle, i.e. as $\theta + \lambda$, and is often used in Strength of Materials and elementary Solid Mechanics analyses.

4.1.3 Sign Convention for Strain

A positive normal strain means that the line element is lengthening. A negative normal strain means the line element is shortening.

For shear strain, one has the following convention: when the two perpendicular line elements are both directed in the positive directions (say x and y), or both directed in the negative directions, then a positive shear strain corresponds to a *decrease* in right angle. Conversely, if one line segment is directed in a positive direction whilst the other is directed in a negative direction, then a positive shear strain corresponds to an *increase* in angle. The four possible cases of shear strain are shown in Fig. 4.1.11a (all four shear strains are positive). A box undergoing a positive shear and a negative shear are also shown, in Figs. 4.1.11b,c.

[2] not to be confused with the term *engineering strain*, i.e. *small strain*, used throughout this Chapter

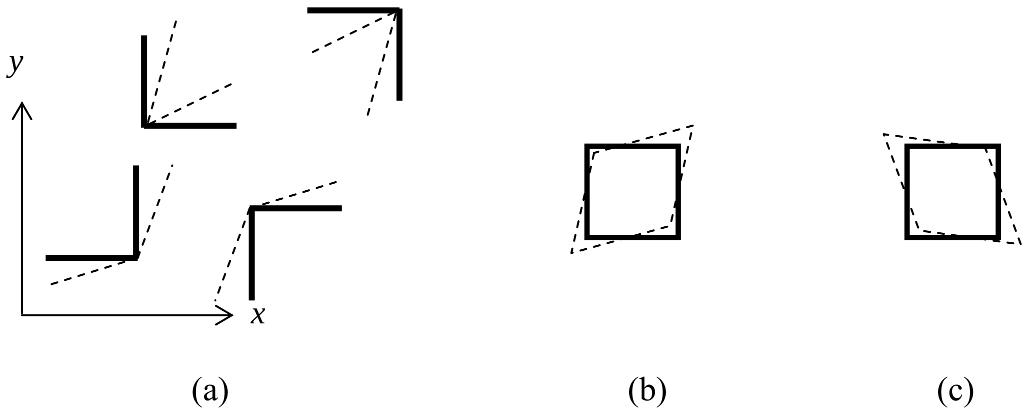


Figure 4.1.11: sign convention for shear strain; (a) line elements undergoing positive shear, (b) a box undergoing positive shear, (c) a box undergoing negative shear

4.1.4 Geometrical Interpretation of the Engineering Strain

Consider a small "box" element and suppose it to be so small that the strain is constant/uniform throughout - one says that the strain is **homogeneous**. This implies that straight lines remain straight after straining and parallel lines remain parallel. A few simple deformations are examined below and these are related to the strains.

A positive normal strain $\varepsilon_{xx} > 0$ is shown in Fig. 4.1.12a. Here the undeformed box element (dashed) has elongated. Knowledge of the strain alone is not enough to determine the position of the strained element, since it is free to move in space as a rigid body. The displacement over some part of the box is usually specified, for example the left hand end has been fixed in Fig. 4.1.12b. A negative normal strain acts in Fig. 4.1.12c and the element has contracted.

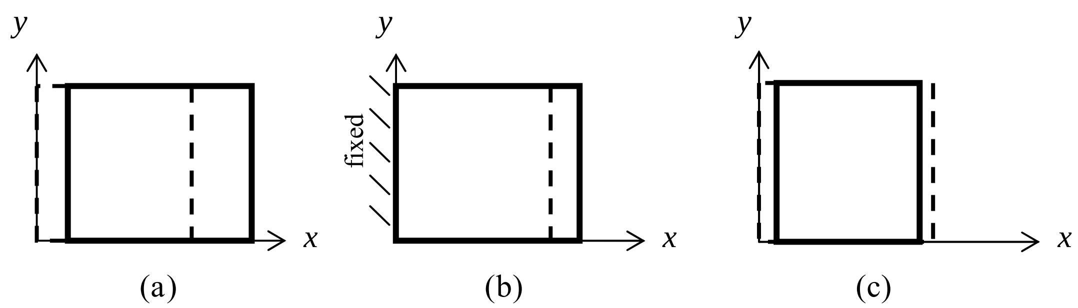


Figure 4.1.12: normal strain; (a) positive normal strain, (b) positive normal strain with the left-hand end fixed in space, (c) negative normal strain

A case known as **simple shear** is shown in Fig. 4.1.13a, and that of **pure shear** is shown in Fig. 4.1.13b. In both illustrations, $\varepsilon_{xy} > 0$. A pure (rigid body) **rotation** is shown in Fig. 4.1.13c (zero strain).

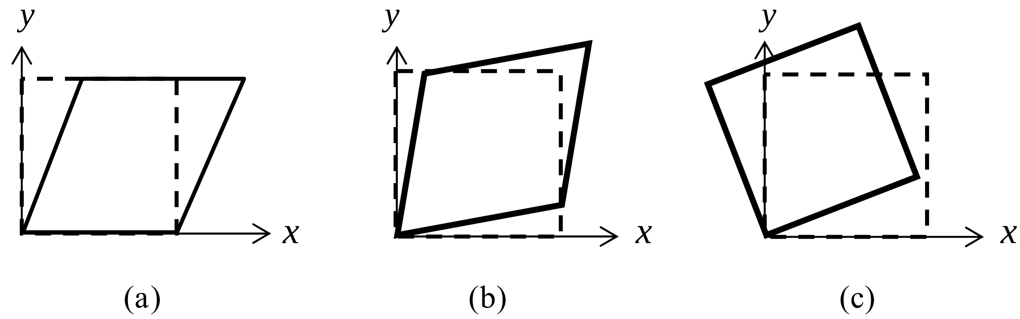


Figure 4.1.13: (a) simple shear, (b) pure shear, (c) pure rotation

Any shear strain can be decomposed into a pure shear and a pure rotation, as illustrated in Fig. 4.1.14.

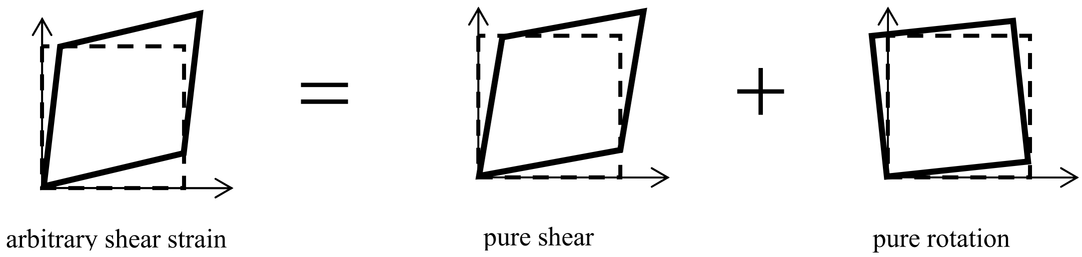


Figure 4.1.14: shear strain decomposed into a pure shear and a pure rotation

4.1.5 Large Rotations and the Small Strain

The example in section 4.2 above illustrated that the small strain approximation is good, provided the deformations are small. However, this is provided also that any *rigid body rotations are small*. To illustrate this, consider a square material element which undergoes a pure rigid body rotation of θ, Fig. 4.1.15. The exact strains remain zero. The small shear strain remains zero also. However, the small normal strains are seen to be $\varepsilon_{xx} = \varepsilon_{yy} = \cos\theta - 1$. Using a Taylor series expansion, this is equal to $\varepsilon_{xx} = \varepsilon_{yy} \approx -\theta^2/2 + \theta^4/24 - \cdots$. Thus, when θ is small, the rotation-induced strains are of the magnitude/order θ^2. If θ is of the same order as the strains themselves, i.e. in the range $10^{-6} - 10^{-2}$, then θ^2 will be very much smaller than θ and the rotation-induced strains will not introduce any inaccuracy; the small strains will be a good approximation to the actual strains. If, however, the rotation is large, then the engineering normal strains will be wildly inaccurate. For example, when $\theta = 45^\circ$, the rotation-induced normal strains are ≈ -0.3, and will likely be larger than the actual strains occurring in the material.

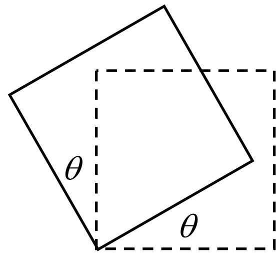


Figure 4.1.15: an element undergoing a rigid body rotation

As an example, consider a cantilevered beam which undergoes large bending, Fig. 4.1.16. The shaded element shown might well undergo small normal and shear strains. However, because of the large rotation of the element, additional spurious engineering normal strains are induced. Use of the precise definition, Eqn. 4.1.7, is required in cases such as this.

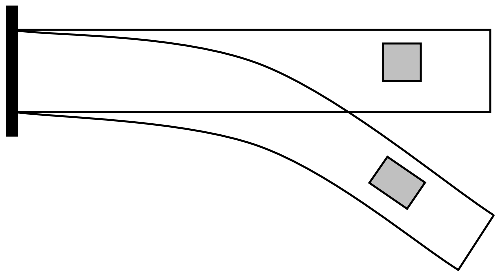


Figure 4.1.16: Large rotations of an element in a bent beam

4.1.6 Three Dimensional Strain

The above can be generalized to three dimensions. In the general case, there are three normal strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, and three shear strains, $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$. The ε_{zz} strain corresponds to a change in length of a line element initially lying along the *z* axis. The ε_{yz} strain corresponds to half the change in the originally right angle of two perpendicular line elements aligned with the *y* and *z* axes, and similarly for the ε_{zx} strain. Straining in the $y-z$ plane ($\varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yz}$) is illustrated in Fig. 4.1.17 below.

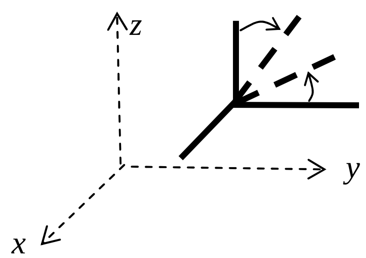


Figure 4.1.17: strains occurring in the $y-z$ plane

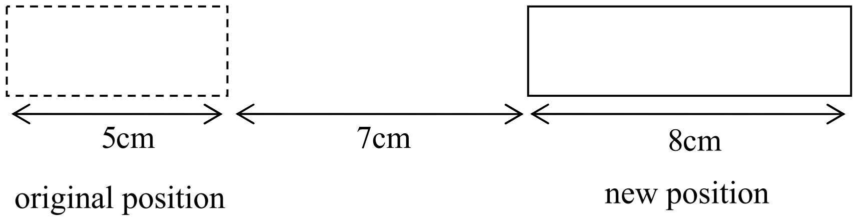
The 3D strains can be represented in the (symmetric) matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{4.1.11}$$

As with the stress (see Eqn. 3.4.4), there are nine components in 3D, with 6 of them being independent.

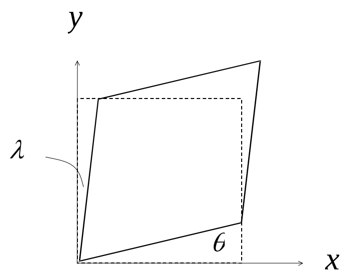
4.1.7 Problems

1. Consider a rod which moves and deforms (uniformly) as shown below.
   (a) What is the displacement of the left-hand end of the rod?
   (b) What is the engineering strain at the left-hand end of the rod



2. A slender rod of initial length 2cm is extended (uniformly) to a length 4cm. It is then compressed to a length of 3cm.
   (a) Calculate the engineering strain and the true strain for the extension
   (b) Calculate the engineering strain and the true strain for the compression
   (c) Calculate the engineering strain and the true strain for one step, i.e. an extension from 2cm to 3cm.
   (d) From your calculations in (a,b,c), which of the strain measures is additive?

3. An element undergoes a homogeneous strain, as shown. There is no normal strain in the element. The angles are given by $\lambda = 0.001$ and $\theta = 0.002$ radians. What is the (tensorial) shear strain in the element?

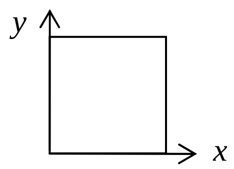


4. In a fixed $x-y$ reference system established for the test of a large component, three points *A*, *B* and *C* on the component have the following coordinates before and after loading (see figure):

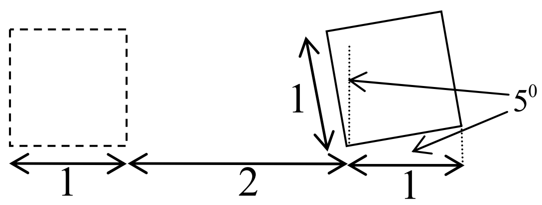
$$C : (0.0000, 1.5000) \qquad C' : (-0.0025, 1.5030)$$
$$A : (0.0000, 0.0000) \qquad A' : (0.0000, 0.0000)$$
$$B : (2.0000, 0.0000) \qquad B' : (2.0045, 0.0000)$$

Determine the actual strains and the small strains (at/near point *A*). What is the error in the small strain compared to the actual strains?

5. Sketch the deformed shape for the material shown below under the following strains (*A*, *B* constant):
   (i) $\varepsilon_{xx} = A > 0$ (taking $\varepsilon_{yy} = \varepsilon_{xy} = 0$) – assume that the right-hand edge is fixed
   (ii) $\varepsilon_{yy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{xy} = 0$) – assume that the lower edge is fixed
   (iii) $\varepsilon_{xy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{yy} = 0$) – assume that the left-hand edge is fixed



6. The element shown below undergoes the change in position and dimensions shown (dashed square = undeformed). What are the three engineering strains $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$?



4.2 Plane Strain

A state of plane strain is defined as follows:

Plane Strain: If the strain state at a material particle is such that the only non-zero strain components act in one plane only, the particle is said to be in plane strain.

The axes are usually chosen such that the $x-y$ plane is the plane in which the strains are non-zero, Fig. 4.2.1.

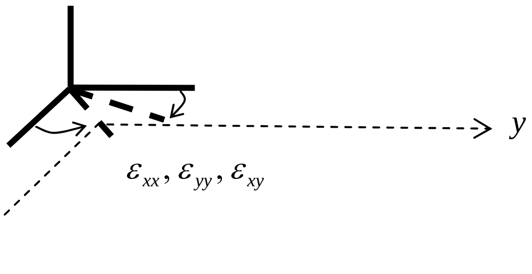


Figure 4.2.1: non-zero strain components acting in the $x-y$ plane

Then $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$. The fully three dimensional strain matrix reduces to a two dimensional one:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.2.1}$$

4.2.1 Analysis of Plane Strain

Stress transformation formulae, principal stresses, stress invariants and formulae for maximum shear stress were presented in §4.4-§4.5. The strain is very similar to the stress. They are both mathematical objects called tensors, having nine components, and all the formulae for stress hold also for the strain. All the equations in section 3.5.2 are valid again in the case of plane strain, with σ replaced with ε. This will be seen in what follows.

Strain Transformation Formula

Consider two perpendicular line-elements lying in the coordinate directions x and y, and suppose that it is known that the strains are $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$, Fig. 4.2.2. Consider now a second coordinate system, with axes x', y', oriented at angle θ to the first system, and consider line-elements lying along these axes. Using some trigonometry, it can be shown that the line-elements in the second system undergo strains according to the following (two dimensional) **strain transformation equations** (see the Appendix to this section, §4.2.5, for their derivation):

$$\boxed{\begin{aligned} \varepsilon'_{xx} &= \cos^2\theta\varepsilon_{xx} + \sin^2\theta\varepsilon_{yy} + \sin 2\theta\varepsilon_{xy} \\ \varepsilon'_{yy} &= \sin^2\theta\varepsilon_{xx} + \cos^2\theta\varepsilon_{yy} - \sin 2\theta\varepsilon_{xy} \\ \varepsilon'_{xy} &= \sin\theta\cos\theta(\varepsilon_{yy} - \varepsilon_{xx}) + \cos 2\theta\varepsilon_{xy} \end{aligned}}$$

Strain Transformation Formulae (4.2.2)

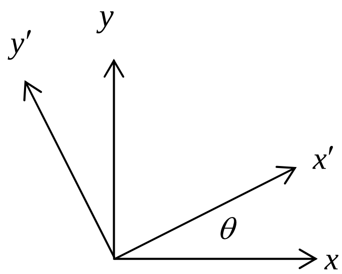


Figure 4.2.2: A rotated coordinate system

Note the similarity between these equations and the stress transformation formulae, Eqns. 3.4.8. Although they have the same structure, the stress transformation equations were derived using Newton's laws, whereas no physical law is used to derive the strain transformation equations 4.2.2, just trigonometry.

Eqns. 4.2.2 are valid only when the strains are small (as can be seen from their derivation in the Appendix to this section), and the engineering/small strains are assumed in all which follows. The exact strains, Eqns. 4.1.7, do not satisfy Eqn. 4.2.2 and for this reason they are rarely used in 2D analyses – when the strains are large, other strain measures, such as those in Eqns. 4.1.4, are used.

Principal Strains

Using exactly the same arguments as used to derive the expressions for principal stress, there is always at least one set of perpendicular line elements which stretch and/or contract, but which do not undergo angle changes. The strains in this special coordinate system are called **principal strains**, and are given by (compare with Eqns. 3.5.5)

$$\boxed{\begin{aligned} \varepsilon_1 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2} \\ \varepsilon_2 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) - \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2} \end{aligned}}$$

Principal Strains (4.2.3)

Further, it can be shown that ε_1 is the maximum normal strain occurring at the point, and that ε_2 is the minimum normal strain occurring at the point.

The **principal directions**, that is, the directions of the line elements which undergo the principal strains, can be obtained from (compare with Eqns. 3.5.4)

$$\tan 2\theta = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{4.2.4}$$

Here, θ is the angle at which the principal directions are oriented with respect to the x axis, Fig. 4.2.2.

Maximum Shear Strain

Analogous to Eqn. 3.5.9, the maximum shear strain occurring at a point is

$$\varepsilon_{xy}\Big|_{\max} = \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right) \tag{4.2.5}$$

and the perpendicular line elements undergoing this maximum angle change are oriented at 45° to the principal directions.

Example (of Strain Transformation)

Consider the block of material in Fig. 4.2.3a. Two sets of perpendicular lines are etched on its surface. The block is then stretched, Fig. 4.2.3b.

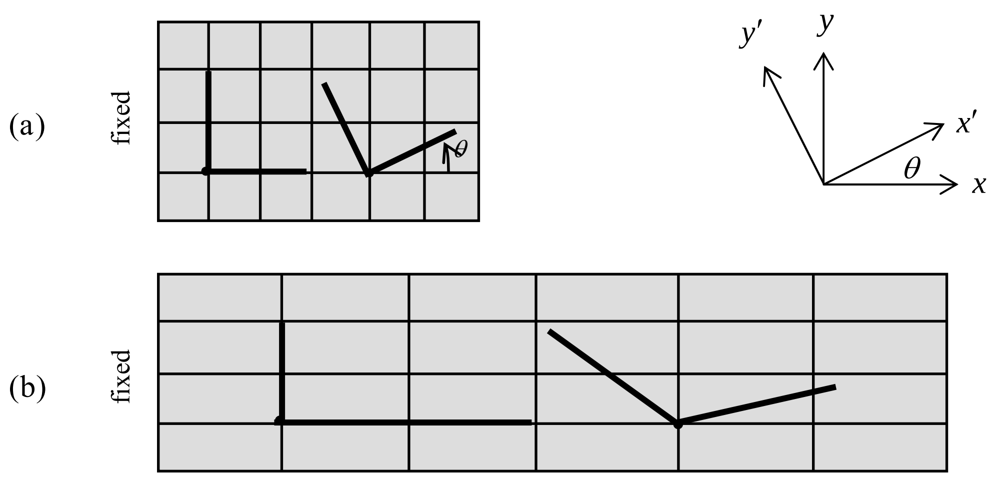


Figure 4.2.3: A block with strain measured in two different coordinate systems

This is a homogeneous deformation, that is, the strain is the same at all points. However, in the $x-y$ description, $\varepsilon_{xx} > 0$ and $\varepsilon_{yy} = \varepsilon_{xy} = 0$, but in the $x'-y'$ description, none of the strains is zero. The two sets of strains are related through the strain transformation equations.

■

Example (of Strain Transformation)

As another example, consider a square material element which undergoes a pure shear, as illustrated in Fig. 4.2.4, with

$$\varepsilon_{xx} = \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = 0.01$$

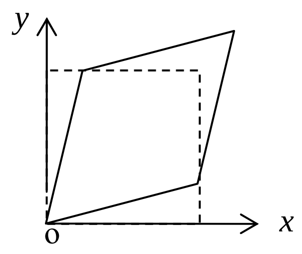


Figure 4.2.4: A block under pure shear

From Eqn. 4.2.3, the principal strains are $\varepsilon_1 = +0.01, \varepsilon_2 = -0.01$ and the principal directions are obtained from Eqn. 4.2.4 as $\theta = \pm 45°$. To find the direction in which the maximum normal strain occurs, put $\theta = +45°$ in the strain transformation formulae to find that $\varepsilon_1 = \varepsilon'_{xx} = +0.01$, so the deformation occurring in a piece of material whose sides are aligned in these principal directions is as shown in Fig. 4.2.5.

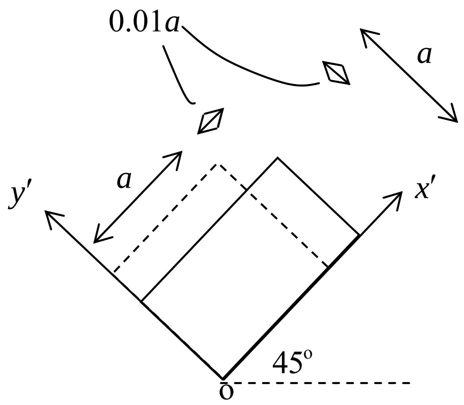


Figure 4.2.5: Principal strains for the block in pure shear

The strain as viewed along the principal directions, and using the $x - y$ system, are as shown in Fig. 4.2.6.

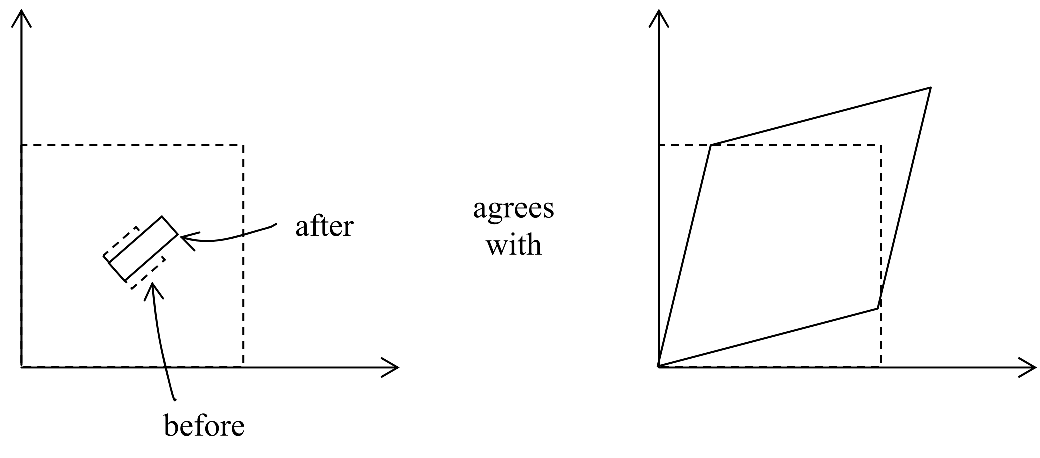


Figure 4.2.6: Strain viewed from two different coordinate systems

Note that, since the original $x-y$ axes were oriented at $\pm 45°$ to the principal directions, these axes are those of maximum shear strain – the original $\varepsilon_{xy} = 0.01$ is the maximum shear strain occurring at the material particle.

■

4.2.2 Thick Components

It turns out that, just as the state of plane stress often arises in thin components, a state of plane strain often arises in very thick components.

Consider the three dimensional block of material in Fig. 4.2.7. The material is constrained from undergoing normal strain in the z direction, for example by preventing movement with rigid immovable walls – and so $\varepsilon_{zz} = 0$.

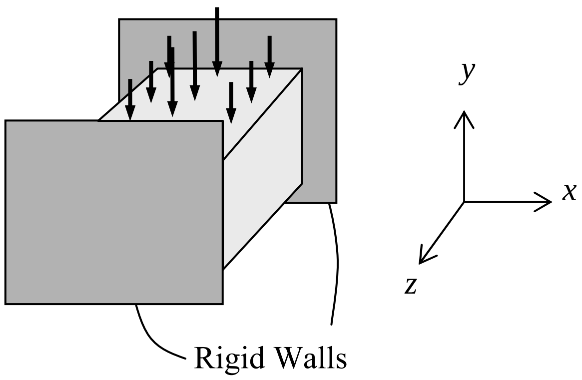


Figure 4.2.7: A block of material constrained by rigid walls

If, in addition, the loading is as shown in Fig. 4.2.7, i.e. it is the same on all cross sections parallel to the $y-z$ plane (or $x-z$ plane) – then the line elements shown in Fig. 4.2.8 will remain perpendicular (although they might move out of plane).

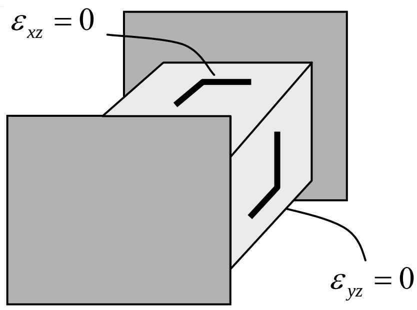


Figure 4.2.8: Line elements etched in a block of material – they remain perpendicular in a state of plane strain

Then $\varepsilon_{xz} = \varepsilon_{yz} = 0$. Thus a state of plane strain will arise.

The problem can now be analysed using the three independent strains, which simplifies matters considerable. Once a solution is found for the deformation of one plane, the solution has been found for the deformation of the whole body, Fig. 4.2.9.

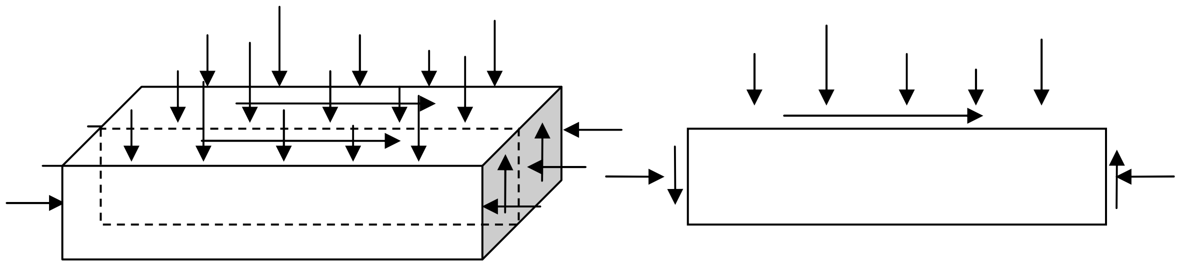


Figure 4.2.9: three dimensional problem reduces to a two dimensional one for the case of plane strain

Note that reaction stresses σ_{zz} act over the ends of the large mass of material, to prevent any movement in the z direction, i.e. ε_{zz} strains, Fig. 4.2.10.

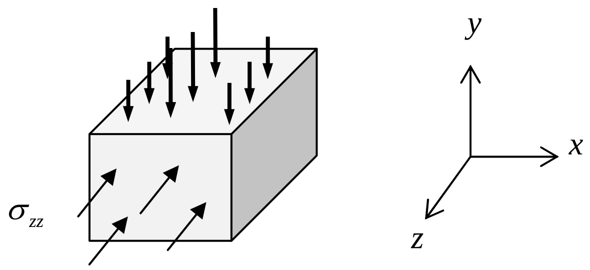


Figure 4.2.10: end-stresses required to prevent material moving in the *z* direction

A state of plane strain will also exist in thick structures without end walls. Material towards the centre is constrained by the mass of material on either side and will be (approximately) in a state of plane strain, Fig. 4.2.10.

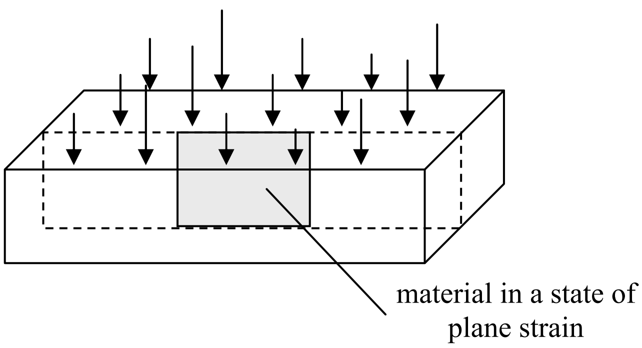


Figure 4.2.10: material in an approximate state of plane strain

The concept of Plane Strain is useful when solving many types of problem involving thick components, even when the ends of the mass of material are allowed to move (as in Fig. 4.2.10); this idea will be explored in the context of **generalised plane strain** and associated topics in Book II.

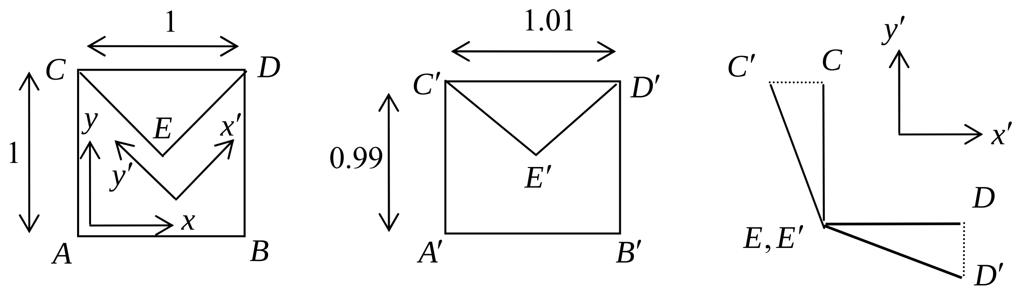
4.2.3 Mohr's Circle for Strain

Because of the similarity between the stress transformation equations 3.4.8 and the strain transformation equations 4.2.2, Mohr's Circle for strain is identical to Mohr's Cirlce for stress, section 3.5.5, with σ replaced by ε (and τ replaced by ε_{xy}).

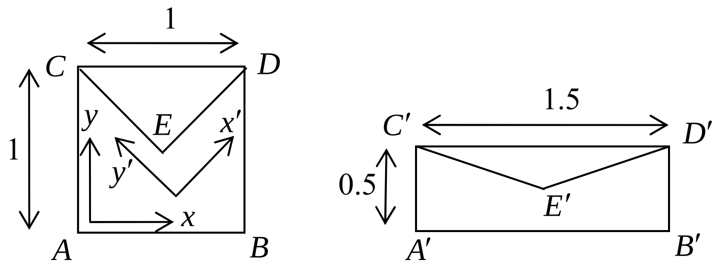
4.2.4 Problems

1. In Fig. 4.2.3, take $\theta = 30^\circ$ and $\varepsilon_{xx} = 0.02$.
   (a) Calculate the strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$.
   (b) What are the principal strains?
   (c) What is the maximum shear strain?
   (d) Of all the line elements which could be etched in the block, at what angle θ to the *x* axis are the perpendicular line elements which undergo the largest angle change from the initial right angle?

2. Consider the undeformed rectangular element below left which undergoes a uniform strain as shown centre.
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$. Hint: use the two half-diagonals *EC* and *ED* sketched; by superimposing points E, E' (to remove the rigid body motion of *E*), it will be seen that point *D* moves straight down and *C* moves left, when viewed along the x', y' axes, as shown below right.
   (c) Use the strain transformation formulae 4.2.2 and your results from (a) to check your results from (b). Are they the same?
   (d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



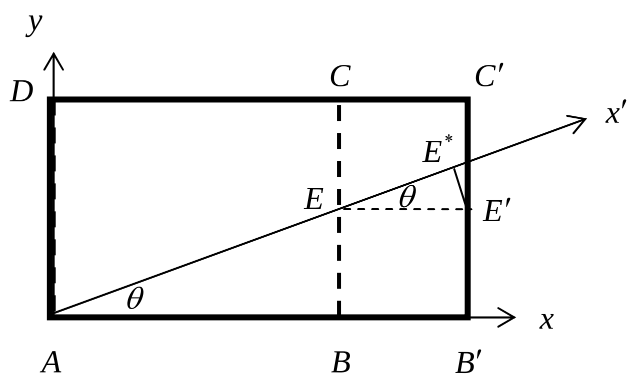
3. Repeat problem 2 only now consider the larger deformation shown below:
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$
   (c) Use the strain transformation formulae 2.4.2 and your results from (a) to check your results from (b). Are they accurate?
   (d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



4.2.5 Appendix to §4.2

Derivation of the Strain Transformation Formulae

Consider an element $ABCD$ undergoing a strain ε_{xx} with $\varepsilon_{yy} = \varepsilon_{xy} = 0$ to $AB'C'D$ as shown in the figure below.



In the $x-y$ coordinate system, by definition, $\varepsilon_{xx} = BB'/AB$. In the $x'-y'$ system, AE moves to AE', and one has

$$\varepsilon'_{xx} = \frac{EE^*}{AE} = \frac{\cos\theta\, EE'}{AB/\cos\theta} = \cos^2\theta \frac{BB'}{AB}$$

which is the first term of the first of Eqn. 4.2.2. The remainder of the transformation formulae can be derived in a similar manner.

4.3 Volumetric Strain

The volumetric strain is the unit change in volume due to a deformation. It is an important measure of deformation and is discussed in what follows.

4.3.1 Two-Dimensional Volumetric Strain

Analogous to Eqn 3.5.1, the strain invariants are

$$\boxed{\begin{aligned} I_1 &= \varepsilon_{xx} + \varepsilon_{yy} \\ I_2 &= \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2 \end{aligned}} \qquad \textbf{Strain Invariants} \qquad (4.3.1)$$

Using the strain transformation formulae, Eqns. 4.2.2, it will be verified that these quantities remain unchanged under any rotation of axes.

The first of these has a very significant physical interpretation. Consider the deformation of the material element shown in Fig. 4.3.1a. The unit change in volume, called the **volumetric strain**, is

$$\begin{aligned} \frac{\Delta V}{V} &= \frac{(a+\Delta a)(b+\Delta b) - ab}{ab} \\ &= (1+\varepsilon_{xx})(1+\varepsilon_{yy}) - 1 \\ &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy} \end{aligned} \qquad (4.3.2)$$

If the strains are small, the term $\varepsilon_{xx}\varepsilon_{yy}$ will be very much smaller than the other two terms, and the volumetric strain in that case is given by

$$\boxed{\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy}} \qquad \textbf{Volumetric Strain} \qquad (4.3.3)$$

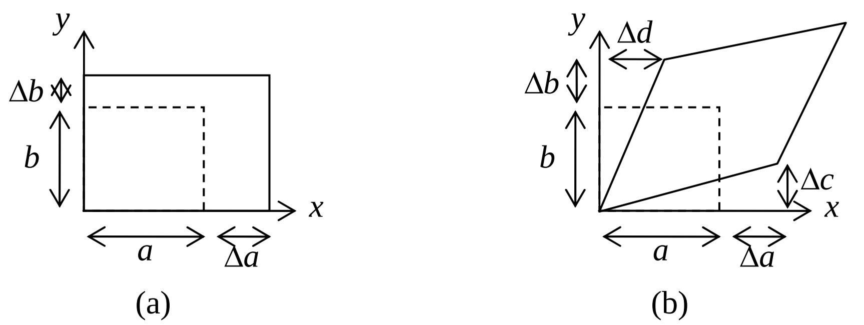


Figure 4.3.1: deformation of a material element; (a) normal deformation, (b) with shearing

Since by Eqn. 4.3.1 the volume change is an invariant, the normal strains in any coordinate system may be used in its evaluation. This makes sense: the volume change cannot depend on the particular axes we choose to measure it. In particular, the principal strains may be used:

$$\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 \tag{4.3.4}$$

The above calculation was carried out for stretching in the *x* and *y* directions, but the result is valid for any arbitrary deformation. For example, for the general deformation shown in Fig. 4.3.1b, the volumetric strain is $\Delta V / V = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy} - (\Delta c / a)(\Delta d / b)$, which again reduced to Eqns 4.3.3, 4.3.4, for small strains.

An important consequence of Eqn. 4.3.3 is that *normal strains induce volume changes*, whereas *shear strains induce a change of shape but no volume change*.

4.3.2 Three Dimensional Volumetric Strain

A slightly different approach will be taken here in the three dimensional case, so as not to simply repeat what was said above.

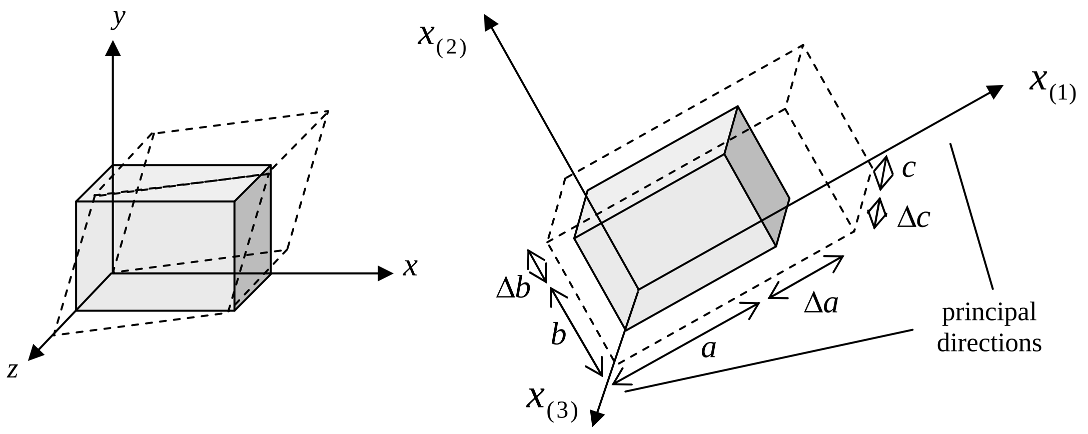
Consider the element undergoing strains $\varepsilon_{xx}, \varepsilon_{xy}$, etc., Fig. 4.3.2a. The same deformation is viewed along the principal directions in Fig. 4.3.2b, for which only normal strains arise.

The volumetric strain is:

$$\begin{aligned} \frac{\Delta V}{V} &= \frac{(a+\Delta a)(b+\Delta b)(c+\Delta c) - abc}{abc} \\ &= (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 \\ &\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \end{aligned} \tag{4.3.5}$$

and the squared and cubed terms can be neglected because of the small-strain assumption.

Since any elemental volume such as that in Fig. 4.3.2a can be constructed out of an infinite number of the elemental cubes shown in Fig. 4.3.3b, this result holds for any elemental volume irrespective of shape.



(a) (b)

Figure 4.3.2: A block of deforming material; (a) subjected to an arbitrary strain; (a) principal strains

5 Material Behaviour and Mechanics Modelling

In this Chapter, the real physical response of various types of material to different types of loading conditions is examined. The means by which a mathematical model can be developed which can predict such real responses is considered.

5.1 Mechanics Modelling

5.1.1 The Mechanics Problem

Typical questions which mechanics attempts to answer were given in Section 1.1. In the examples given, one invariably knows (some of) the forces (or stresses) acting on the material under study, be it due to the wind, water pressure, the weight of the human body, a moving train, and so on. One also often knows something about the displacements along some portion of the material, for example it might be fixed to the ground and so the displacements there are zero. A schematic of such a generic material is shown in Fig. 5.1.1 below.

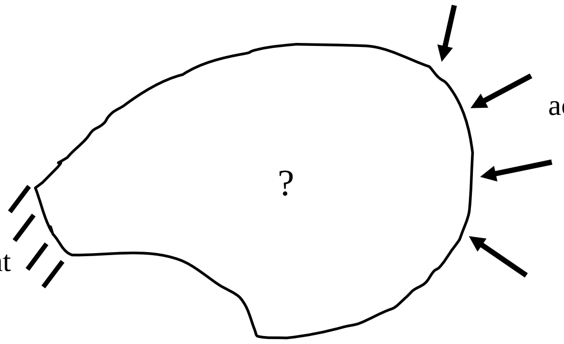


Figure 5.1.1: a material component; force and displacement are known along some portion of the boundary

The basic problem of mechanics is to determine what is happening *inside* the material. This means: what are the stresses and strains inside the material? With this information, one can answer further questions: Where are the stresses high? Where will the material first fail? What can we change to make the material function better? Where will the component move to? What is going on inside the material, at the microscopic level? Generally speaking, what is happening and what will happen?

One can relate the loads on the component to the stresses inside the body using equilibrium equations and one can relate the displacement to internal strains using kinematics relations. For example, consider again the simple rod subjected to tension forces examined in Section 3.3.1, shown again in Fig. 5.1.2. The internal normal stress σ_N on any plane oriented at an angle θ to the rod cross-section is related to the external force F through the equilibrium equation 3.3.1: $\sigma_N = F\cos^2\theta / A$, where A is the cross-sectional area. Similarly, if the ends undergo a separation/displacement of $\Delta = l - l_0$, Fig. 5.1.2b, the strain of any internal line element, at orientation θ, is $\varepsilon_N = \Delta\cos^2\theta / l_0$. However, there is no relationship between this internal stress and internal strain: for any given force, there is no way to determine the internal strain (and hence displacement of the rod); for any given displacement of the rod, there is no way to determine the internal stress (and hence force applied to the rod). The required relationship between stress and strain is discussed next.

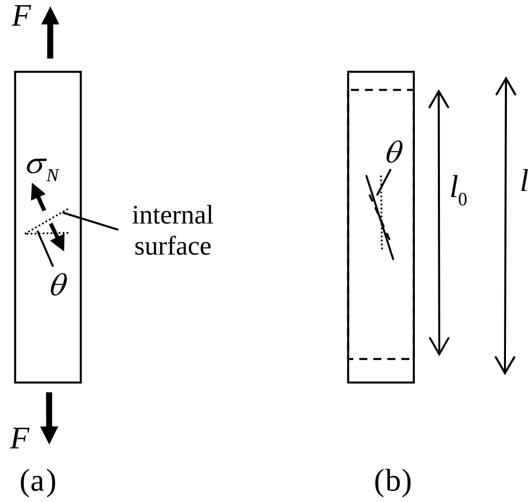


Figure 5.1.2: a slender rod; (a) internal stress due to external force, (b) internal strain due to gross displacement of rod (dotted = before straining)

5.1.2 Constitutive Equation

Stress was discussed in Chapter 3 and strain in Chapter 4. In all that discussion, no mention was made of the particular material under study, be it metallic, polymeric, biological or foodstuff (apart from the necessity that the strain be small when using the engineering strain). The concept of stress and the resulting theory of stress transformation, principal stresses and so on, are based on physical principles (Newton's Laws), which apply to *all* materials. The concept of strain is based, essentially, on geometry and trigonometry; again, it applies to all materials. However, it is the relationship *between* stress and strain which differs from material to material.

The relationship between the stress and strain for any particular material will depend on the microstructure of that material – what constitutes that material. For this reason, the stress-strain relationship is called the **constitutive relation**, or **constitutive law**. For example, metals consist of a closely packed lattice of atoms, whereas a rubber consists of a tangled mass of long-chain polymer molecules; for this reason, the strain in a metal will be different to that in rubber, for any given stress.

The constitutive equation allows the mechanics problem to be solved – this is shown schematically in Fig. 5.1.3.

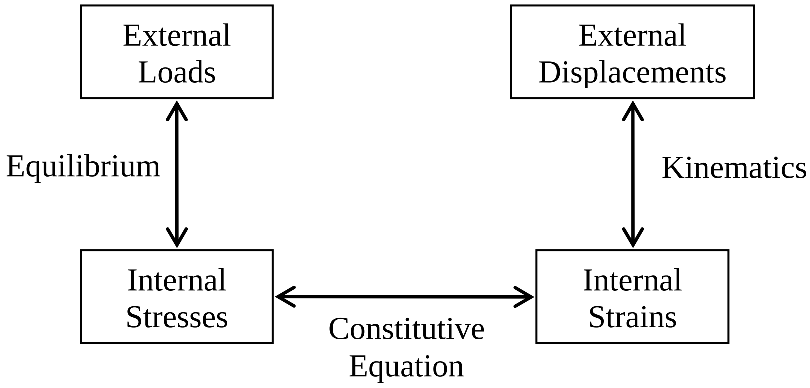


Figure 5.1.3: the role of the constitutive equation in the equations of mechanics

Example Constitutive Equations

A constitutive equation will be of the general form

$$\sigma = f(\varepsilon). \tag{5.1.1}$$

The simplest constitutive equation is a **linear elastic** relation, in which the stress is proportional to the strain:

$$\sigma \propto \varepsilon. \tag{5.1.2}$$

Although no real material satisfies precisely Eqn. 5.1.2, many do so approximately – this type of relation will be discussed in Chapters 6-8. More complex relations can involve the *rate* at which a material is strained or stressed; these types of relation will be discussed in Chapter 10.

More on constitutive equations will follow in Section 5.3.

5.1.3 Mathematical Model

Some of the questions asked earlier can be answered using experimentation. For example, one could use a car-crash test to determine the weakest points in a car. However, one cannot carry our multiple tests for each and every possible scenario – different car speeds, different obstacles into which it crashes, and so on; it would be too time-consuming and too expensive. The only practical way in which these questions can be answered is to develop a **mathematical model**. This model consists of the various equations of equilibrium and the kinematics, the constitutive relation, equations describing the shape of the material, etc. (see Fig. 5.1.3). The mathematical model will have many approximations to reality associated with it. For example, it might be assumed that the material is in the shape of a perfect sphere, when in fact it only resembles a sphere. It may be assumed that a load is applied at a "point" when in fact it is applied over a region of the material's surface. Another approximation in the mathematical model is the constitutive equation itself; the relation between stress and strain in any material can be extremely complex, and the constitutive equation can only be an approximation of the reality.

Once the mathematical model has been developed, the various equations can be solved and the model can then be used to *make a prediction*. The prediction of the model can now be tested against reality: a set of well-defined experiments can be carried out – does the material really move to where the model says it will move?

Simple models (simple constitutive relations) should be used as a first step. If the predictions of the model are wildly incorrect, the model can be adjusted (made more complicated), and the output tested again.

The equations associated with simple models can often be solved analytically, i.e. using a pen and paper. More complex models result in complex sets of equations which can only be solved approximately (though, hopefully, accurately) using a computer.

5.2 The Response of Real Materials

The constitutive equation was introduced in the previous section. The means by which the constitutive equation is determined is by carrying out experimental tests on the material in question. This topic is discussed in what follows.

5.2.1 The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching. A typical specimen would have diameter about 1cm and length 5cm, and larger ends so that it can be easily gripped, Fig. 5.2.1a. Specialised machines are used, for example the Instron testing machine shown in Fig. 5.2.1b.

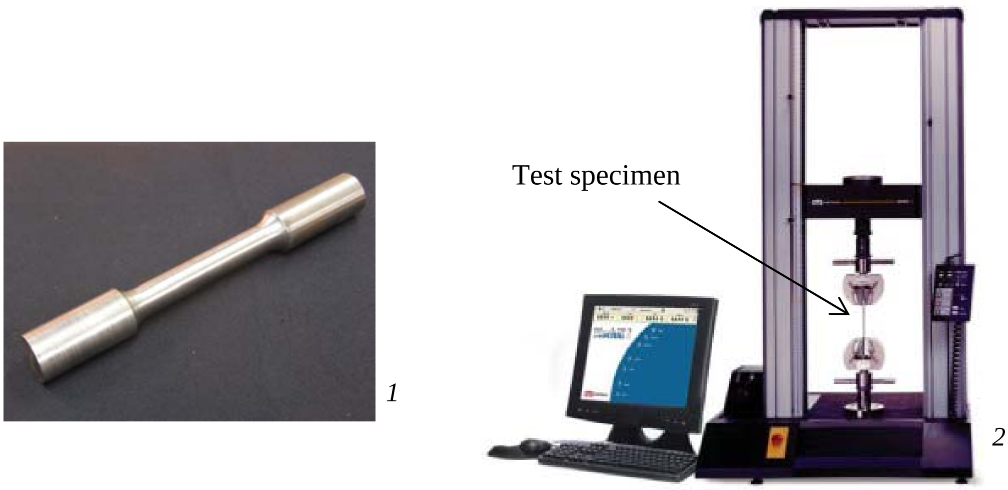


Figure 5.2.1: the tension test; (a) test specimen, (b) testing machine

As the specimen is stretched, the force required to hold the specimen at a given displacement/stretch is recorded[1].

The Engineering Materials

For many of the (hard) engineering materials, the force/displacement curve will look something like that shown in Fig. 5.2.2. It will be found that the force is initially proportional to displacement as with the linear portion OA in Fig. 5.2.2. The following observations will also be made:

(1) if the load has not reached point A, and the material is then unloaded, the force/displacement curve will trace back along the line OA down to zero force and zero displacement; further loading and unloading will again be up and down OA.

(2) the loading curve remains linear up to a certain force level, the **elastic limit** of the material (point A). Beyond this point, **permanent deformations** are induced[2]; on

[1] the very precise details of how the test should be carried out are contained in the special standards for materials testing developed by the American Society for Testing and Materials (ASTM)

unloading to zero force (from point *B* to *C*), the specimen will have a permanent elongation. An example of this response (although not a tension test) can be seen with a paper clip – gently bend the clip and it will “spring back” (this is the *OA* behaviour); bend the clip too much (*AB*) and it will stay bent after you let go (*C*).

(3) above the elastic limit (from *A* to *B*), the material **hardens**, that is, the force required to maintain further stretching, unsurprisingly, keeps increasing. (However, some materials can **soften**, for example granular materials such as soils).

(4) the rate (speed) at which the specimen is stretched makes little difference to the results observed (at least if the speed and/or temperature is not too high).

(5) the strains up to the elastic limit are small, less than 1% (see below for more on strains).

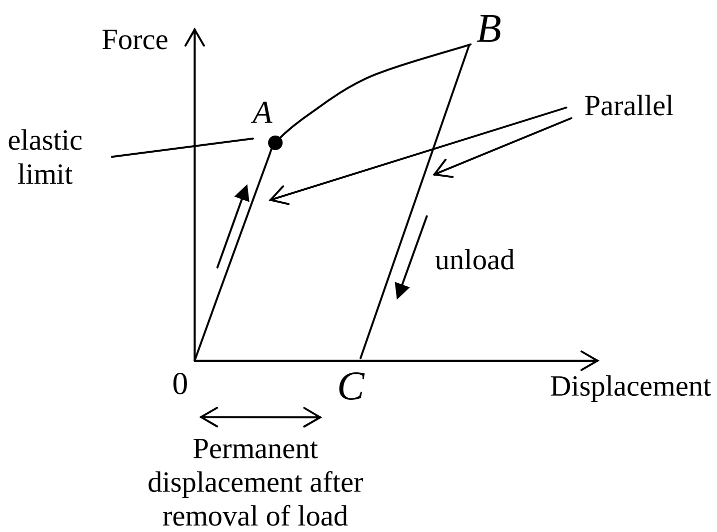


Figure 5.2.2: force/displacement curve for the tension test; typical response for engineering materials

Stress-Strain Curve

There are two definitions of stress used to describe the tension test. First, there is the force divided by the *original* cross sectional area of the specimen A_0; this is the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{5.2.1}$$

Alternatively, one can evaluate the force divided by the (smaller) *current* cross-sectional area *A*, leading to the **true stress**

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[2] if the tension tests are carried out extremely carefully, one might be able to distinguish between a point where the stress-strain curve ceases to be linear (the **proportional limit**) and the elastic limit (which will occur at a slightly higher stress)

$$\sigma = \frac{F}{A} \tag{5.2.2}$$

in which F and A are both changing with time. For small elongations, within the linear range OA, the cross-sectional area of the material undergoes negligible change and both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. As discussed in Section 4.1.1, one can use the engineering strain

$$\varepsilon = \frac{l - l_0}{l_0} \tag{5.2.3}$$

or the true strain

$$\varepsilon_t = \ln\left(\frac{l}{l_0}\right) \tag{5.2.4}$$

Here, l_0 is the original specimen length and l is the current length. Again, at small deformations, the difference between these two strain measures is negligible.

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 5.2.3. The shape of the nominal stress/strain diagram, Fig. 5.2.3a, is of course the same as the graph of force versus displacement. C here denotes the point at which the maximum force the specimen can withstand has been reached. The nominal stress at C is called the **Ultimate Tensile Strength** (UTS) of the material.

After the UTS is reached, the specimen "necks", that is, the specimen begins to deform locally – with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen breaks, as indicated by the asterisk in Fig. 5.2.3. The appearance of a test specimen at each of these stages of the stress-strain curve is shown top of Fig. 5.2.3a.

For many materials, it will be observed that there is little or no volume change during the permanent deformation phase, so $A_0 l_0 = Al$ and $\sigma = \sigma_N(1+\varepsilon)$. This nominal stress to true stress conversion formula will only be valid up to the point of necking.

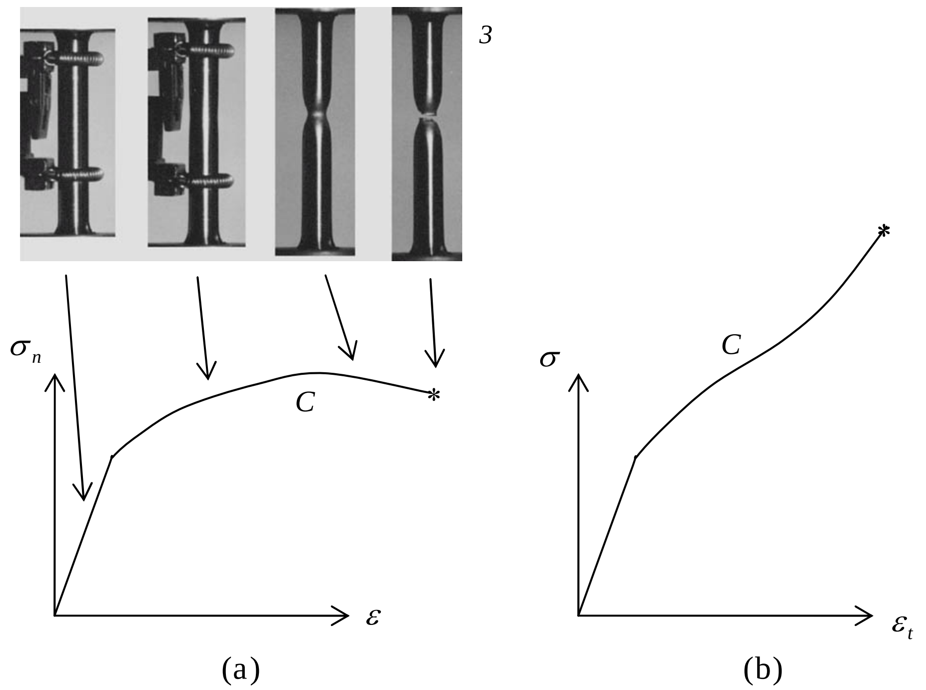


Figure 5.2.3: typical stress-strain curve for an engineering material; (a) engineering stress and strain, (b) true stress and strain

The stress-strain curves for mild steel and aluminium are shown in Fig. 5.2.4. For mild steel, the stress at first increases after reaching the elastic limit, but then decreases. The curve contains a distinct **yield point**; this is where a large increase in strain begins to occur with little increase in required stress[3], i.e. little hardening. There is no distinct yield point for aluminium (or, in fact, for most materials), Fig. 5.2.4b. In this case, it is useful to define a **yield strength** (or **offset yield point**). This is the maximum stress that can be applied without exceeding a specified value of permanent strain. This offset strain is usually taken to be 0.1 or 0.2% and the yield strength is found by following a line parallel to the linear portion until it intersects the stress-strain curve.

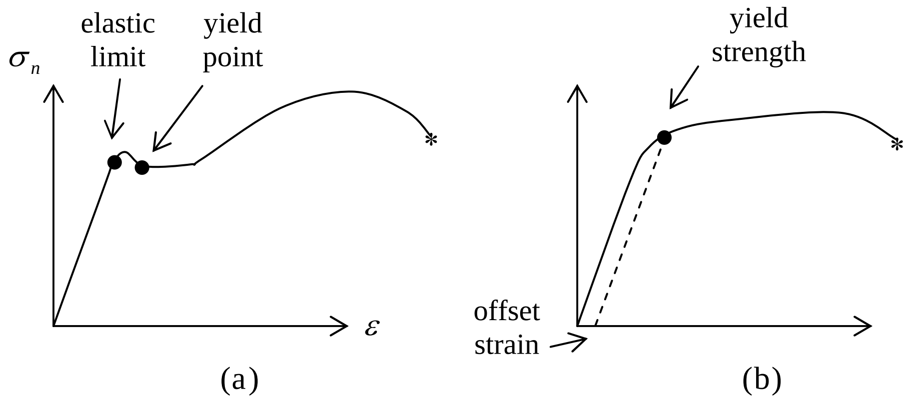


Figure 5.2.4: typical stress-strain curves for (a) mild steel, (b) aluminium

[3] this is also called the **lower yield point**; the **upper yield point** is then the higher stress value just above the elastic limit

The Young's Modulus

The slope of the stress-strain curve over the linear region, before the elastic limit is reached, is the **Young's Modulus** E:

$$E = \frac{\sigma}{\varepsilon} \tag{5.2.5}$$

The Young's Modulus has units of stress and is a measure of how "stiff" a material is.

Eqn. 5.2.5 is a constitutive relation (see Eqn. 5.1.2); it is the **one-dimensional linear elastic** constitutive relation.

Use of the Tension Test Data

What is the data from the tension test used for? First of all, it is of direct use in many structural applications. Many structures, such as bridges, buildings and the human skeleton, are composed in part of relatively long and slender components. In service, these components undergo tension and/or compression, very much like the test specimen in the tension test. The tension test data (the Young's Modulus, the Yield Strength and the UTS) then gives direct information on the amount of stress that these components can safely handle, before undergoing dangerous straining or all-out failure.

More importantly, the tension test data (and similar test data – see below) can be used to predict what will happen when a component of complex three-dimensional shape is loaded in a complex way, nothing like as in the simple tension test. This can be put another way: one must be able to predict the world around us without having to resort to complex, expensive, time-consuming materials testing – one should be able to use the test data from the tension test (and similar simple tests) to achieve this. How this is actually done is a major theme of mechanics modelling and these Books.

Test data for a number of metals are listed in Table 5.2.1 below. Note that although some materials can have similar stiffnesses, for example Nickel and Steel, their relative strengths can be very different.

	Young's Modulus E (GPa)	0.2% Yield Strength (MPa)	Ultimate Tensile Strength (MPa)
Ni	200	70	400
Mild steel	203	220	430
Steel (AISI 1144)	210	540	840
Cu	120	60	400
Al	70	40	200
Al Alloy (2014-T651)	73	415	485

Table 5.2.1: Tensile test data for some metals (at room temperature)

Data as listed above should be treated with caution – it should be used only as a rough guide to the actual material under study; the data can vary wildly depending on the purity and precise nature of the material. For example, the tensile strength of glass as found in a typical glass window is about 50MPa. For fine glass fibres as used in fibre-reinforced plastics and composite materials, the tensile strength can be 4000MPa. In fact, glass is a good reminder as to why the tensile values differ from material to material – it is due to the difference in microstructure. The glass window has many very fine flaws and cracks in it, invisible to the naked eye, and so this glass is not very strong; very fine slivers of glass have no such flaws and are extremely strong – hence their use in engineering applications.

The Poisson's Ratio

Another useful material parameter is the **Poisson's ratio** ν.[4] As the material stretches in the tension test, it gets thinner; the Poisson's ratio is a measure of the ease with which it thins:

$$\nu = -\frac{\Delta w / w_0}{\Delta l / l_0} = -\frac{\varepsilon_w}{\varepsilon} \tag{5.2.6}$$

Here, $\Delta w = w - w_0$, w_0 are the change in thickness and original thickness of the specimen, Fig. 5.2.5; $\Delta l = l - l_0$, l_0 are the change in length and original length of the specimen; $\varepsilon_w = (w - w_0)/w_0$ is the strain in the thickness direction. A negative sign is included because Δw is negative, making the Poisson's ratio a positive number. (It is implicitly assumed here that the material is getting thinner by the same amount in all directions; see below in the context of anisotropy for when this is not the case.)

Most engineering materials have a Poisson's Ratio of about 0.3. Values for a range of materials are listed in Table 5.2.2 further below.

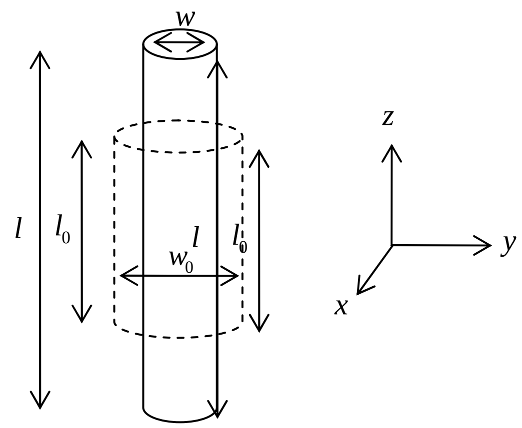


Figure 5.2.5: Change in dimensions of a test specimen

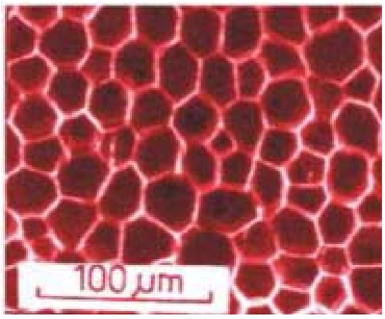
Recall from Section 4.3 that the volumetric strain is given by the sum of the normal strains. There is no harm in re-calculating this for the tensile test specimen of Fig. 5.2.5. One has $\Delta V / V = w^2 l / w_0^2 l_0 - 1$, so that, assuming the strains are small so that the terms $\varepsilon\varepsilon_w$, ε_w^2 and $\varepsilon\varepsilon_w^2$ can be neglected, $\Delta V / V = \varepsilon + 2\varepsilon_w$ (this is the sum of the normal

[4] this is the Greek letter *nu*, not the letter "v"

strains, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$, Fig. 5.2.4). Using the definition of the Poisson's ratio, Eqn. 5.2.6, one has

$$\frac{\Delta V}{V} = \varepsilon(1-2\nu) \tag{5.2.7}$$

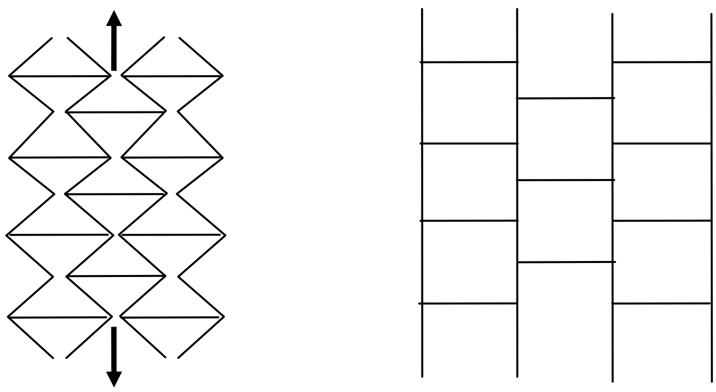
A material which undergoes little volume change thus has a Poisson's Ratio close to 0.5; rubber and other soft tissues, for example biological materials, have Poisson's Ratios very close to 0.5. A material which undergoes zero volume change ($\nu = 0.5$) is called **incompressible** (see more on incompressibility in Section 5.2.4 below). At the other extreme, materials such as cork can have Poisson's Ratios close to zero. The reason for this can be seen from the microstructure of cork shown in Fig. 5.2.6; when tested in compression, the hexagonal honeycomb structure simply folds down, with no necessary lateral expansion.



[4]

Figure 5.2.6: Microstructure of Cork

Auxetic materials are materials which have a negative Poisson's Ratio; when they are stretched, they get thicker. Examples can be found amongst polymers, foams, rocks and biological materials. These materials obviously have a very particular microstructure. A typical example is the network microstructure shown in Fig. 5.2.7.



(a) (b)

Figure 5.2.7: Auxetic material (a) before loading, (b) after loading

Ductile and Brittle Materials

The engineering materials can be grouped into two broad classes: the ductile materials and the brittle materials. The ductile materials undergo large permanent deformations, stretching and necking before failing[5]. The term ductile **rupture** is usually reserved for materials which fail in this way. The separate pieces of the specimen pull away from each other gradually, leaving rough surfaces. A simple measure of ductility is the engineering strain at failure. The brittle materials are generally more stiff and strong, but fail without undergoing much permanent deformation – the tension specimen undergoes a sudden clean break – a **fracture**. The UTS in the case of a brittle material is the same as the failure/fracture stress. Ceramics and glasses are extremely brittle – they fractures suddenly without undergoing any permanent deformation. The difference is illustrated schematically in Fig. 5.2.8 below.

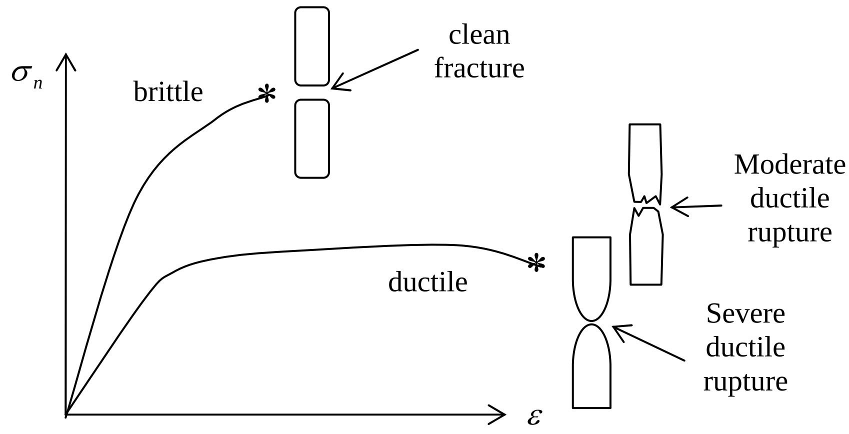


Figure 5.2.8: the difference between ductile and brittle materials

Ductility will depend on temperature – a very cold metal will tend to shatter suddenly, whereas it will stretch more easily when hot.

Soft Materials

Tension test data for non-engineering materials can be very different to that given above. For example, the typical response of a "soft" material, such as rubber, is shown in Fig. 5.2.9. For many soft materials, the elastic limit (or yield strength) can be very high on the stress-strain curve, close to failure. Most of the curve is elastic, meaning that when one unloads the material, the unloading curve traces over the loading curve back down to zero stress and zero strain: the material does not undergo any permanent deformation[6]. Note that the stress-strain curve is non-linear (curved), unlike the straight line elastic portion for a typical metal, Fig. 5.2.2-4, so these materials do not have a single Young's Modulus through which their response can be described.

[5] the term ductile is used for a specimen in tension; the analogous term for compression is **malleability** – a malleable material is easily "squashed"
[6] here, as elsewhere, these statements should not be taken literally; a real rubber will undergo *some* permanent deformation, only it will often be so small that it can be discounted, and an unload curve will never "exactly" trace over a loading curve

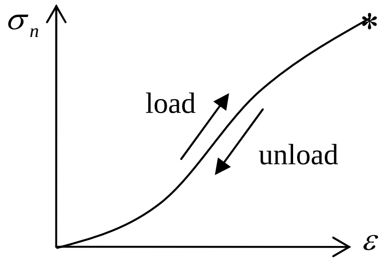
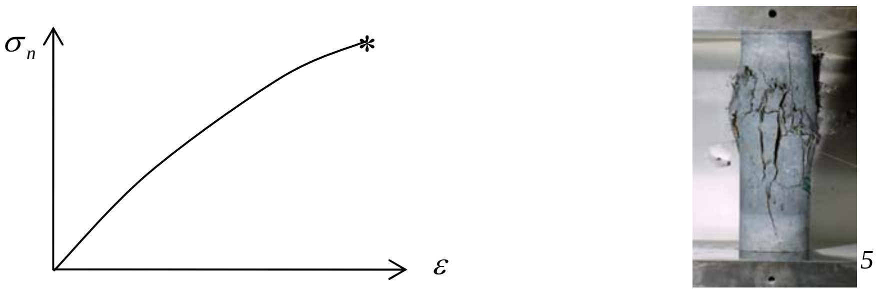


Figure 5.2.9: typical load/unload curves for rubber

5.2.2 Compression Tests

Many materials are used, or designed for use, in compression only, for example soils and concrete. These materials are tested in compression. A common testing method for concrete is to place a cylindrical specimen between two parallel plates and bring the plates together. The typical response of concrete is shown in Fig. 5.2.10a; at failure, the concrete crushes catastrophically, as in the specimen shown in Fig. 5.2.10b. Nominal stresses in the region 20-70MPa are typical and a good concrete would strain to much less than 1% at failure.



(a) (b)

Figure 5.2.10: typical compressive response of concrete; (a) stress-strain curve, (b) specimen at failure

For many materials, e.g. metals, a compression test will lead to similar results as the tensile stress. The yield strength in compression will be approximately the same as (the negative of) the yield strength in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), and the two curves more or less coincide, this would indicate that the behaviour of the material under compression is broadly similar to that under tension. However, if one were to use the nominal stress and strain, then the two curves would not coincide even if the real tensile/compressive behaviour was similar (although they would of course in the small-strain linear region); this is due to the definition of the engineering strain/stress.

5.2.3 Shear Tests

In the **shear test**, the material is subjected to a shear strain $\gamma \equiv 2\varepsilon_{xy}$ by applying a shear stress[7] $\tau \equiv \sigma_{xy}$, Fig. 5.2.11a. The resulting shear stress-strain curve will be similar to the tensile stress-strain curve, Fig. 5.2.11b. The shear stress at failure, the **shear strength**, can be greater or smaller than the UTS. The shear yield strength, on the other hand, is usually in the region of 0.5-0.75 times the tensile yield strength. In the linear small-strain region, the shear stress will be proportional to the shear strain; the constant of proportionality is the **shear modulus** G:

$$G = \frac{\tau}{\gamma} \tag{5.2.8}$$

For many of the engineering materials, $G \approx 0.4E$.

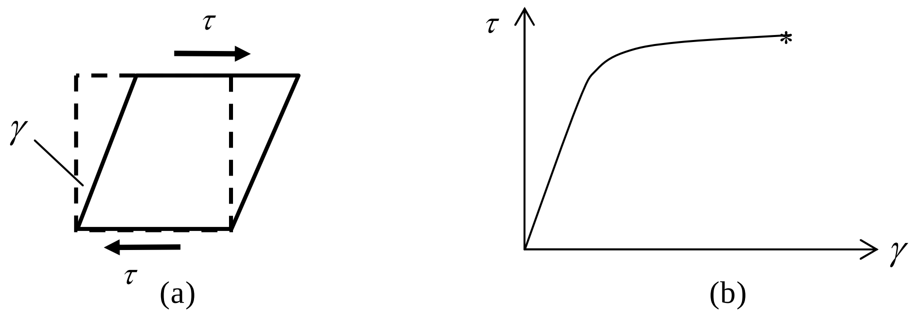


Figure 5.2.11: the shear test; (a) specimen subjected to shear stress and shear strain (dotted = undeformed), (b) shear stress-strain curve

5.2.4 Compressibility

In the **confined compression test**, a sample is placed in a container and a piston is used to compress it at some pressure p, Fig. 5.2.12a. This test can be used to determine how compressible a material is. When a material is compressed by equal pressures on all sides, the ratio of applied pressure p to (unit) volume change, i.e. volumetric strain $\Delta V / V$, is called the **Bulk Modulus** K, Fig. 5.2.12b (this is not quite the situation in Fig. 5.2.12a – the reaction pressures on the side walls will only be about half the applied surface pressure p; see Section 6.2):

$$K = -\frac{p}{\Delta V / V} \tag{5.2.9}$$

The negative sign is included since a positive pressure implies a negative volumetric strain, so that the Bulk Modulus is a positive value.

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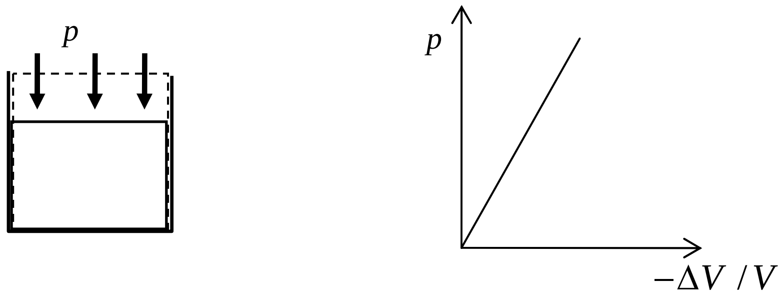
[7] there are many ways that this can be done, for example by pushing blocks of the material over each other, or using more sophisticated methods such as twisting thin tubes of the material (see Section 7.2)



(a) (b)

Figure 5.2.12: the confined compression test; (a) specimen subjected to confined compression, (b) pressure plotted against volume change

A material which can be easily compressed has a low Bulk Modulus. As mentioned earlier, a material which cannot be compressed at all is called incompressible ($K \to \infty$).

No real material is incompressible, but some can be regarded as incompressible so as to make the mechanics modelling easier. For example, the Shear Modulus of rubber is very much smaller than its Bulk Modulus, Table 5.2.2. Essentially, this means that the shape of rubber can be easily changed as compared to its volume. Thus, in applications where a rubber component is being deformed or subjected to arbitrary stressing, it is perfectly reasonable to simply assume that rubber is incompressible. The same applies, only more so, to water; the Shear Modulus is effectively zero and there is no resistance to change in shape (which will be observed on pouring a glass of water on to the ground); it is thus regarded almost always as completely incompressible. On the other hand, even though the Bulk Modulus of the metals and other engineering materials is very much *larger* than that of water or rubber, they are still regarded as compressible in applications – the extremely small changes in volume are significant.

	Young's Modulus E (GPa)	Shear Modulus G (GPa)	Bulk Modulus K (GPa)	Poissons Ratio
Ni	200	76	180	0.31
Mild steel	203	78	138	0.30
Steel (AISI 1144)	210	80	140	0.31
Cu	120	46	142	0.34
Al	70	26	76	0.35
Rubber	14.9×10^{-4}	5×10^{-4}	1	0.49
Water	$\approx 10^{-14}$	$\approx 10^{-14}$	2.2	

Table 5.2.2: Moduli and Poisson's Ratios for a number of materials

5.2.5 Cyclic Tests

Many materials are subjected to complex loading regimes when in service, not simply a one-off stretching, shearing or compression. A classic example are the wings of an aircraft which are continually loaded in tension, then compression, then tension and so on, as in Fig. 5.2.13. Anything moving back and forward is likely to be subjected to this tension/compression-type cyclic loading. Another example would be the stresses experienced by cardiac tissue in a pumping heart.

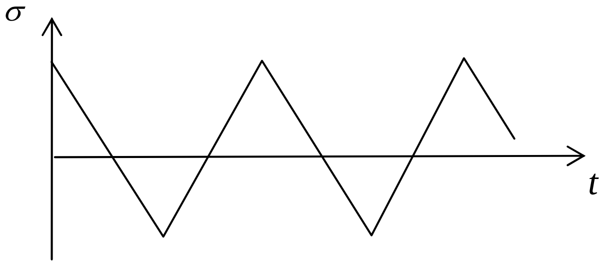
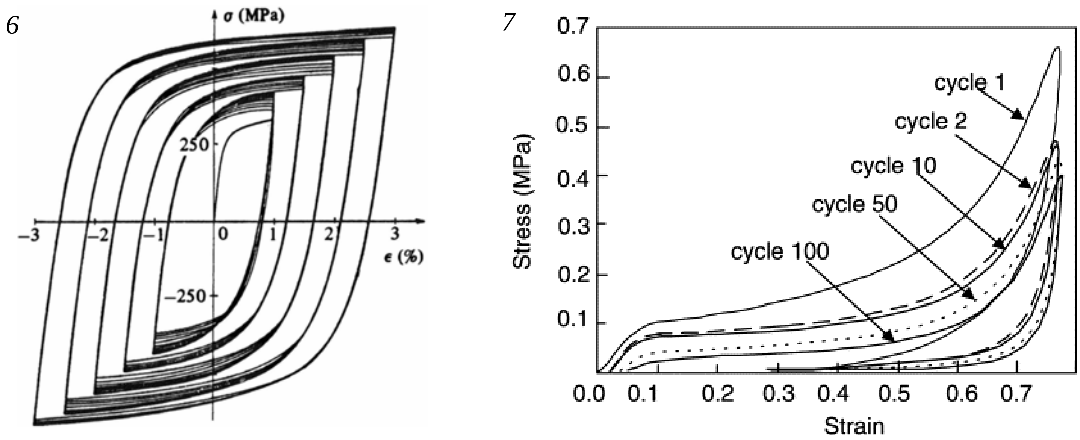


Figure 5.2.13: cyclic loading; alternating between tension (positive stress) and compression (negative stress) over time *t*

Cyclic tests can be carried out to determine the response of materials to such loading cycles. An example is shown in Fig. 5.2.14a, the stress-strain response of a Stainless Steel. The Steel is first cycled between two strain values (one positive, one negative, differing only in sign) a number of times. The stress is seen to increase on each successive cycle. The strain is then increased for a number of further cycles, and so on.

One does not have to move from tension to compression; many materials cycle in only tension or compression. For example, the response to cyclic (compressive) loading of polyurethane foam is shown in Fig. 5.2.14b (note how the loading curve is similar to that in 5.2.9).



(a) (b)

Figure 5.2.14: cyclic loading; (a) cyclic straining of a Stainless Steel, (b) cyclic loading (in compression) of a polyurethane foam

5.2.6 Other Tests

There are other important tests, for example the Vickers and Brinell **hardness tests**, and the **three-point bending test**. The hardness tests will be discussed in Book II. The bending test is discussed in section 7.4.9, in the context of beam theory. Another two very important tests, the **creep test** and the **stress relaxation test**, will be discussed in Chapter 10.

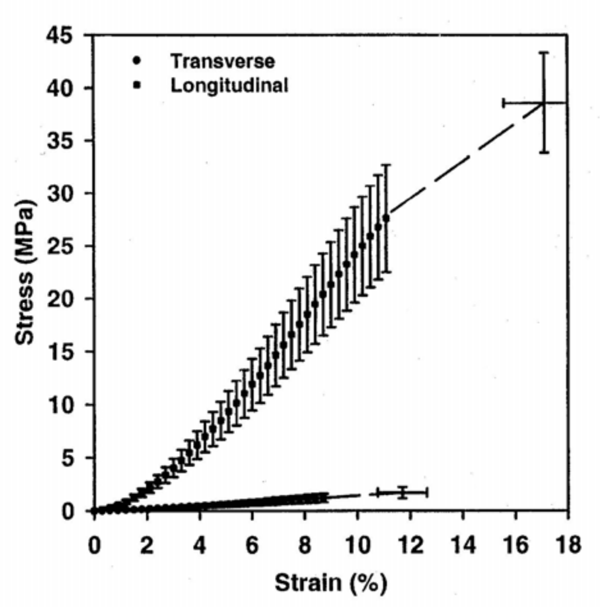
5.2.7 Isotropy and Anisotropy

Many materials have a strong direction-dependence. The classic example is wood, which has a clear structure –along the grain, along which fine lines can be seen, and against the grain, Fig. 5.2.15. The wood is stiffer and stronger along the grain than against the grain. A material which has this direction-dependence of mechanical (and physical) properties is called **anisotropic**.

8

Figure 5.2.15: Wood

Fig. 5.2.16 shows stress-strain curves for human ligament tissue; in one test, the ligament is stretched along its length (the **longitudinal** direction), in the second, across the width of the ligament (the **transverse** direction). It can be seen that the stiffness is much higher in the longitudinal direction. Another example is bone – it is much stiffer along the length of the bone than across the width of the bone. In fact, many biological materials are strongly anisotropic.



9

Figure 5.2.16: Anisotropic response of human ligament

A material whose properties are the same in all directions is called **isotropic**. In particular, the relationship between stress and strain *at any single location* in a material is the same in all directions. This implies that if a specimen is cut from an isotropic material and subjected to a load, it would not matter in which orientation the specimen is cut, the resulting deformation would be the same – as illustrated in Fig. 5.2.17. Most metals and ceramics can be considered to be isotropic (see Section 5.4).

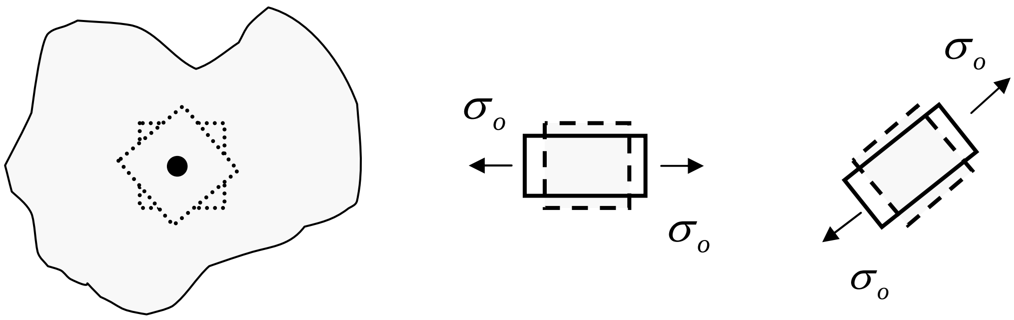


Figure 5.2.17: Illustration of Isotropy; the stress-strain response is the same no matter in what "direction" the test specimen is cut from the material

Anisotropy will be examined in more detail in §6.3. It will be shown there, for example, that an anisotropic material can have a Poisson's ratio greater than 0.5.[8]

5.2.8 Homogeneous Materials

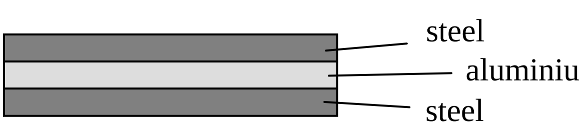
The term homogeneous means that the mechanical properties are the same at each point throughout the material. In other words, the relationship between stress and strain is the same for all material particles. Most materials can be assumed to be homogeneous.

In engineering applications, it is sometimes beneficial to design materials/components which are specifically not homogeneous, i.e. **inhomogeneous**. Such materials whose properties vary gradually throughout are called **Functionally Graded Materials**, and have been gaining popularity since the 1980s-90s in advanced technologies.

Note that a material can be homogeneous and not isotropic, and *vice versa* – homogeneous refers to different locations whereas isotropy refers to the same location.

5.2.9 Problems

1. Steel and aluminium can be considered to be isotropic and homogeneous materials. Is the composite sandwich-structure shown here isotropic and/or homogeneous? Everywhere in the sandwich?



[8] cork was mentioned earlier and it was pointed out that it has a near-zero Poisson's Ratio; actually, cork is quite anisotropic and the Poisson's Ratio in other "directions" will be different (close to 1.0)

Images used:

1. http://site.metacos.com/main/3108/index.asp?pageid=84386&t=&AlbumID=0&page=2
2. http://travisarp.wordpress.com/2012/01/24/how-do-they-do-that-tenderness/
3. http://www.ara.com/Projects/SVO/popups/weld_geometry.html
4. http://10minus9.wordpress.com/2010/03/23/10minus9-interview-philip-moriarty-part-2/
5. http://info.admet.com/blog/?Tag=Compression%20Test
6. Chaboche JL, On some modifications of kinematic hardening to improve the description of ratcheting effects, Int. J. Plasticity 7(7), 661-678, 1991.
7. Shen Y, Golnaraghi F, Plumtree A, Modelling compressive cyclic stress-strain behaviour of structural foam, Int J Fatigue, 23(6), 491-497, 2001.
8. http://www.sufaparket.com/solid-parquet/product-profile-solid/sungkai
9. Quapp KM, Weiss JA, Material characterization of human medial collateral ligament, J Biomech Engng, 120(6), 757-763, 1998.

5.3 Material Models

The response of real materials to various loading conditions was discussed in the previous section. Now comes the task of creating mathematical models which can predict this response. To this end, it is helpful to categorise the material responses into ideal models. There are four broad **material models** which are used for this purpose: (1) the **elastic model**, (2) the **viscoelastic model**, (3) the **plastic model**, and (4) the **viscoplastic model**. These models will be discussed briefly in what follows, and in more depth throughout the rest of this book.

5.3.1 The Elastic Model

An ideal elastic material has the following characteristics:

(i) the unloading stress-strain path is the same as the loading path
(ii) there is no dependence on the rate of loading or straining
(iii) it does not undergo permanent deformation; it returns to its precise original shape when the loads are removed

Typical stress-strain curves for an ideal elastic model subjected to a tension (or (compression) test are shown in Fig. 5.3.1. The response of a **linear elastic material**, where the stress is *proportional* to the strain, is shown in Fig. 5.3.1a and that for a **non-linear elastic material** is shown in Fig. 5.3.1b.

From the discussion in the previous section, the linear elastic model will well represent the engineering materials up to their elastic limit (see, for example, Figs. 5.2.2-4). It will also represent the complete stress-strain response up to the point of fracture of many very brittle materials. The model can also be used to represent the response of almost any material, provided the stresses are sufficiently small.

The non-linear elastic model is useful for predicting the response of soft materials like rubber and biological soft tissue (see, for example Fig. 5.2.9).

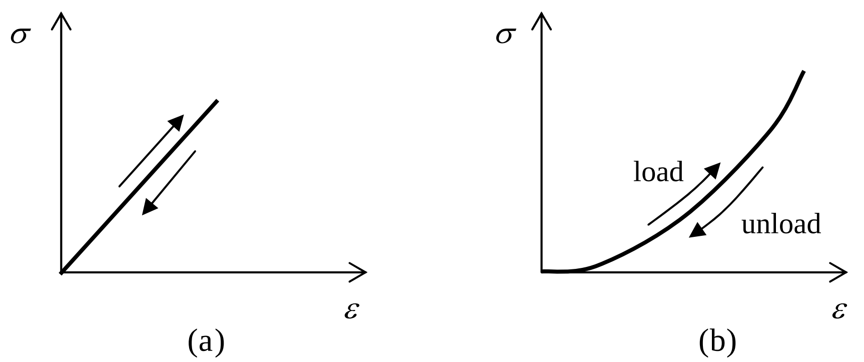


Figure 5.3.1: The Elastic Model; (a) linear elastic, (b) non-linear elastic

It goes without saying that there is no such thing as a purely elastic material. All materials will undergo at least some permanent deformations, even at low loads; no material's response will be exactly the same when stretched at different speeds, and so on.

However, if these occurrences and differences are small enough to be neglected, the ideal elastic model will be useful.

Note also that a prediction of a material's response may be made with accuracy using the elastic model in some circumstances, but not in others. An example would be metal; the elastic model might well be able to predict the response right up to high stress levels when the metal is cold, but not so well when the temperature is high, when inelastic effects may not be so easily disregarded (see below).

5.3.2 Viscoelasticity

When solid materials have some "fluid-like" characteristics, they are said to be viscoelastic. A fluid is something which flows easily when subjected to loading – it cannot keep to any particular shape. If a fluid is one (the "viscous") extreme and the elastic solid is at the other extreme, then the viscoelastic material is somewhere in between.

The typical response of a viscoelastic material is sketched in Fig. 5.3.2. The following will be noted:

(i) the loading and unloading curves do not coincide, Fig. 5.3.2a, but form a **hysteresis loop**
(ii) there is a dependence on the rate of straining $d\varepsilon / dt$, Fig. 5.3.2b; the faster the stretching, the larger the stress required
(iii) there may or may not be some permanent deformation upon complete unloading, Fig. 5.3.2a

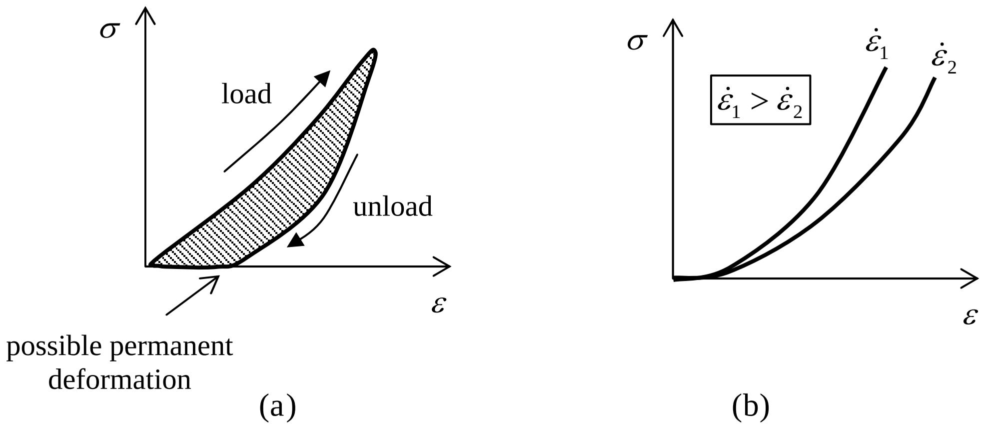


Figure 5.3.2: Response of a Viscoelastic material in the Tension test; (a) loading and unloading with possible permanent deformation (non-zero strain at zero stress), (b) different rates of stretching

The effect of *rate* of stretching shows that the viscoelastic material *depends on time*. This contrasts with the elastic material; it makes no difference whether an elastic material is loaded to some given stress level for one second or one day, or quickly or slowly, the resulting strain will be the same. This rate effect can be seen when you push your hand through water – it is easier to do so when you push slowly than when you push fast.

Depending on how "fluid-like" or "solid-like" a material is, it can be considered to be a **viscoelastic fluid**, for example blood or toothpaste, or a **viscoelastic solid**, for example Silly Putty™ or foam. That said, the model for both and the theory behind each will be similar.

Viscoelastic materials will be discussed in detail in Chapter 10.

5.3.3 Plasticity

Plasticity has the following characteristics:

(i) The loading is elastic up to some threshold limit, beyond which permanent deformations occur

(ii) The permanent deformation, i.e. the **plasticity**, is time independent

This plasticity can be seen in Figs. 5.2.2-4. The threshold limit – the elastic limit – can be quite high but it can also be extremely small, so small that significant permanent deformations occur at almost any level of loading. The plasticity model is particularly useful in describing the permanent deformations which occur in metals, soils and other engineering materials. It will be discussed in further detail in Chapter 11.

5.3.4 Viscoplasticity

Finally, the viscoplastic model is a combination of the viscoelastic and plastic models. In this model, the plasticity is rate-dependent. One of the main applications of the model is in the study of metals at high temperatures, but it is used also in the modeling of a huge range of materials and other applications, for example asphalt, concrete, clay, paper pulp, biological cells growth, etc. This model will be discussed in Chapter 12.

5.4 Continuum Models and Micromechanics

The models mentioned in the previous section are **continuum models**. What this means is explained in what follows.

5.4.1 Stress and Scale

In the definition of the traction vector, §3.3.1, it was assumed that the ratio of force over area would reach some definite limit as the area ΔS of the surface upon which the force ΔF acts was shrunk to zero. This issue can be explored further by considering Fig. 5.4.1. Assume first that the plane upon which the force acts is fairly large; it is then shrunk and the ratio F/S tracked. A schematic of this ratio is shown in Fig. 5.4.2. At first (to the right of Fig. 5.4.2) the ratio F/S undergoes change, assuming the stress to vary within the material, as it invariably will if the material is loaded in some complex way. Eventually the plane will be so small that the ratio changes very little, perhaps with some small variability ε. If the plane is allowed to get too small, however, down below some distance h^* say and down towards the atomic level, where one might encounter "intermolecular space", there will be large changes in the ratio and the whole concept of a force acing on a single surface breaks down.

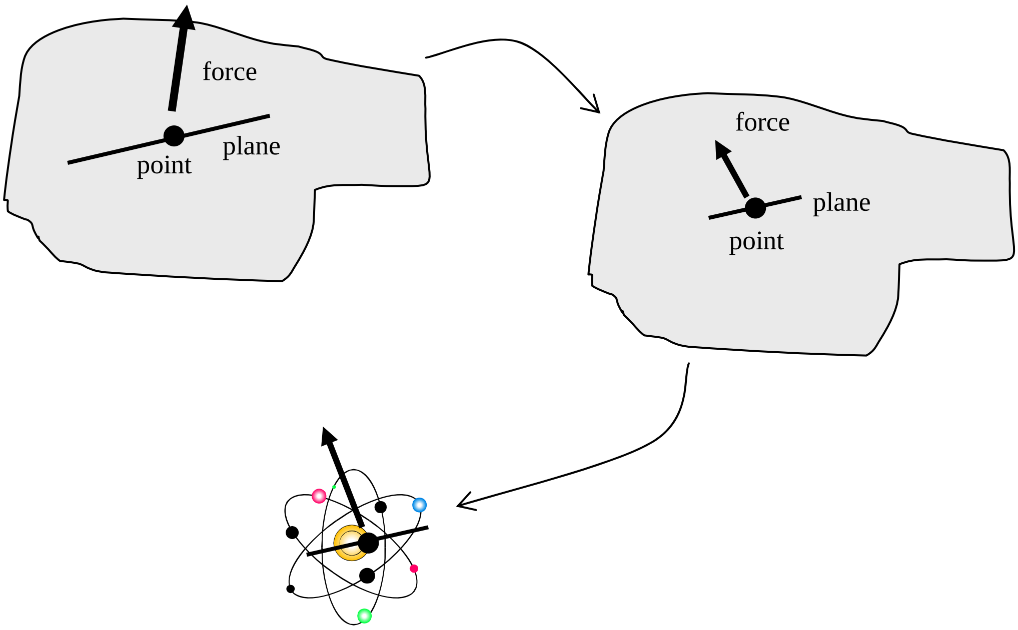


Figure 5.4.1: A force acting on an internal surface; allowing the plane on which the force acts to get progressively smaller

In a continuum model, it is assumed that the ratio F/S follows the dotted path shown in Fig. 5.4.2; a definite limit is reached as the plane shrinks to *zero size*. It should be kept in mind that the traction in a *real* material should be evaluated through

$$\mathbf{t} = \lim_{\Delta S \to (h^*)^2} \frac{\Delta F}{\Delta S} \tag{5.4.1}$$

where h^* is some minimum dimension below which there is no acceptable limit. On the other hand, it is necessary to take the limit to zero in the *mathematical* modelling of materials since that is the basis of calculus (this will become more necessary in Book II, where calculus is used more extensively).

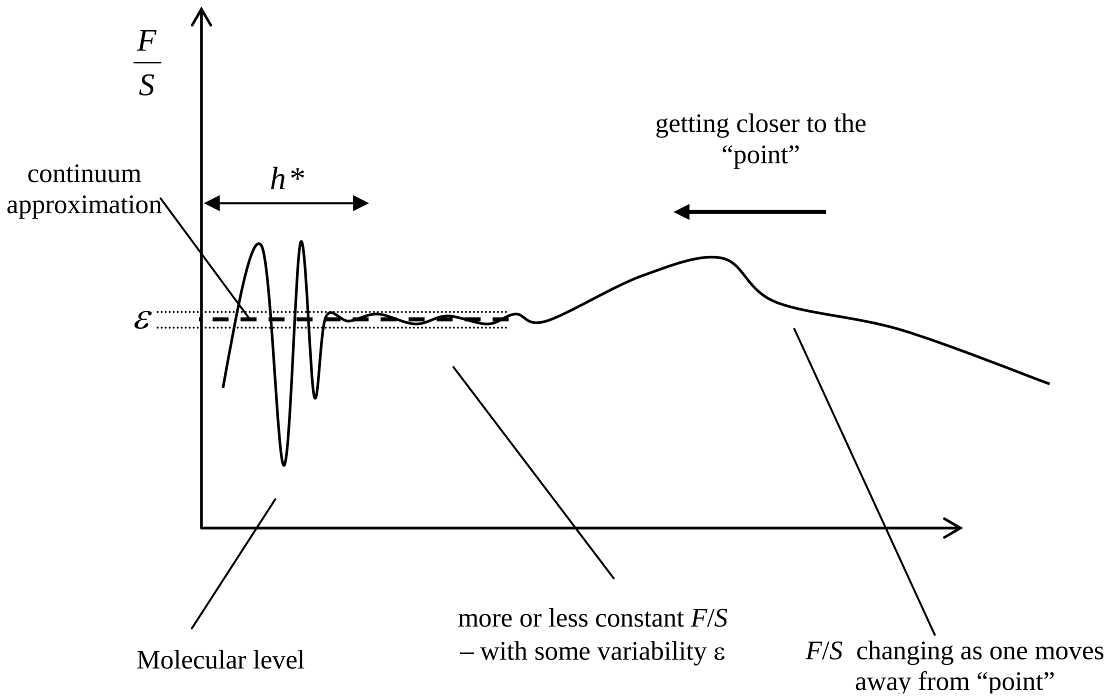


Figure 5.4.2: the change in traction as the plane upon which a force acts is reduced in size

In a continuum model, then, there is a minimum sized element one can consider, say of size $\Delta V = (h^*)^3$. When one talks about the stress on this element, the mass of this element, the density, velocity and acceleration of this element, one means the average of these quantities throughout or over the surface of the element – the discrete atomic structure within the element is ignored and is "smeared" out into a **continuum element**.

The continuum element is also called a **representative volume element** (RVE), an element of material large enough for the heterogeneities to be replaced by homogenised mean values of their properties. The order of the dimensions of RVE's for some common engineering materials would be approximately (see the metal example which follows)

Metal:	0.1mm
Polymers/composites:	1mm
Wood:	10mm
Concrete:	100mm

One does not have any information about what is happening inside the continuum element – it is like a "black box". The scale of the element (and higher) is called the **macroscale** – continuum mechanics is mechanics on the macroscale. The scale of entities within the element is termed the **microscale** – continuum models cannot give any information about what happens on the microscale.

5.4.2 Example: Metal

Metal, from a distance, appears fairly uniform. With the help of a microscope, however, it will be seen to consist of many individual grains of metal. For example, the metal shown in Fig. 5.4.3 has grains roughly 0.05mm across, and each one has very individual properties (the crystals in each grain are aligned in different directions).

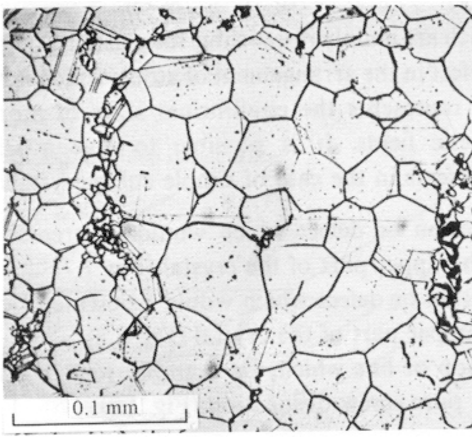


Figure 5.4.3: metal grains

If one is interested in the gross deformation of a moderately sized component of this metal, it would be sufficient to consider deformations that are averaged over volumes which are large compared to individual grains, but small compared to the whole component. A minimum dimension of, say, $h^* = 0.5\text{mm}$ for the metal of Fig. 5.4.3 would suffice, and this would be the macro/micro-scale boundary, with a minimum surface area of dimension $(h^*)^2$ for the definition of stress.

When one measures physical properties of the metal "at a point", for example the density, one need only measure an average quantity over an element of the order, say, $(0.5\text{mm})^3$ or higher. It is not necessary to consider the individual grains of metal – these are inside the "black box". The model will return valuable information about the deformation of the gross material, but it will not be able to furnish any information about movement of individual grains.

It was shown how to evaluate the Young's Modulus and other properties of a metal in Section 5.2.1. The test specimens used for such tests are vastly larger than the continuum elements discussed above. Thus the test data is perfectly adequate to describe the response of the metal, on the macroscale.

What if the response of individual grains to applied loads is required? In that case a model would have to be constructed which accounted for the different mechanical properties of each grain. The metal could no longer be considered to be a uniform material, but a complex one with many individual grains, each with different properties and orientation. The macro/micro boundary could be set at about $h^* = 0.1\mu\text{m}$. There are now two problems which need to be dealt with: (1) experiments such as the tensile test would have to be conducted on specimens much smaller than the grain size in order to provide data for any mathematical model, and (2) the mathematical model will be more complex and difficult to solve.

5.4.3 Micromechanical Models

Consider the schematic of a continuum model shown in Fig. 5.4.4 below. One can determine the material's properties, such as the Young's modulus *E*, through experimentation, and the resulting mathematical continuum model can be used to make predictions about the material's response. With the improved power of computers, especially since the 1990s, it has now become possible to complement continuum models with **micromechanical models**. These models take into account more fine detail of the material's structure (for example of the individual grains of the metal discussed earlier). Usually, one will have a micromechanical model of a small (typical) RVE of material. This then provides information regarding the properties of the RVE to be included in a continuum model (rather than having a micromechanical model of the *complete* material, which is in most cases still not practical). The means by which the properties at the micro scale are averaged (for example into a "smeared out" single *E* value) and passed "up" to the continuum model is through **homogenisation theory**. Such micromechanical models can provide further insight into material behaviour than the simpler continuum model.

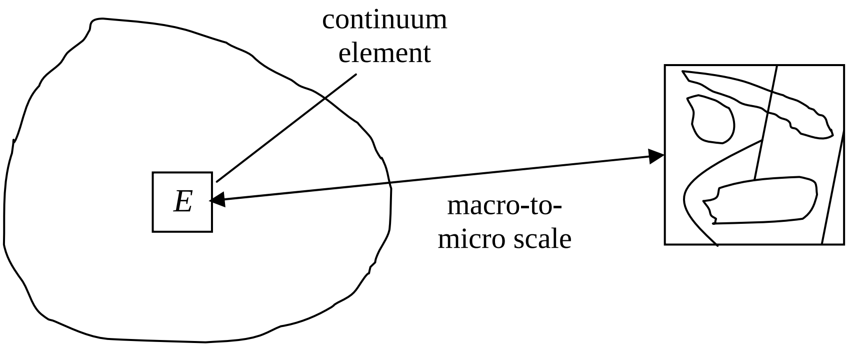


Figure 5.4.4: continuum model and micromechanical model

5.4.4 Problems

1. You want to evaluate the stiffness *E* of a metal for inclusion in a mechanics model. What *minimum* size specimen would you use in your test - $10\mu\text{m}$, 0.1mm, 5mm or 5cm?

2. Individual rice grains are separate solid particles. However, rice flowing down a chute at a food processing plant can be considered to be a fluid, and the flow of rice can be solved using the equations of mechanics. What minimum dimension h^* should be employed for measurements in this case to ensure the validity of a continuum model of flowing rice?

6 Linear Elasticity

The simplest constitutive law for solid materials is the linear elastic law, which assumes a linear relationship between stress and engineering strain. This assumption turns out to be an excellent predictor of the response of components which undergo small deformations, for example steel and concrete structures under large loads, and also works well for practically any material at a sufficiently small load.

The linear elastic model is discussed in this chapter and some elementary problems involving elastic materials are solved. Anisotropic elasticity is discussed in Section 6.3.

6.1 The Linear Elastic Model

6.1.1 The Linear Elastic Model

Repeating some of what was said in Section 5.3: the Linear Elastic model is used to describe materials which respond as follows:

(i) the strains in the material are small[1]
(ii) the stress is proportional to the strain, $\sigma \propto \varepsilon$ (**linear**)
(iii) the material returns to its original shape when the loads are removed, and the unloading path is the same as the loading path (**elastic**)
(iv) there is no dependence on the rate of loading or straining

From the discussion in the previous chapter, this model well represents the engineering materials up to their elastic limit. It also models well almost any material provided the stresses are sufficiently small.

The stress-strain (loading and unloading) curve for the Linear Elastic solid is shown in Fig. 6.1.1a. Other possible responses are shown in Figs. 6.1.1b,c. Fig. 6.1.1b shows the typical response of a rubbery-type material and many biological tissues; these are **non-linear elastic** materials (see Book IV). Fig. 6.1.1c shows the typical response of **viscoelastic** materials (see Chapter 10) and that of many plastically and viscoplastically deforming materials (see Chapters 11 and 12).

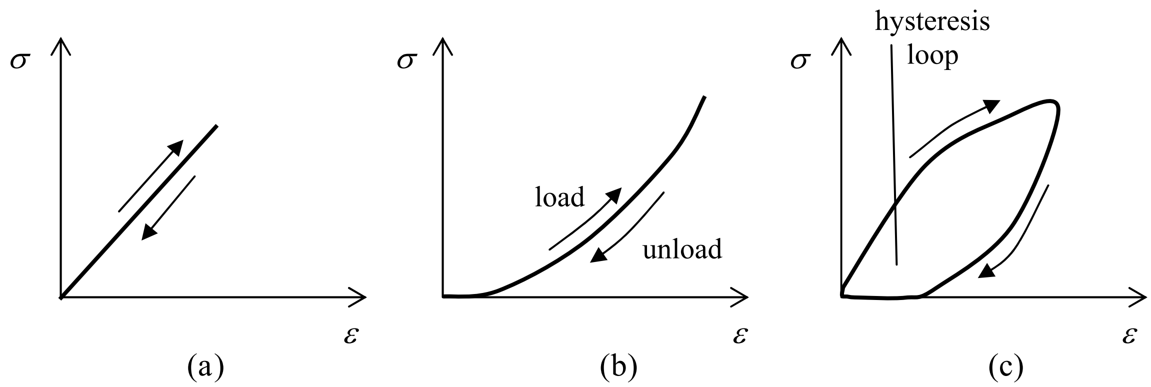


Figure 6.1.1: Different stress-strain relationships; (a) linear elastic, (b) non-linear elastic, (c) viscoelastic/plastic/viscoplastic

It will be assumed at first that the material is isotropic and homogeneous. The case of an anisotropic elastic material is discussed in Section 6.3.

[1] if the small-strain approximation is not made, the stress-strain relationship will be inherently non-linear; the actual strain, Eqn. 4.1.7, involves (non-linear) squares and square-roots of lengths

6.1.2 Stress-Strain Law

Consider a cube of material subjected to a uniaxial tensile stress σ_{xx}, Fig. 6.1.2a. One would expect it respond by extending in the *x* direction, $\varepsilon_{xx} > 0$, and to contract laterally, so $\varepsilon_{yy} = \varepsilon_{zz} < 0$, these last two being equal because of the isotropy of the material. With stress proportional to strain, one can write

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} \tag{6.1.1}$$

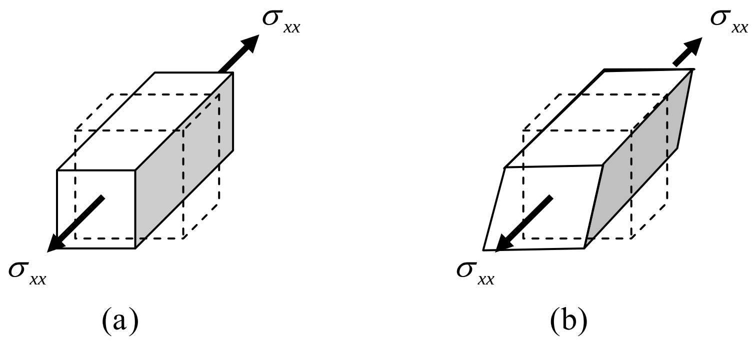


Figure 6.1.2: an element of material subjected to a uniaxial stress; (a) normal strain, (b) shear strain

The constant of proportionality between the normal stress and strain is the Young's Modulus, Eqn. 5.2.5, the measure of the stiffness of the material. The material parameter ν is the Poisson's ratio, Eqn. 5.2.6. Since $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$, it is a measure of the contraction relative to the normal extension.

Because of the isotropy/symmetry of the material, the shear strains are zero, and so the deformation of Fig. 6.1.2b, which shows a non-zero ε_{xy}, is not possible – shear strain can arise if the material is not isotropic.

One can write down similar expressions for the strains which result from a uniaxial tensile σ_{yy} stress and a uniaxial σ_{zz} stress:

$$\begin{aligned} \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy}, \quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{zz} &= \frac{1}{E}\sigma_{zz}, \quad \varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{zz} \end{aligned} \tag{6.1.2}$$

Similar arguments can be used to write down the shear strains which result from the application of a shear stress:

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz} \tag{6.1.3}$$

The constant of proportionality here is the Shear Modulus μ, Eqn. 5.2.8, the measure of the resistance to shear deformation (the letter *G* was used in Eqn. 5.2.8 – both *G* and μ are used to denote the Shear Modulus, the latter in more "mathematical" and "advanced" discussions) .

The strain which results from a combination of all six stresses is simply the sum of the strains which result from each[2]:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \quad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz}, \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right], \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz} \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]
\end{aligned} \tag{6.1.4}$$

These equations involve three material parameters. It will be proved in §6.3 that an isotropic linear elastic material can have only *two* independent material parameters and that, in fact,

$$\mu = \frac{E}{2(1+\nu)} \tag{6.1.5}$$

This relation will be verified in the following example.

Example: Verification of Eqn. 6.1.5

Consider the simple shear deformation shown in Fig. 6.1.3, with $\varepsilon_{xy} > 0$ and all other strains zero. With the material linear elastic, the only non-zero stress is $\sigma_{xy} = 2\mu\varepsilon_{xy}$.

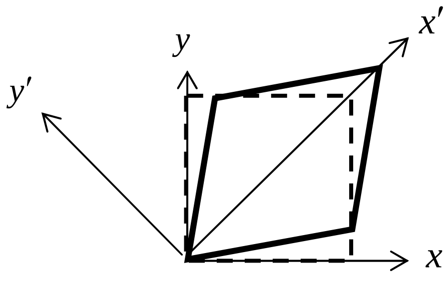


Figure 6.1.3: a simple shear deformation

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[2] this is called the **principle of linear superposition**: the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause". For a linear relation, e.g. $\sigma = E\varepsilon$, the effects of two causes σ_1, σ_2 are $E\varepsilon_1$ and $E\varepsilon_2$, and the effect of the sum of the causes $\sigma_1 + \sigma_2$ is indeed equal to the sum of the individual effects: $E(\varepsilon_1 + \varepsilon_2) = E\varepsilon_1 + E\varepsilon_2$. This is not true of a non-linear relation, e.g. $\sigma = E\varepsilon^2$, since $E(\varepsilon_1 + \varepsilon_2)^2 \neq E\varepsilon_1^2 + E\varepsilon_2^2$

Using the strain transformation equations, Eqns. 4.2.2, the only non-zero strains in a second coordinate system $x' - y'$, with x' at $\theta = 45^\circ$ from the x axis (see Fig. 6.1.3), are $\varepsilon'_{xx} = +\varepsilon_{xy}$ and $\varepsilon'_{yy} = -\varepsilon_{xy}$. Because the material is isotropic, Eqns 6.1.4 hold also in this second coordinate system and so the stresses in the new coordinate system can be determined by solving the equations

$$\begin{aligned}
\varepsilon'_{xx} &= +\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{xx} - \nu\left(\sigma'_{yy} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{xy} = 0 = \frac{1}{2\mu}\sigma'_{xy}, \quad \varepsilon'_{xz} = 0 = \frac{1}{2\mu}\sigma'_{xz}, \\
\varepsilon'_{yy} &= -\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{yy} - \nu\left(\sigma'_{xx} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{yz} = 0 = \frac{1}{2\mu}\sigma'_{yz} \\
\varepsilon'_{zz} &= 0 \quad = \frac{1}{E}\left[\sigma'_{zz} - \nu\left(\sigma'_{xx} + \sigma'_{yy}\right)\right]
\end{aligned} \tag{6.1.6}$$

which results in

$$\sigma'_{xx} = +\frac{E}{1+\nu}\varepsilon_{xy}, \quad \sigma'_{yy} = -\frac{E}{1+\nu}\varepsilon_{xy} \tag{6.1.7}$$

But the stress transformation equations, Eqns. 3.4.8, with $\sigma_{xy} = 2\mu\varepsilon_{xy}$, give $\sigma'_{xx} = +2\mu\varepsilon_{xy}$ and $\sigma'_{yy} = -2\mu\varepsilon_{xy}$ and so Eqn. 6.1.5 is verified.

■

Relation 6.1.5 allows the Linear Elastic Solid stress-strain law, Eqn. 6.1.4, to be written as

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned}} \quad \textbf{Stress-Strain Relations} \tag{6.1.8}$$

This is known as **Hooke's Law**. These equations can be solved for the stresses to get

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \\
\sigma_{xz} &= \frac{E}{1+\nu}\varepsilon_{xz} \\
\sigma_{yz} &= \frac{E}{1+\nu}\varepsilon_{yz}
\end{aligned}
$$

Stress-Strain Relations (6.1.9)

Values of E and ν for a number of materials are given in Table 6.1.1 below (see also Table 5.2.2).

Material	*E* (GPa)	ν
Grey Cast Iron	100	0.29
A316 Stainless Steel	196	0.3
A5 Aluminium	68	0.33
Bronze	130	0.34
Plexiglass	2.9	0.4
Rubber	0.001-2	0.4-0.49
Concrete	23-30	0.2
Granite	53-60	0.27
Wood (pinewood) fibre direction transverse direction	 17 1	 0.45 0.79

Table 6.1.1: Young's Modulus *E* and Poisson's Ratio ν for a selection of materials at 20$^{\mathrm{o}}$C

Volume Change

Recall that the volume change in a material undergoing small strains is given by the sum of the normal strains (see Section 4.3). From Hooke's law, normal stresses cause normal strain and shear stresses cause shear strain. It follows that *normal stresses produce volume changes* and *shear stresses produce distortion* (change in shape), but no volume change.

6.1.3 Two Dimensional Elasticity

The above three-dimensional stress-strain relations reduce in the case of a two-dimensional stress state or a two-dimensional strain state.

Plane Stress

In plane stress (see Section 3.5), $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, Fig. 6.1.5, so the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left[\nu\varepsilon_{xx} + \varepsilon_{yy}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Stress) (6.1.10)

with

$$\begin{aligned}
&\varepsilon_{zz} = -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right], \quad \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
&\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \tag{6.1.11}$$

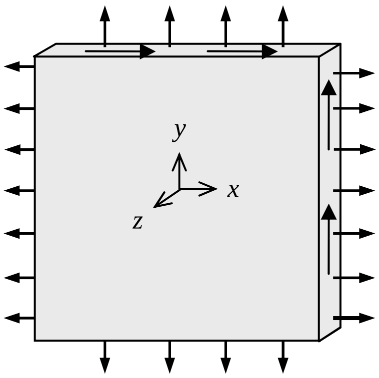


Figure 6.1.5: Plane stress

Note that the ε_{zz} strain is *not* zero. Physically, ε_{zz} corresponds to a change in thickness of the material perpendicular to the direction of loading.

Plane Strain

In plane strain (see Section 4.2), $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$, Fig. 6.1.6, and the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Strain) (6.1.12)

with

$$\begin{gathered}
\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
\sigma_{zz} = \nu\left[\sigma_{xx} + \sigma_{yy}\right], \quad \sigma_{xz} = \sigma_{yz} = 0
\end{gathered} \tag{6.1.13}$$

Again, note here that the stress component σ_{zz} is *not* zero. Physically, this stress corresponds to the forces preventing movement in the *z* direction.

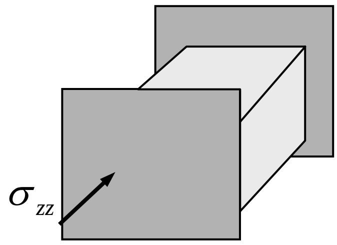


Figure 6.1.6 Plane strain - a thick component constrained in one direction

Similar Solutions

The expressions for plane stress and plane strain are very similar. For example, the plane strain constitutive law 6.1.12 can be derived from the corresponding plane stress expressions 6.1.10 by making the substitutions

$$E = \frac{E'}{1-\nu'^2}, \quad \nu = \frac{\nu'}{1-\nu'} \tag{6.1.14}$$

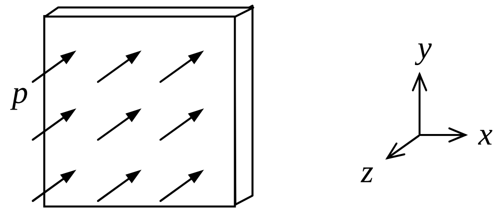
in 6.1.10 and then dropping the primes. The plane stress expressions can be derived from the plane strain expressions by making the substitutions

$$E = E'\frac{1+2\nu'}{(1+\nu')^2}, \quad \nu = \frac{\nu'}{1+\nu'} \tag{6.1.15}$$

in 6.1.12 and then dropping the primes. Thus, if one solves a plane stress problem, one has automatically solved the corresponding plane strain problem, and *vice versa*.

6.1.4 Problems

1. Consider a very thin sheet of material subjected to a normal pressure p on one of its large surfaces. It is fixed along its edges. This is an example of a **plate** problem, an important branch of elasticity with applications to boat hulls, aircraft fuselage, etc.
   (a) write out the complete three dimensional stress-strain relations for this case (both cases, stress in terms of strain, strain in terms of stress) - simplify the relations using the fact that the sheet is thin, the stress boundary condition on the large face and the coordinate system shown (just substitute in appropriate values for σ_{xz}, σ_{yz} and σ_{zz})
   (b) assuming that the through thickness change in the sheet can be neglected, show that
$$p = -\nu(\sigma_{xx} + \sigma_{yy})$$



2. A strain gauge at a certain point on the surface of a thin A5 Aluminium component (loaded in-plane) records strains of $\varepsilon_{xx} = 60\,\mu\text{m}$, $\varepsilon_{yy} = 30\,\mu\text{m}$, $\varepsilon_{xy} = 15\,\mu\text{m}$. Determine the principal stresses. (See Table 6.1.1 for the material properties.)

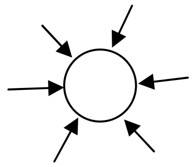
3. Use the stress-strain relations to prove that, for a linear elastic solid,
$$\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}$$
and, indeed,
$$\frac{2\sigma_{xz}}{\sigma_{xx} - \sigma_{zz}} = \frac{2\varepsilon_{xz}}{\varepsilon_{xx} - \varepsilon_{zz}}, \quad \frac{2\sigma_{yz}}{\sigma_{yy} - \sigma_{zz}} = \frac{2\varepsilon_{yz}}{\varepsilon_{yy} - \varepsilon_{zz}}$$
Note: from Eqns. 3.5.4 and 4.2.4, these show that the principal axes of stress and strain coincide for an *isotropic* elastic material

4. Consider the case of hydrostatic pressure in a linearly elastic solid:
$$[\sigma_{ij}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$
as might occur, for example, when a spherical component is surrounded by a fluid under high pressure, as illustrated in the figure below. Show that the volumetric strain is equal to

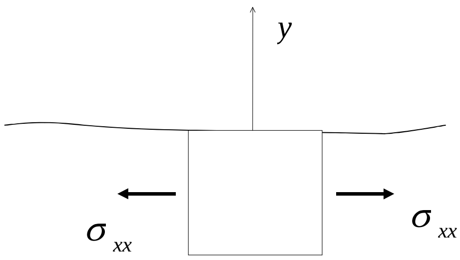
$$-p\frac{3(1-2\nu)}{E},$$

so that the Bulk Modulus, Eqn. 5.2.9, is

$$K=\frac{E}{3(1-2\nu)}$$



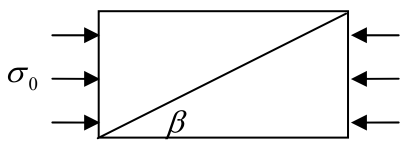
5. Consider again Problem 2 from §3.5.7.
   (a) Assuming the material to be linearly elastic, what are the strains? Draw a second material element (superimposed on the one shown below) to show the deformed shape of the square element – assume the displacement of the box-centre to be zero and that there is no rotation. Note how the free surface moves, even though there is no stress acting on it.
   (b) What are the principal strains ε_1 and ε_2? You will see that the principal directions of stress and strain coincide (see Problem 3) – the largest normal stress and strain occur in the same direction.



6. A thin linear elastic plate is subjected to a uniform compressive stress σ_0 as shown below. Show that the slope of the plate diagonal shown after deformation is given by

$$\tan(\beta+\delta\beta)=\frac{b}{a}\left(\frac{1+\nu\sigma_0/E}{1-\sigma_0/E}\right)$$

What is the magnitude of $\delta\beta$ for a steel plate ($E=210\text{GPa}$, $\nu=0.3$) of dimensions $20\times 20\ \text{cm}^2$ with $\sigma_0=1\text{MPa}$?



6.2 Homogeneous Problems in Linear Elasticity

A **homogeneous** stress (strain) field is one where the stress (strain) is the same at all points in the material. Homogeneous conditions will arise when the geometry is simple and the loading is simple.

6.2.1 Elastic Rectangular Cuboids

Hooke's Law, Eqns. 6.1.8 or 6.1.9, can be used to solve problems involving homogeneous stress and deformation. Hoooke's law is 6 equations in 12 unknowns (6 stresses and 6 strains). If some of these unknowns are given, the rest can be found from the relations.

Example

Consider the block of linear elastic material shown in Fig. 6.2.1. It is subjected to an equi-biaxial stress of $\sigma_{xx} = \sigma_{yy} = \overline{\sigma} > 0$.

Since this is an isotropic elastic material, the shears stresses and strains will be all zero for such a loading. One thus need only consider the three normal stresses and strains.

There are now 3 equations (the first 3 of Eqns. 6.1.8 or 6.1.9) in 6 unknowns. One thus needs to know *three* of the normal stresses and/or strains to find a solution. From the loading, one knows that $\sigma_{xx} = \overline{\sigma}$ and $\sigma_{yy} = \overline{\sigma}$. The third piece of information comes from noting that the surfaces parallel to the $x-y$ plane are free surfaces and so $\sigma_{zz} = 0$.

From Eqn. 6.1.8 then, the strains are

$$\varepsilon_{xx} = \varepsilon_{yy} = (1-\nu)\frac{\overline{\sigma}}{E}, \quad \varepsilon_{zz} = -2\nu\frac{\overline{\sigma}}{E}, \quad \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

As expected, $\varepsilon_{xx} = \varepsilon_{yy}$ and $\varepsilon_{zz} < 0$.

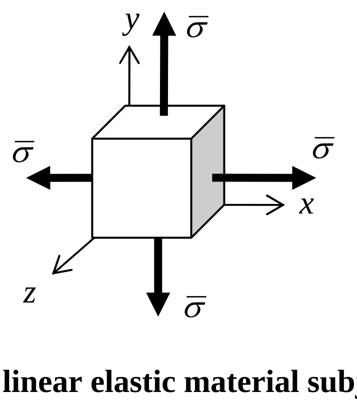


Figure 6.2.1: A block of linear elastic material subjected to an equi-biaxial stress

■

6.2.2 Problems

1. A block of isotropic linear elastic material is subjected to a compressive normal stress σ_o over two opposing faces. The material is constrained (prevented from moving) in one of the direction normal to these faces. The other faces are free.
   (a) What are the stresses and strains in the block, in terms of σ_o, ν, E ?
   (b) Calculate three maximum shear stresses, one for each plane (parallel to the faces of the block). Which of these is the overall maximum shear stress acting in the block?

2. Repeat problem 1a, only with the free faces now fixed also.

7 Applications of Elasticity

The linear elastic model was introduced in the previous chapter and some elementary problems involving elastic materials were solved there (in particular in section 6.2). In this Chapter, five important, practical, theories are presented concerning elastic materials; they all have specific geometries and are subjected to particular types of load. In §7.1, the geometry is that of a long slender bar and the load is one which acts along the length of the bar; in §7.2, the geometry is that of a long slender circular bar and the load is one which twists the bar; in §7.3 the geometry is that of a thin-walled cylindrical or spherical component, and the load is normal to these walls; in §7.4 the geometry is that of a long and slender beam, and the load is transverse to the beam length. Finally, in §7.5, the geometry is a column, fixed at one end and loaded at the other so that it deflects. These five particular situations allow for simplifications (or approximations) to be made to the full three-dimensional linear elastic stress-strain relations; this allows one to write down simple expressions for the stress and strain and so solve some important practical problems analytically.

7.1 One Dimensional Axial Deformations

In this section, a specific simple geometry is considered, that of a long and thin straight component loaded in such a way that it deforms in the axial direction only. The *x*-axis is taken as the longitudinal axis, with the cross-section lying in the $x-y$ plane, Fig. 7.1.1.

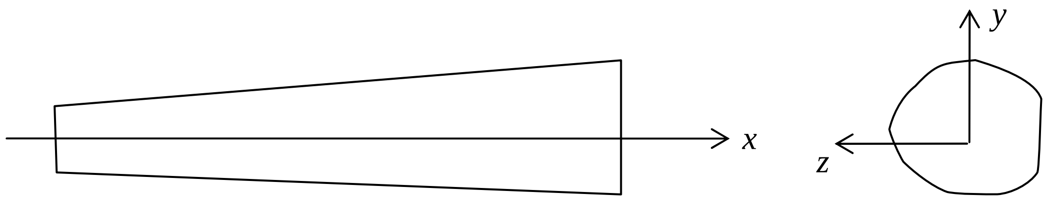


Figure 7.1.1: A slender straight component; (a) longitudinal axis, (b) cross-section

7.1.1 Basic relations for Axial Deformations

Any static analysis of a structural component involves the following three considerations:
(1) constitutive response
(2) kinematics
(3) equilibrium

In this Chapter, it is taken for (1) that the material responds as an isotropic linear elastic solid. It is assumed that the only significant stresses and strains occur in the axial direction, and so the stress-strain relations 6.1.8-9 reduce to the one-dimensional equation $\sigma_{xx} = E\varepsilon_{xx}$ or, dropping the subscripts,

$$\sigma = E\varepsilon \tag{7.1.1}$$

Kinematics (2), the study of deformation, was the subject of Chapter 4. In the theory developed here, known as **axial deformation**, it is assumed that the axis of the component remains straight and that cross-sections that are initially perpendicular to the axis remain perpendicular after deformation. This implies that, although the strain might vary along the axis, it remains *constant over any cross section*. The axial strain occurring over any section is defined by Eqn. 4.1.2,

$$\varepsilon = \frac{L - L_0}{L_0} \tag{7.1.2}$$

This is illustrated in Fig. 7.1.2, which shows a (shaded) region undergoing a compressive (negative) strain.

Recall that individual particles/points undergo displacements whereas regions/line-elements undergo strain. In Fig. 7.1.2, the particle originally at *A* has undergone a displacement $u(A)$ whereas the particle originally at *B* has undergone a displacement $u(B)$. From Fig. 7.1.2, another way of expressing the strain in the shaded region is (see Eqn. 4.1.3)

$$\varepsilon = \frac{u(B) - u(A)}{L_0} \tag{7.1.3}$$

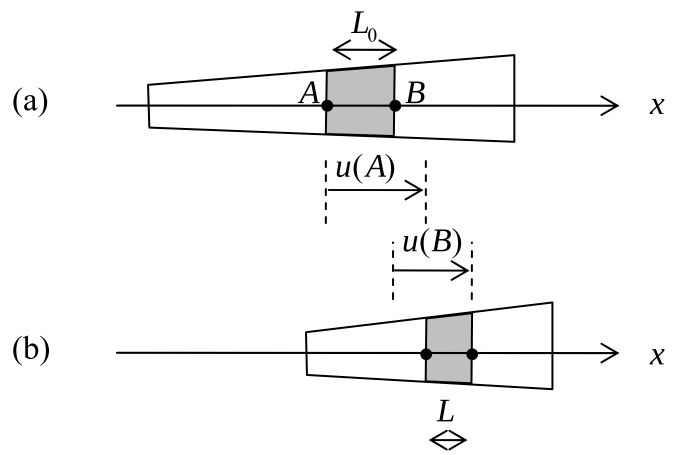


Figure 7.1.2: axial strain; (a) before deformation, (b) after deformation

Both displacements $u(A)$ and $u(B)$ of Fig. 7.1.2 are *positive*, since the particles displace in the positive *x* direction – if they moved to the left, for consistency, one would say they underwent *negative* displacements. Further, positive stresses are as shown in Fig. 7.1.3a and negative stresses are as shown in Fig. 7.1.3b. From Eqn. 7.1.1, a positive stress implies a positive strain (lengthening) and a compressive stress implies a negative strain (contracting)

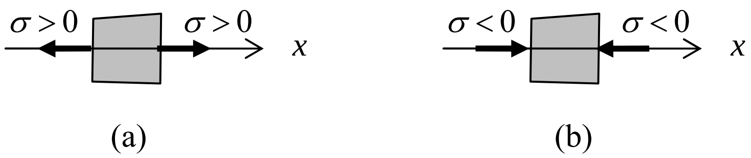


Figure 7.1.3: Stresses arising in the slender component; (a) positive (tensile) stress, (b) negative (compressive) stress

Equilibrium, (3), will be considered in the individual examples below.

Note that, in the previous Chapter, problems were solved using only the stress-strain law (1). Kinematics (2) and equilibrium (3) were not considered, the reason being the problems were so simple, with uniform (homogeneous) stress and strain (as indeed also in the first example which follows). Whenever more complex problems are encountered, with non-uniform stress and strains, (3) and perhaps (2) need to be considered to solve for the stress and strain.

7.1.2 Structures with Uniform Members

A uniform axial member is one with cross-section *A* and modulus *E* constant along its length, and loaded with axial forces at its ends only.

Example

Consider the bar of initial length L shown in Fig. 7.1.4, subjected to equal and opposite end-forces F. The free-body (equilibrium) diagram of a section of the bar shown in Fig. 7.1.4b shows that the internal force is also F everywhere along the bar. The stress is thus everywhere $\sigma = F / A$ and the strain is everywhere

$$\varepsilon = \frac{F}{EA} \tag{7.1.4}$$

and, from Eq. 7.1.2, the bar extends in length by an amount

$$\Delta = \frac{FL}{EA} \tag{7.1.5}$$

Note that although the force acting on the left-hand end is negative (acting in the $-x$ direction), the stress there is positive (see Fig. 7.1.3).

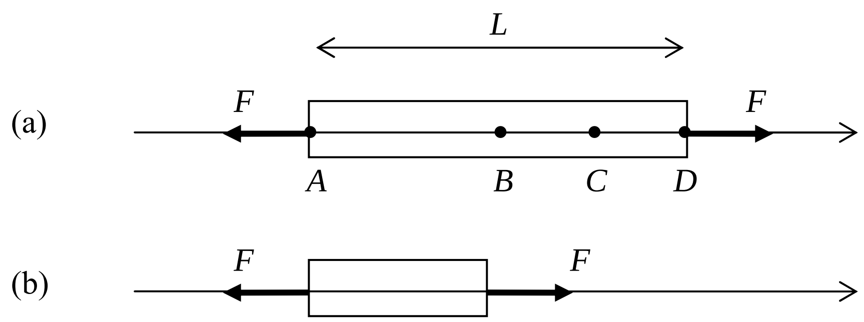


Figure 7.1.4: A uniform axial member; (a) subjected to axial forces *F*, (b) free-body diagram

Displacements need to be calculated relative to some datum displacement[1]. For example, suppose that the displacement at the centre of the bar is zero, $u(B) = 0$, Fig. 7.1.4. Then, from Eqn. 7.1.3,

$$\begin{aligned} u(C) &= u(B) + \varepsilon(C - B) = \frac{F}{EA}\frac{L}{4} \\ u(D) &= u(B) + \varepsilon(D - B) = \frac{F}{EA}\frac{L}{2} \\ u(A) &= u(B) + \varepsilon(A - B) = -\frac{F}{EA}\frac{L}{2} \end{aligned} \tag{7.1.6}$$

■

[1] which is another way of saying that one can translate the bar left or right as a rigid body without affecting the stress or strain – but it does affect the displacements

Example

Consider the two-element structure shown in Fig. 7.1.5. The first element is built-in at end *A*, is of length L_1, cross-sectional area A_1 and Young's modulus E_1. The second element is attached at *B* and has properties L_2, A_2, E_2. External loads F and *P* are applied at *B* and *C* as shown. An unknown reaction force *R* acts at *A*. This can be determined from the force equilibrium equation for the structure:

$$R - F + P = 0 \tag{7.1.7}$$

As usual, the reaction is first assumed to act in the positive (*x*) direction. With *R* known, the stress $\sigma^{(1)}$ in the first element can be evaluated using the free-body diagram 7.1.5b, and $\sigma^{(1)}$ using Fig. 7.1.5c:

$$\sigma^{(1)} = \frac{P - F}{A_1}, \qquad \sigma^{(2)} = \frac{P}{A_2} \tag{7.1.8}$$

and so the strain is

$$\varepsilon^{(1)} = \frac{P - F}{E_1 A_1}, \qquad \varepsilon^{(2)} = \frac{P}{E_2 A_2} \tag{7.1.9}$$

Note that the stress and strain are *discontinuous* at *B* [2].

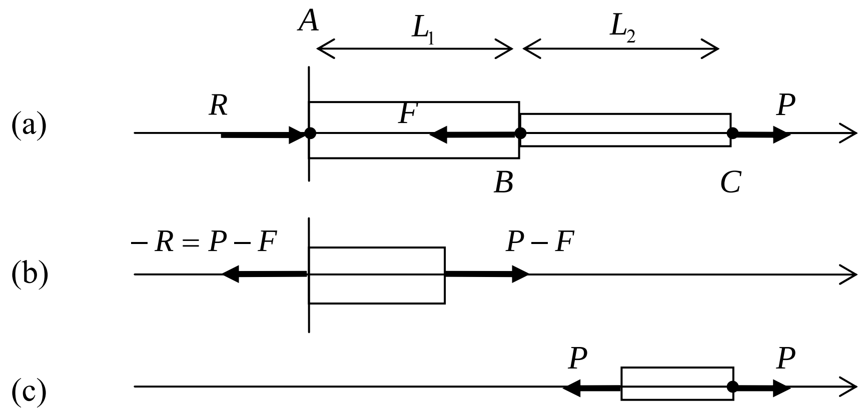


Figure 7.1.5: A two-element structure (a) subjected to axial forces *F* and *P*, (b,c) free-body diagrams

For each element, the total elongations Δ_i are

[2] this result, which can be viewed as a violation of equilibrium at *B*, is a result of the one-dimensional approximation of what is really a two-dimensional problem

$$\begin{aligned}\Delta_1 &= u(B) - u(A) = \frac{(P-F)L_1}{E_1 A_1} \\ \Delta_2 &= u(C) - u(B) = \frac{PL_2}{E_2 A_2}\end{aligned} \tag{7.1.10}$$

If $P > F$, then $\Delta_1 > 0$ as expected, with $R < 0$ and $\sigma > 0$.

Thus far, the stress and strain (and elongations) have been obtained. If one wants to evaluate the displacements, then one needs to ensure that the strains in each of the two elements are **compatible**, that is, that the elements fit together after deformation just like they did before deformation. In this example, the displacements at *B* and *C* are

$$u(B) = u(A) + \Delta_1, \quad u(C) = u(B) + \Delta_2 \tag{7.1.11}$$

A **compatibility condition**, bringing together the separate relations in 7.1.11, is then

$$u(C) = u(A) + \frac{(P-F)L_1}{E_1 A_1} + \frac{PL_2}{E_2 A_2} \tag{7.1.12}$$

ensuring that $u(B)$ is unique. As in the previous example, the displacements can now be calculated if the displacement at any one (datum) point is known. Indeed, it is known that $u(A) = 0$.

■

Example

Consider next the similar situation shown in Fig. 7.1.6. Here, both ends of the two-element structure are built-in and there is only one applied force, *F*, at *B*. There are now two reaction forces, at ends *A* and *C*, but there is only one equilibrium equation to determine them:

$$R_A + F + R_C = 0 \tag{7.1.13}$$

Any structure for which there are more unknowns than equations of equilibrium, so that the stresses cannot be determined without considering the deformation of the structure, is called a **statically indeterminate** structure[3].

[3] See the end of §2.3.3

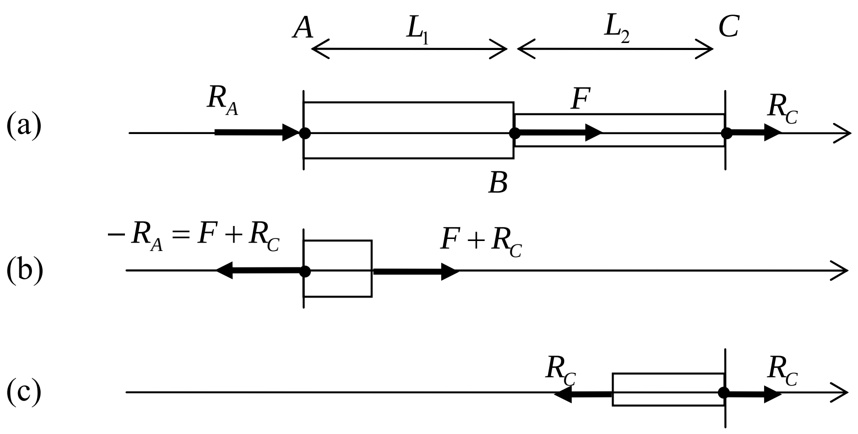


Figure 7.1.6: A two-element structure built-in and both ends; (a) subjected to an axial force *F*, (b,c) free-body diagrams

In terms of the unknown reactions, the strains are

$$\varepsilon^{(1)} = \frac{\sigma^{(1)}}{E_1} = -\frac{R_A}{E_1 A_1} = \frac{F + R_C}{E_1 A_1}, \qquad \varepsilon^{(2)} = \frac{\sigma^{(2)}}{E_2} = \frac{R_C}{E_2 A_2} \tag{7.1.14}$$

and, for each element, the total elongations are

$$\Delta_1 = \frac{R_A L_1}{E_1 A_1}, \qquad \Delta_2 = \frac{R_C L_2}{E_2 A_2} \tag{7.1.15}$$

Finally, compatibility of both elements implies that the total elongation $\Delta_1 + \Delta_2 = 0$. Using this relation with Eqn. 7.1.13-14 then gives

$$R_A = +F \frac{L_2 E_1 A_1}{L_1 E_2 A_2 + L_2 E_1 A_1}, \qquad R_C = -F \frac{L_1 E_2 A_2}{L_1 E_2 A_2 + L_2 E_1 A_1} \tag{7.1.16}$$

The displacements can now be evaluated, for example,

$$u(B) = +F \frac{1}{E_1 A_1 / L_1 + E_2 A_2 / L_2} \tag{7.1.17}$$

so that a positive *F* displaces *B* to the right and a negative *F* displaces *B* to the left.

■

Note the general solution procedure in this last example, known as the **basic force method**:

Equilibrium + Compatibility of Strain in terms of unknown Forces
→ Solve equations for unknown Forces

The Stiffness Method

The **stiffness method** (also known as the **displacement method**) is a slight modification of the above solution procedure, where the final equations to be solved involve known forces and unknown displacements only:

Equilibrium in terms of Displacement
→ Solve equations for unknown Displacements

If one deals in displacements, one does not need to ensure compatibility (it will automatically be satisfied); compatibility only needs to be considered when dealing in strains (as in the previous example).

Example (The Stiffness Method)

Consider a series of three bars of cross-sectional areas A_1, A_2, A_3, Young's moduli E_1, E_2, E_3 and lengths L_1, L_2, L_3, Fig. 7.1.7. The first and third bars are built-in at points *A* and *D*, bars one and two meet at *B* and bars two and three meet at *C*. Forces P_B and P_C act at *B* and *C* respectively.

The force is constant in each bar, and for each bar there is a relation between the force F_i, and elongation, Δ_i, Eqn. 7.1.5:

$$F_i = k_i \Delta_i \quad \text{where} \quad k_i = \frac{A_i E_i}{L_i} \tag{7.1.18}$$

Here, k_i is the effective **stiffness** of each bar. The elongations are related to the displacements, $\Delta_1 = u_B - u_A$ etc., so that, with $u_A = u_D = 0$,

$$F_1 = k_1 u_B, \qquad F_2 = k_2 (u_C - u_B), \qquad F_3 = -k_3 u_C \tag{7.1.19}$$

There are two **degrees of freedom** in this problem, that is, two nodes are free to move. One therefore needs two equilibrium equations. One could use any two of

$$-F_1 + P_B + P_C + F_3 = 0, \qquad -F_1 + P_B + F_2 = 0, \qquad -F_2 + P_C + F_3 = 0 \tag{7.1.20}$$

In the stiffness method, one uses the second and third of these; the second is the "node *B*" equation and the third is the "node *C*" equation. Substituting Eqns. 7.1.19 into 7.1.20 leads to the system of two equations

$$\begin{aligned} -(k_1 + k_2)u_B + k_2 u_C &= -P_B \\ +k_2 u_B - (k_2 + k_3)u_C &= -P_C \end{aligned} \tag{7.1.21}$$

which can be solved for the two unknown nodal displacements

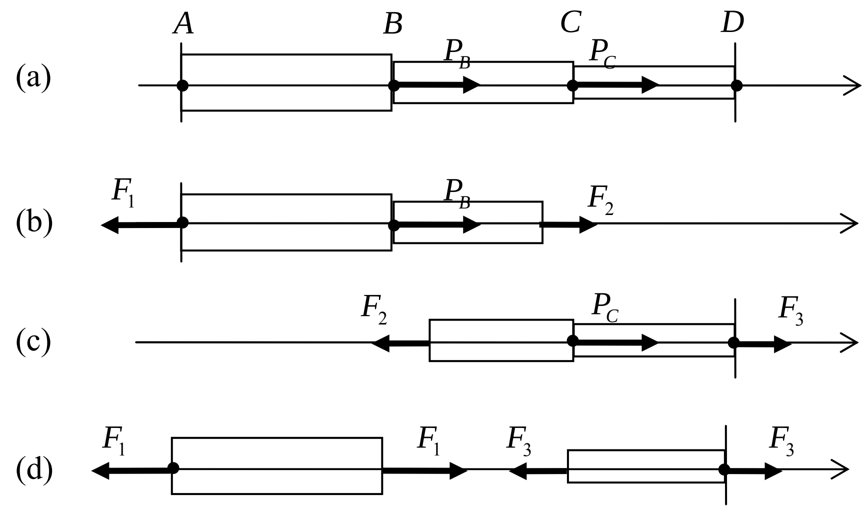


Figure 7.1.7: three bars in series; (a) subjected to external loads, (b,c,d) free-body diagrams

Note that it was not necessary to evaluate the reactions to obtain a solution. Once the forces have been found, the reactions can be found using the free-body diagram of Fig. 7.1.7d.

The stiffness method is a very systematic procedure. It can be used to solve for structures with many elements, with the two equations 7.1.21 replaced by a large system of equations which can be solved numerically using a computer.

7.1.3 Structures with Non-uniform Members

Consider the structure shown in Fig. 7.1.8, an axial bar consisting of two separate components bonded together. The components have Young's moduli E_1, E_2 and cross-sectional areas A_1, A_2. The bar is subjected to equal and opposite forces *F* as shown, in such a way that axial deformations occur, that is, the cross-sections remain perpendicular to the *x* axis throughout the deformation.

Since there are only axial deformations, the strain is constant over a cross-section. However, the stress is not uniform, with $\sigma_1 = E_1\varepsilon$ and $\sigma_2 = E_2\varepsilon$; on any cross-section, the stress is higher in the stiffer component. The resultant force acting on each component is $F_1 = E_1 A_1 \varepsilon$ and $F_2 = E_2 A_2 \varepsilon$. Since $F_1 + F_2 = F$, the total elongation is

$$\Delta = \frac{FL}{E_1 A_1 + E_2 A_2} \qquad (7.1.22)$$

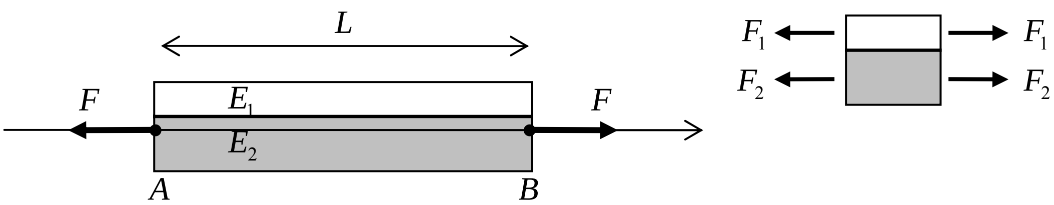


Figure 7.1.8: A bar consisting of two separate materials bonded together

7.1.4 Resultant Force and Moment

Consider the force and moments acting over any cross-section, Fig. 7.1.9. The resultant force is the integral of the stress times elemental area over the cross section, Eqn. 3.1.2,

$$F = \int_A \sigma dA \tag{7.1.23}$$

There are two moments; the moment M_y about the *y* axis and M_z about the *z* axis,

$$M_y = \int_A z\sigma dA, \quad M_z = -\int_A y\sigma dA \tag{7.1.24}$$

Positive moments are defined through the **right hand rule**, i.e. with the thumb of the right hand pointing in the positive *y* direction, the closing of the fingers indicates the positive M_y; the negative sign in Eqn. 7.1.24b is due to the fact that a positive stress with $y > 0$ would lead to a negative moment M_z.

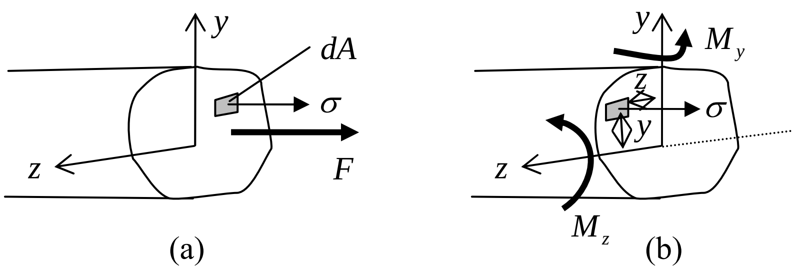


Figure 7.1.9: Resultants on a cross-section; (a) resultant force, (b) resultant moments

Consider now the case where *the stress is constant over a cross-section*. Since it is assumed that the strain is constant over the cross-section, from Eqn. 7.1.1 this will occur when the Young's modulus is constant. In that case, Eqns. 7.1.23-24 can be re-written as

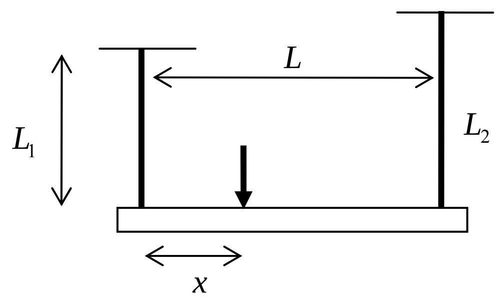
$$F = \sigma A, \quad M_y = \sigma \int_A z dA, \quad M_z = -\sigma \int_A y dA \tag{7.1.25}$$

The quantities $\int_A z dA$ and $\int_A y dA$ are the **first moments of area** about, respectively, the *y* and *z* axes. These are equal to $\bar{z}A$ and $\bar{y}A$, where $(\bar{y}, \bar{z})$ are the coordinates of the centroid of the section (see Eqn. 3.2.2). Taking the *x* axis to run through the centroid, $\bar{y} = \bar{z} = 0$ results in $M_y = M_z = 0$. Thus, a resultant axial force which acts through the centroid of the cross-section ensures that there is no moment/rotation of that cross-section, the main assumption of this section.

For the non-uniform member of Fig. 7.1.8, since the resultant of a constant stress over an area is a force acting through the centroid of that area, the forces F_1, F_2 act through the centroids of the respective areas A_1, A_2. The precise location of the total resultant force *F* can be determined by taking the moments of the forces F_1, F_2 about the *y* and *z* axes, and equating this to the moment of the force *F* about these axes.

7.1.5 Problems

1. Consider the *rigid* beam supported by two deformable bars shown below. The bars have properties L_1, A_1 and L_2, A_1 and have the same Young's modulus *E*. They are separated by a distance *L*. The beam supports an arbitrary load at position *x*, as shown. What is *x* if the beam is to remain horizontal after deformation.



7.2 Torsion

In this section, the geometry to be considered is that of a long slender circular bar and the load is one which twists the bar. Such problems are important in the analysis of twisting components, for example lug wrenches and transmission shafts.

7.2.1 Basic relations for Torsion of Circular Members

The theory of torsion presented here concerns **torques**[1] which twist the members but which *do not induce any warping*, that is, cross sections which are perpendicular to the axis of the member remain so after twisting. Further, radial lines remain straight and radial as the cross-section rotates – they merely rotate with the section.

For example, consider the member shown in Fig. 7.2.1, built-in at one end and subject to a torque *T* at the other. The *x* axis is drawn along its axis. The torque shown is positive, following the right-hand rule (see §7.1.4). The member twists under the action of the torque and the radial plane $ABCD$ moves to $ABC'D$.

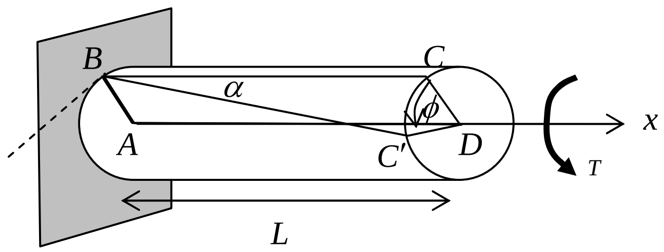


Figure 7.2.1: A cylindrical member under the action of a torque

Whereas in the last section the measure of deformation was elongation of the axial members, here an appropriate measure is the amount by which the member twists, the rotation angle ϕ. The rotation angle will vary along the member – the sign convention is that ϕ is positive in the same direction as positive T as indicated by the arrow in Fig. 7.2.1. Further, whereas the measure of strain used in the previous section was the normal strain ε_{xx}, here it will be the engineering shear strain γ_{xy} (twice the tensorial shear strain ε_{xy}). A relationship between γ (dropping the subscripts) and ϕ will next be established.

As the line BC deforms into BC', Fig. 7.2.1, it undergoes an angle change α. As defined in §4.1.2, the shear strain γ is the change in the original right angle formed by BC and a tangent at *B* (indicated by the dotted line – this is the *y* axis to be used in γ_{xy}). If α is small, then

$$\gamma = \alpha \approx \tan\alpha = \frac{CC'}{BC} \approx \frac{R\phi(L)}{L} \tag{7.2.1}$$

[1] the term torque is usually used instead of moment in the context of twisting shafts such as those considered in this section

where *L* is the length, *R* the radius of the member and $\phi(L)$ means the magnitude of ϕ at *L*. Note that the strain is constant along the length of the member although ϕ is not. Considering a general cross-section within the member, as in Fig. 7.2.2, one has

$$\gamma = \alpha \approx \frac{R\phi(x)}{x} \tag{7.2.2}$$

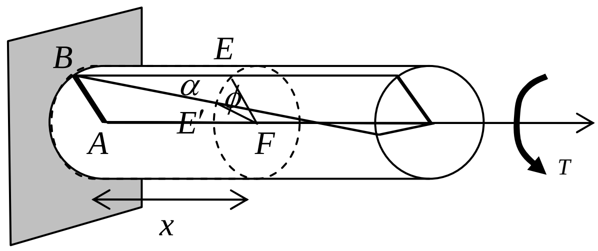


Figure 7.2.2: A section of a twisting cylindrical member

The shear strain at an arbitrary radial location *r*, $0 < r < R$, is

$$\gamma(r) = \frac{r\phi(x)}{x} \tag{7.2.3}$$

showing that the shear strain varies from zero at the centre of the shaft to a maximum $R\phi(L)/L \ \left(= R\phi(x)/x\right)$ on the outer surface of the shaft.

Considering a free-body diagram of any portion of the shaft of Fig. 7.2.1, a torque *T* acts on all cross-sections. This torque must equal the resultant of the shear stresses acting over the section, as schematically illustrated in Fig. 7.2.3a. The elemental force acting over an element with sides dr and $rd\theta$ is $\tau dA = \tau r dr d\theta$, Fig. 7.2.3b, and so the resultant moment about $r = 0$ is

$$T = \int_0^{2\pi}\int_0^R r^2\tau(r)drd\theta = 2\pi\int_0^R r^2\tau(r)dr \tag{7.2.4}$$

Hooke's law is

$$\tau = G\gamma \tag{7.2.5}$$

where *G* is the shear modulus (the μ of Eqn. 6.1.5). But γ / r is a constant and so therefore also is τ / r (provided *G* is) and Eqn. 7.2.4 can be re-written as

$$T = \frac{\tau(r)}{r}\left[2\pi\int_0^R r^3 dr\right] = \frac{\tau(r)J}{r} \tag{7.2.6}$$

The quantity in square brackets is called the **polar moment of inertia** of the cross-section (also called the **polar second moment of area**) and is denoted by *J*. For this circular cross-section it is given by

$$J = 2\pi \int_0^R r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \tag{7.2.7}$$

where D is the diameter. In general, for a cross-section of arbitrary shape,

$$\boxed{J = \int_A r^2 dA} \quad \textbf{Polar Moment of Area} \tag{7.2.8}$$

where dA is an element of area and the integration is over the complete cross-section.

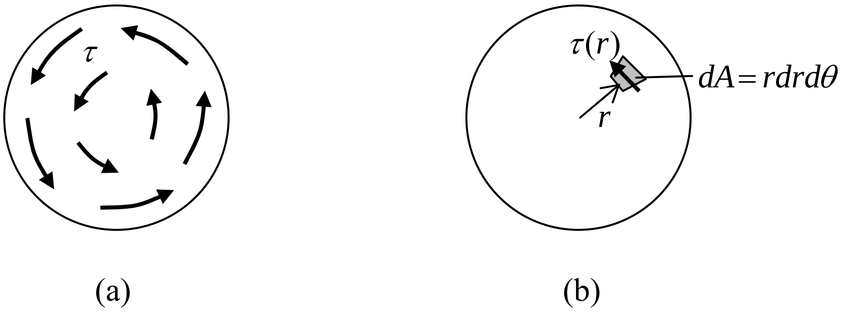


Figure 7.2.3: Shear stresses acting over a cross-section; (a) shear stress, (b) moment for an elemental area

From Eqn. 7.2.6, the shear stress at any radial location is given by

$$\boxed{\tau(r) = \frac{rT}{J}} \tag{7.2.9}$$

From Eqn. 7.2.1, 7.2.5 and 7.2.9, the angle of twist at the end of the member – or the twist at one end relative to that at the other end – is

$$\boxed{\phi = \frac{TL}{GJ}} \tag{7.2.10}$$

Example

Consider the problem shown in Fig.7.2.4, two torsion members of lengths L_1, L_2, diameters d_1, d_2 and shear moduli G_1, G_2, built-in at A and subjected to torques T_B and T_C. Equilibrium of moments can be used to determine the unknown torques acting in each member:

$$-T_1 + T_B + T_C = 0, \qquad -T_2 + T_C = 0 \tag{7.2.11}$$

so that $T_1 = T_B + T_C$ and $T_2 = T_C$.

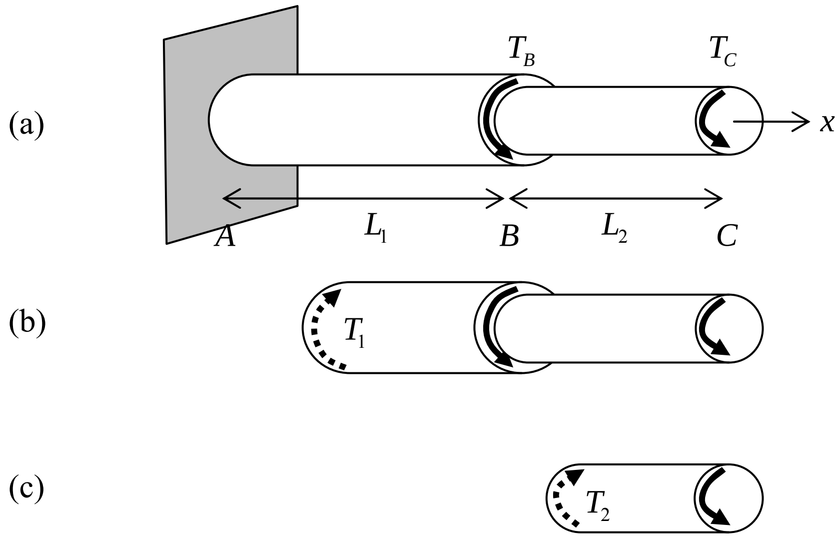


Figure 7.2.4: A structure consisting of two torsion members; (a) subjected to torques T_B and T_C, (b,c) free-body diagrams

The shear stresses in each member are therefore

$$\tau_1 = \frac{r(T_B + T_C)}{J_1}, \qquad \tau_2 = \frac{rT_C}{J_2} \tag{7.2.12}$$

where $J_1 = \pi d_1^4 / 32$ and $J_2 = \pi d_2^4 / 32$.

From Eqn. 7.2.10, the angle of twist at *B* is given by $\phi_B = T_1 L_1 / G_1 J_1$. The angle of twist at *C* is then

$$\phi_C = \frac{T_2 L_2}{G_2 J_2} - \phi_B \tag{7.2.13}$$

■

Statically indeterminate problems can be solved using methods analogous to those used in the section 7.1 for uniaxial members.

Example

Consider the structure in Fig. 7.2.5, similar to that in Fig. 7.2.4 only now both ends are built-in and there is only a single applied torque, T_B.

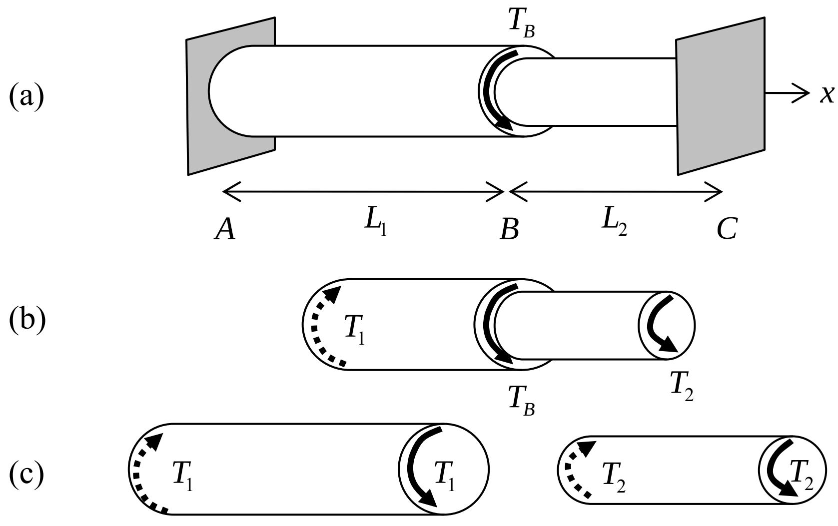


Figure 7.2.5: A structure consisting of two torsion members; (a) subjected to a Torque T_B, (b) free-body diagram, (c) separate elements

Referring to the free-body diagram of Fig. 7.2.5b, there is only one equation of equilibrium with which to determine the two unknown member torques:

$$-T_1 + T_B + T_2 = 0 \tag{7.2.14}$$

and so the deformation of the structure needs to be considered. A systematic way of dealing with this situation is to consider each element separately, as in Fig. 7.2.5c. The twist in each element is

$$\phi_1 = \frac{T_1 L_1}{G_1 J_1}, \quad \phi_2 = \frac{T_2 L_2}{G_2 J_2} \tag{7.2.15}$$

The total twist is zero and so $\phi_1 + \phi_2 = 0$ which, with Eqn. 7.2.14, can be solved to obtain

$$T_1 = +\frac{L_2 G_1 J_1}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B, \quad T_2 = -\frac{L_1 G_2 J_2}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B \tag{7.2.16}$$

The rotation at *B* can now be determined, $\phi_B = \phi_1 = -\phi_2$.

■

7.2.2 Stress Distribution in Torsion Members

The shear stress in Eqn. 7.2.9 is acting over a cross-section of a torsion member. From the symmetry of the stress, it follows that shear stresses act also along the length of the member, as illustrated to the left of Fig. 7.2.6. Shear stresses do not act *on* the surface of the element shown, as it is a free surface.

Any element of material not aligned with the axis of the cylinder will undergo a complex stress state, as shown to the right of Fig. 7.2.6. The stresses acting on an element are given by the stress transformation equations, Eqns. 3.4.8:

$$\sigma'_{xx} = +\sin 2\theta\tau, \qquad \sigma'_{yy} = -\sin 2\theta\tau, \qquad \sigma'_{xy} = +\cos 2\theta\tau \tag{7.2.17}$$

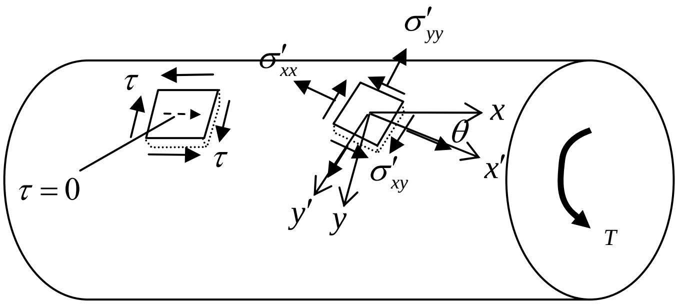
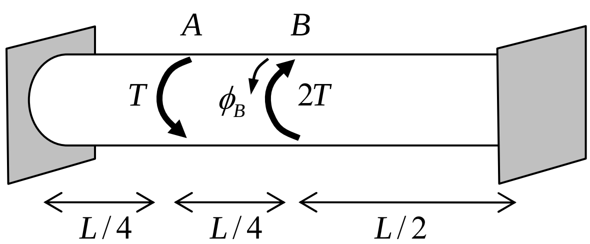


Figure 7.2.6: Stress distribution in a torsion member

From Eqns. 3.5.4-5, the maximum normal (principal) stresses arise on planes at $\theta = \pm 45^o$ and are $\sigma_1 = +\tau$ and $\sigma_2 = -\tau$. Thus the maximum tensile stress in the member occurs at 45^o to the axis and arises at the surface. The maximum shear stress is simply τ, with $\theta = 0$.

7.2.3 Problems

1. A shaft of length *L* and built-in at both ends is subjected to two external torques, T at *A* and $2T$ at *B*, as shown below. The shaft is of diameter *d* and shear modulus *G*. Determine the maximum (absolute value of) shear stress in the shaft and determine the angle of twist at *B*.



7.4 The Elementary Beam Theory

In this section, problems involving long and slender beams are addressed. As with pressure vessels, the geometry of the beam, and the specific type of loading which will be considered, allows for approximations to be made to the full three-dimensional linear elastic stress-strain relations.

The beam theory is used in the design and analysis of a wide range of structures, from buildings to bridges to the load bearing bones of the human body.

7.4.1 The Beam

The term **beam** has a very specific meaning in engineering mechanics: it is a component that is designed to support **transverse loads**, that is, loads that act perpendicular to the longitudinal axis of the beam, Fig. 7.4.1. The beam supports the load by *bending only*. Other mechanisms, for example twisting of the beam, are not allowed for in this theory.

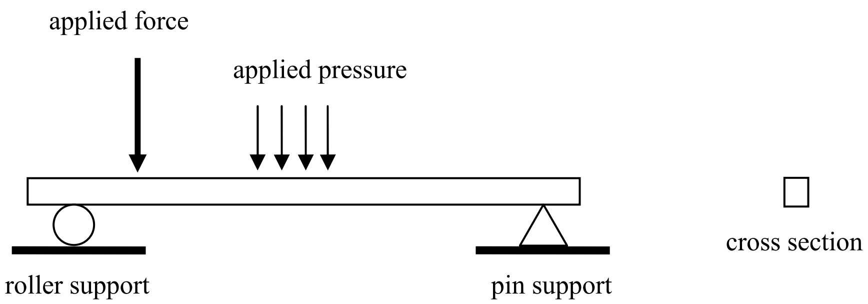


Figure 7.4.1: A supported beam loaded by a force and a distribution of pressure

It is convenient to show a two-dimensional cross-section of the three-dimensional beam together with the beam cross section, as in Fig. 7.4.1. The beam can be supported in various ways, for example by roller supports or pin supports (see section 2.3.3). The cross section of this beam happens to be rectangular but it can be any of many possible shapes.

It will assumed that the beam has a **longitudinal plane of symmetry**, with the cross section symmetric about this plane, as shown in Fig. 7.4.2. Further, it will be assumed that the loading and supports are also symmetric about this plane. With these conditions, the beam has no tendency to twist and will undergo bending only[1].

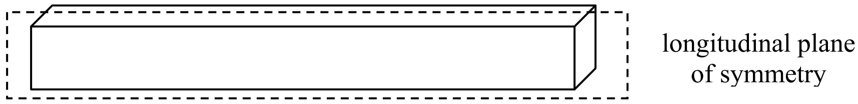


Figure 7.4.2: The longitudinal plane of symmetry of a beam

[1] certain very special cases, where there is *not* a plane of symmetry for geometry and/or loading, can lead also to bending with no twist, but these are not considered here

Imagine now that the beam consists of many fibres aligned longitudinally, as in Fig. 7.4.3. When the beam is bent by the action of downward transverse loads, the fibres near the top of the beam contract in length whereas the fibres near the bottom of the beam extend. Somewhere in between, there will be a plane where the fibres do not change length. This is called the **neutral surface**. The intersection of the longitudinal plane of symmetry and the neutral surface is called the **axis of the beam**, and the deformed axis is called the **deflection curve**.

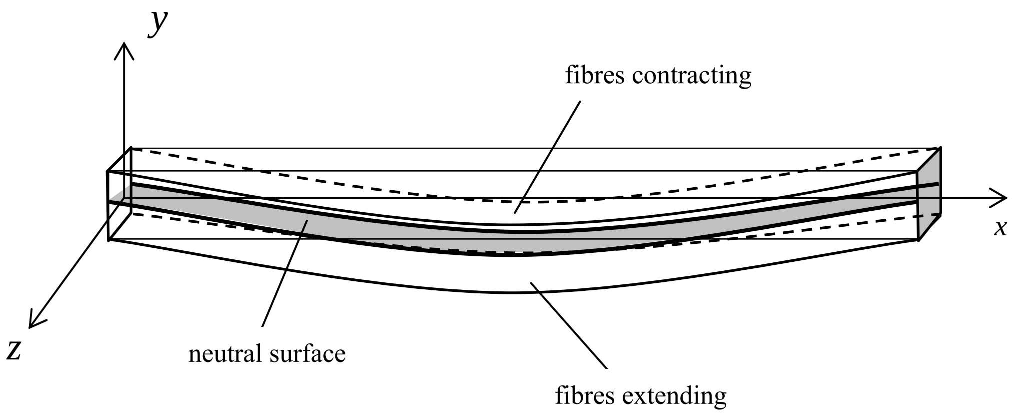


Figure 7.4.3: the neutral surface of a beam

A conventional coordinate system is attached to the beam in Fig. 7.4.3. The *x* axis coincides with the (longitudinal) axis of the beam, the *y* axis is in the transverse direction and the longitudinal plane of symmetry is in the $x-y$ plane, also called the **plane of bending**.

7.4.2 Moments and Forces in a Beam

Normal and shear stresses act over any cross section of a beam, as shown in Fig. 7.4.4. The normal and shear stresses acting on each side of the cross section are equal and opposite for equilibrium, Fig. 7.4.4b. The normal stresses σ will vary over a section during bending. Referring again to Fig. 7.4.3, over one part of the section the stress will be tensile, leading to extension of material fibres, whereas over the other part the stresses will be compressive, leading to contraction of material fibres. This distribution of normal stress results in a moment *M* acting on the section, as illustrated in Fig. 7.4.4c. Similarly, shear stresses τ act over a section and these result in a shear force *V*.

The beams of Fig. 7.4.3 and Fig. 7.4.4 show the normal stress and deflection one would expect when a beam bends downward. There are situations when parts of a beam bend upwards, and in these cases the signs of the normal stresses will be opposite to those shown in Fig. 7.4.4. However, the moments (and shear forces) shown in Fig. 7.4.4 will be regarded as *positive*. This sign convention to be used is shown in Fig. 7.4.5.

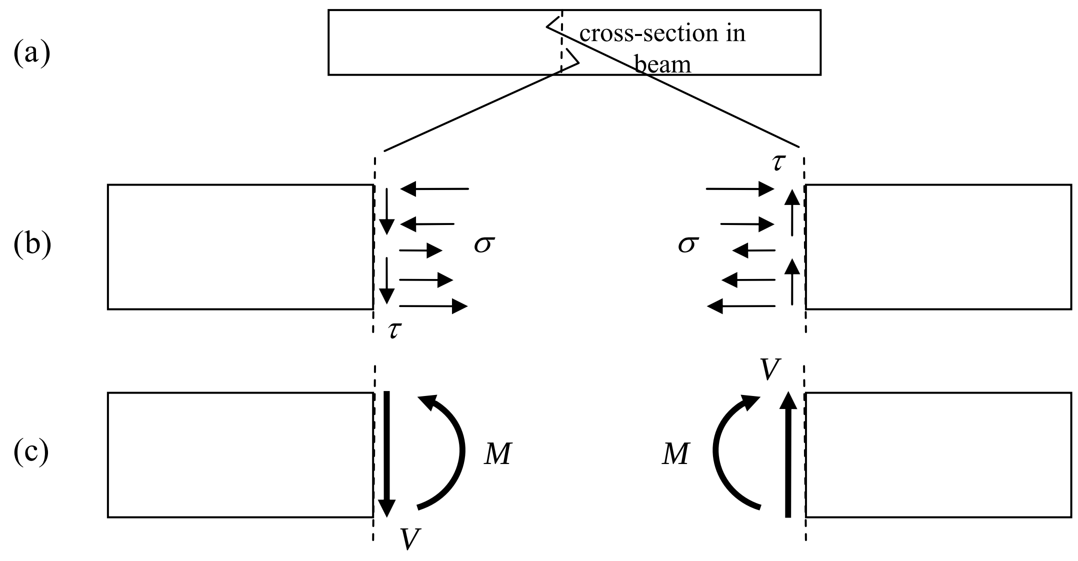


Figure 7.4.4: stresses and moments acting over a cross-section of a beam

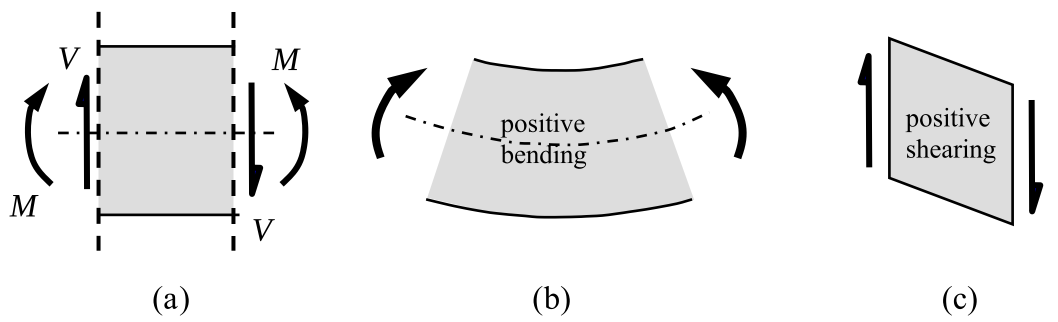


Figure 7.4.5: sign convention for moments and shear forces

Note that the sign convention for the shear stress in the beam theory conflicts with the sign convention for shear stress used in the rest of mechanics, introduced in Chapter 3. This is shown in Fig. 7.4.6.

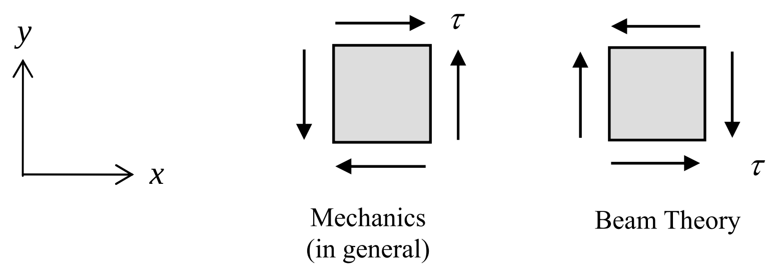


Figure 7.4.6: sign convention for shear stress in beam theory

The moments and forces acting within a beam can in many simple problems be evaluated from equilibrium considerations alone. Some examples are given next.

Example 1

Consider the **simply supported** beam in Fig. 7.4.7. From the loading, one would expect the beam to deflect something like as indicated by the deflection curve drawn. The reaction at the roller support, end *A*, and the vertical reaction at the pin support[2], end *B*, can be evaluated from the equations of equilibrium, Eqns. 2.3.3:

$$R_{Ay} = P/3, \quad R_{By} = 2P/3 \qquad (7.4.1)$$

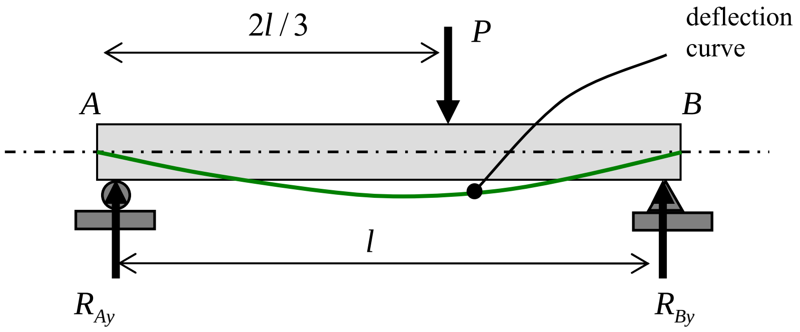


Figure 7.4.7: a simply supported beam

The moments and forces acting *within* the beam can be evaluated by taking free-body diagrams of sections of the beam. There are clearly two distinct regions in this beam, to the left and right of the load. Fig. 7.4.8a shows an arbitrary portion of beam representing the left-hand side. A coordinate system has been introduced, with *x* measured from *A*.[3] An unknown moment *M* and shear force *V* act at the end. A *positive* moment and force have been drawn in Fig. 7.4.8a. From the equilibrium equations, one finds that the shear force is constant but that the moment varies linearly along the beam:

$$V = \frac{P}{3}, \qquad M = \frac{P}{3}x \qquad \left(0 < x < \frac{2l}{3}\right) \qquad (7.4.2)$$

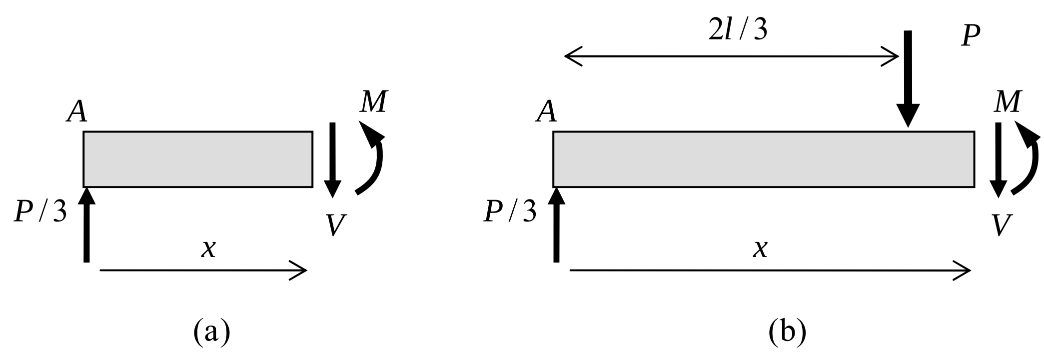


Figure 7.4.8: free body diagrams of sections of a beam

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[2] the horizontal reaction at the pin is zero since there are no applied forces in this direction; the beam theory does not consider such types of load

[3] the coordinate *x* can be measured from any point in the beam; in this example it is convenient to measure it from point *A*

Cutting the beam to the right of the load, Fig. 7.4.8b, leads to

$$V = -\frac{2P}{3}, \qquad M = \frac{2P}{3}(l - x) \qquad \left(\frac{2l}{3} < x < l\right) \tag{7.4.3}$$

The shear force is negative, so acts in the direction opposite to that initially assumed in Fig. 7.4.8b.

The results of the analysis can be displayed in what are known as a **shear force diagram** and a **bending moment diagram**, Fig. 7.4.9. Note that there is a "jump" in the shear force at $x = 2l/3$ equal to the applied force, and in this example the bending moment is everywhere positive.

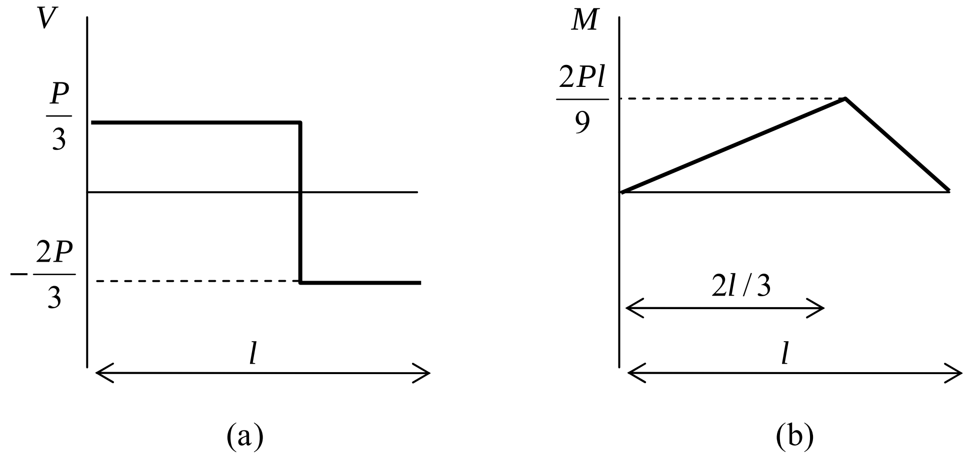


Figure 7.4.9: results of analysis; (a) shear force diagram, (b) bending moment diagram

■

Example 2

Fig. 7.4.10 shows a **cantilever**, that is, a beam supported by clamping one end (refer to Fig. 2.3.8), and loaded by a force at its mid-point and a (negative) moment at its end.

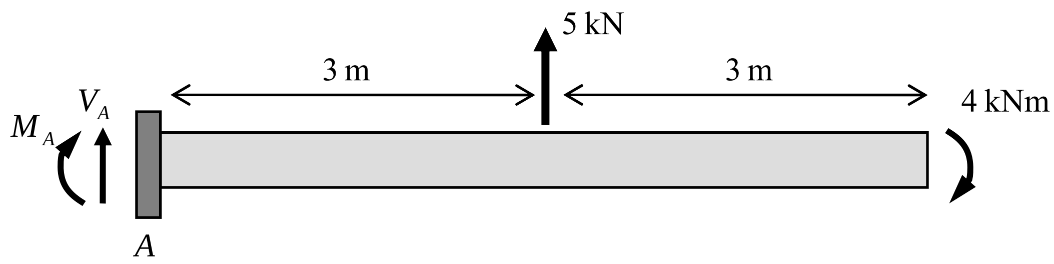


Figure 7.4.10: a cantilevered beam loaded by a force and moment

Again, positive unknown reactions M_A and V_A are considered at the support A. From the equilibrium equations, one finds that

$$M_A = 11\,\text{kNm}, \quad V_A = -5\,\text{kN} \tag{7.4.4}$$

As in the previous example, there are two distinct regions along the beam, to the left and to the right of the applied concentrated force. Again, a coordinate *x* is introduced and the beam is sectioned as in Fig. 7.4.11. The unknown moment *M* and shear force *V* can then be evaluated from the equilibrium equations:

$$\begin{array}{lll} V = -5\,\text{kN}, & M = 11 - 5x \text{ kNm} & (0 < x < 3) \\ V = 0, & M = -4 \text{ kNm} & (3 < x < 6) \end{array} \qquad (7.4.5)$$

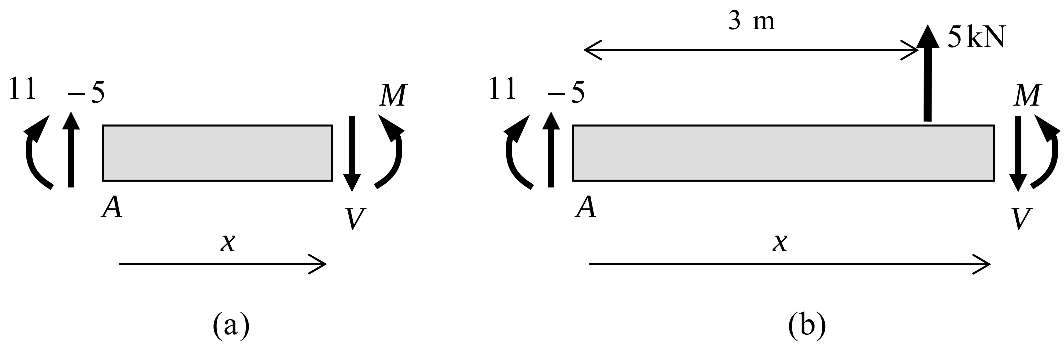


Figure 7.4.11: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.12.

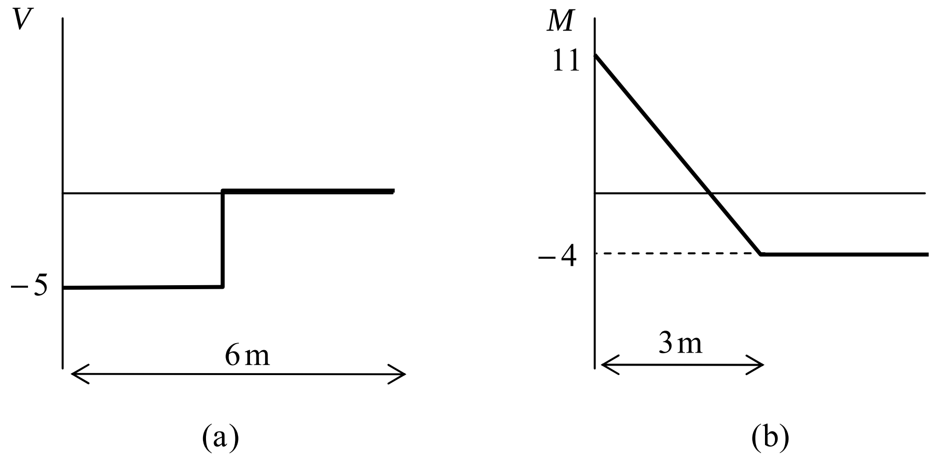


Figure 7.4.12: results of analysis; (a) shear force diagram, (b) bending moment diagram

In this example the beam experiences negative bending moment over most of its length. ■

Example 3

Fig. 7.4.13 shows a simply supported beam subjected to a distributed load (force per unit length). The load is uniformly distributed over half the length of the beam, with a triangular distribution over the remainder.

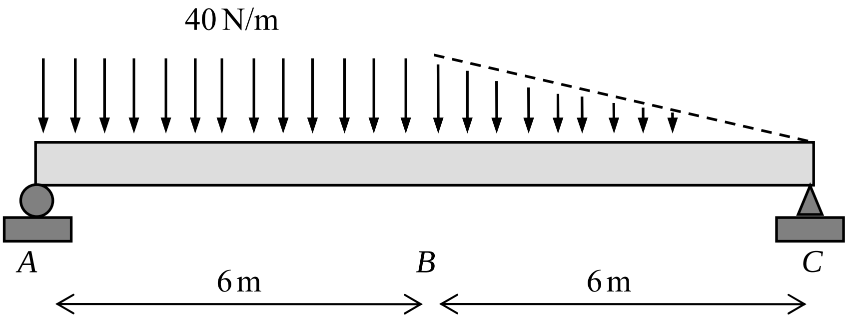


Figure 7.4.13: a beam subjected to a distributed load

The unknown reactions can be determined by replacing the distributed load with statically equivalent forces as in Fig. 7.4.14 (refer to §3.1.2). The equilibrium equations then give

$$R_A = 220 \text{ N}, \quad R_C = 140 \text{ N} \tag{7.4.6}$$

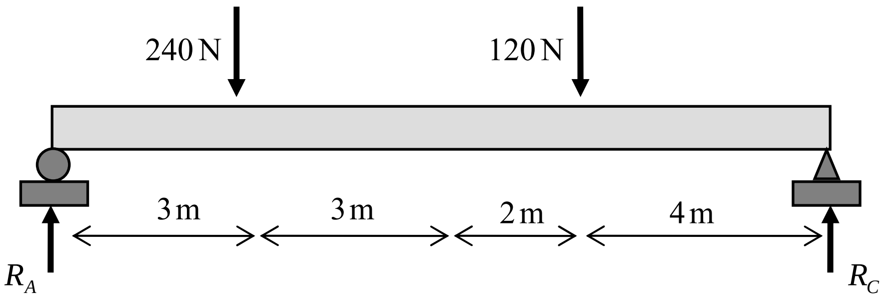


Figure 7.4.14: equivalent forces acting on the beam of Fig. 7.4.13

Referring again to Fig. 7.4.13, there are two distinct regions in the beam, that under the uniform load and that under the triangular distribution of load. The first case is considered in Fig. 7.4.15.

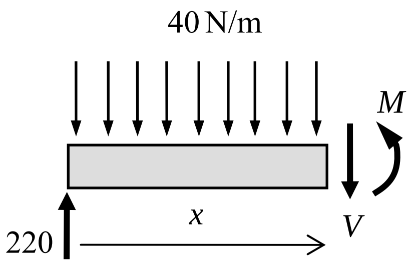


Figure 7.4.15: free body diagram of a section of a beam

The equilibrium equations give

$$V = 220 - 40x, \qquad M = 220x - 20x^2 \qquad (0 < x < 6) \tag{7.4.7}$$

The region beneath the triangular distribution is shown in Fig. 7.4.16. Two possible approaches are illustrated: in Fig. 7.4.16a, the free body diagram consists of the complete length of beam to the left of the cross-section under consideration; in Fig. 7.4.16b, only the portion to the right is considered, with distance measured from the right hand end, as $12 - x$. The problem is easier to solve using the second option. From Fig. 7.4.16b then, with the equilibrium equations, one finds that

$$V = -140 + 10(12 - x)^2 / 3, \quad M = 140(12 - x) - 10(12 - x)^3 / 9 \quad (6 < \mathrm{x} < 12) \tag{7.4.8}$$

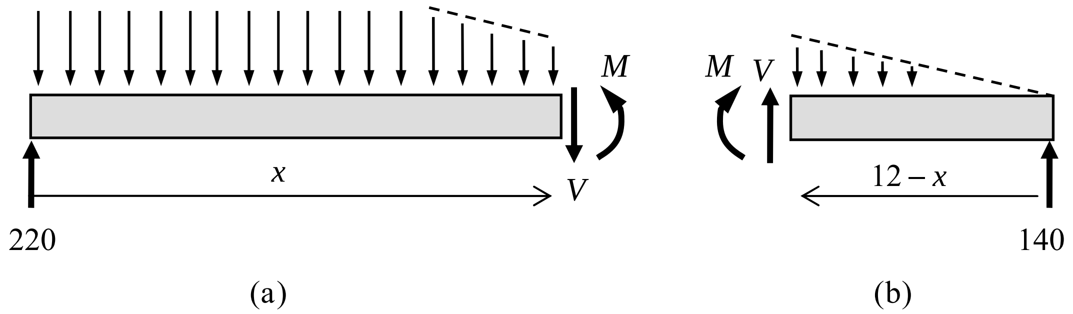


Figure 7.4.16: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.17.

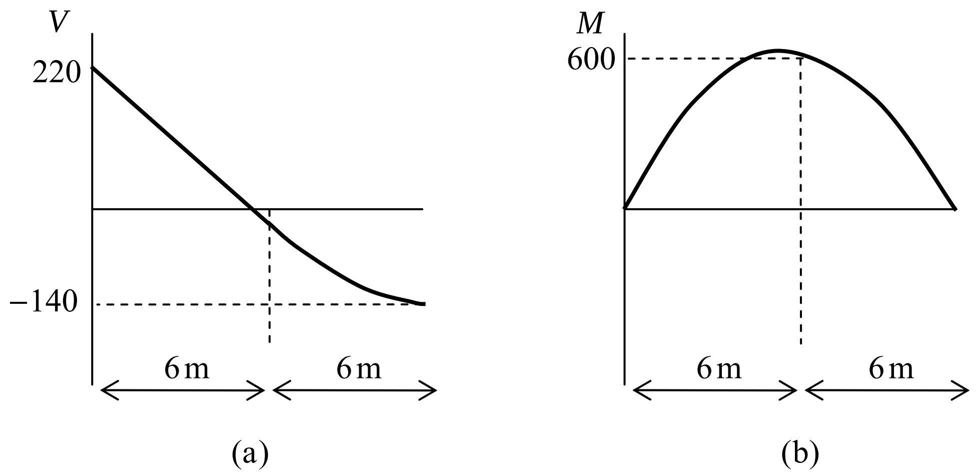


Figure 7.4.17: results of analysis; (a) shear force diagram, (b) bending moment diagram

■

7.4.3 The Relationship between Loads, Shear Forces and Bending Moments

Relationships between the applied loads and the internal shear force and bending moment in a beam can be established by considering a small beam element, of width Δx, and subjected to a distributed load $p(x)$ which varies along the section of beam, and which is *positive upward*, Fig. 7.4.18.

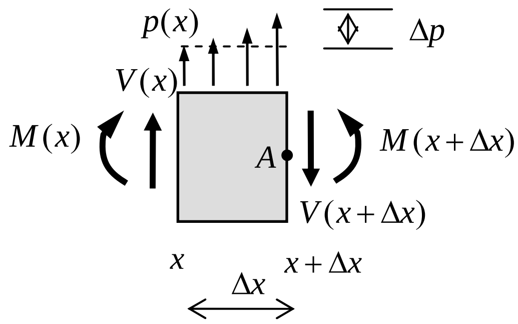


Figure 7.4.18: forces and moments acting on a small element of beam

At the left-hand end of the free body, at position *x*, the shear force, moment and distributed load have values $F(x)$, $M(x)$ and $p(x)$ respectively. On the right-hand end, at position $x+\Delta x$, their values are slightly different: $F(x+\Delta x)$, $M(x+\Delta x)$ and $p(x+\Delta x)$. Since the element is very small, the distributed load, even if it is varying, can be approximated by a *linear* variation over the element. The distributed load can therefore be considered to be a uniform distribution of intensity $p(x)$ over the length Δx together with a triangular distribution, 0 at x and Δp say, a *small* value, at $x+\Delta x$. Equilibrium of vertical forces then gives

$$\begin{aligned} &V(x)+p(x)\Delta x+\frac{1}{2}\Delta p\Delta x-V(x+\Delta x)=0 \\ \to\ &\frac{V(x+\Delta x)-V(x)}{\Delta x}=p(x)+\frac{1}{2}\Delta p \end{aligned} \tag{7.4.9}$$

Now let the size of the element decrease towards zero. The left-hand side of Eqn. 7.4.9 is then the definition of the derivative, and the second term on the right-hand side tends to zero, so

$$\boxed{\frac{dV}{dx}=p(x)} \tag{7.4.10}$$

This relation can be seen to hold in Eqn. 7.4.7 and Fig. 7.4.17a, where the shear force over $0<x<6$ has a slope of -40 and the pressure distribution is uniform, of intensity $-40\,\mathrm{N/m}$. Similarly, over $6<x<12$, the pressure decreases linearly and so does the slope in the shear force diagram, reaching zero slope at the end of the beam.

It also follows from 7.4.10 that the change in shear along a beam is equal to the area under the distributed load curve:

$$\boxed{V(x_2)-V(x_1)=\int_{x_1}^{x_2}p(x)dx} \tag{7.4.11}$$

Consider now moment equilibrium, by taking moments about the point *A* in Fig. 7.4.18:

$$
\begin{aligned}
&-M(x)-V(x)\Delta x+M(x+\Delta x)-p(x)\Delta x\frac{\Delta x}{2}-\frac{1}{2}\Delta p\Delta x\frac{\Delta x}{3}=0 \\
&\to \frac{M(x+\Delta x)-M(x)}{\Delta x}=V(x)+p(x)\frac{\Delta x}{2}+\Delta p\frac{\Delta x}{6}
\end{aligned}
\tag{7.4.12}
$$

Again, as the size of the element decreases towards zero, the left-hand side becomes a derivative and the second and third terms on the right-hand side tend to zero, so that

$$
\boxed{\frac{dM}{dx}=V(x)} \tag{7.4.13}
$$

This relation can be seen to hold in Eqns. 7.4.2-3, 7.4.5 and 7.4.7-8. It also follows from Eqn. 7.4.13 that the change in moment along a beam is equal to the area under the shear force curve:

$$
\boxed{M(x_2)-M(x_1)=\int_{x_1}^{x_2}V(x)dx} \tag{7.4.14}
$$

7.4.4 Deformation and Flexural Stresses in Beams

The moment at any given cross-section of a beam is due to a distribution of normal stress, or **flexural stress** (or **bending stress**) across the section (see Fig. 7.4.4). As mentioned, the stresses to one side of the neutral axis are tensile whereas on the other side of the neutral axis they are compressive. To determine the distribution of normal stress over the section, one must determine the precise location of the neutral axis, and to do this one must consider the *deformation* of the beam.

Apart from the assumption of there being a longitudinal plane of symmetry and a neutral axis along which material fibres do not extend, the following two assumptions will be made concerning the deformation of a beam:

1. cross sections which are plane and are perpendicular to the axis of the undeformed beam remain plane and remain perpendicular to the deflection curve of the deformed beam. In short: “plane sections remain plane”. This is illustrated in Fig. 7.4.19. It will be seen later that this assumption is a valid one provided the beam is sufficiently long and slender.

2. deformation in the vertical direction, i.e. the transverse strain ε_{yy}, may be neglected in deriving an expression for the longitudinal strain ε_{xx}. This assumption is summarised in the deformation shown in Fig. 7.4.20, which shows an element of length *l* and height *h* undergoing transverse and longitudinal strain.

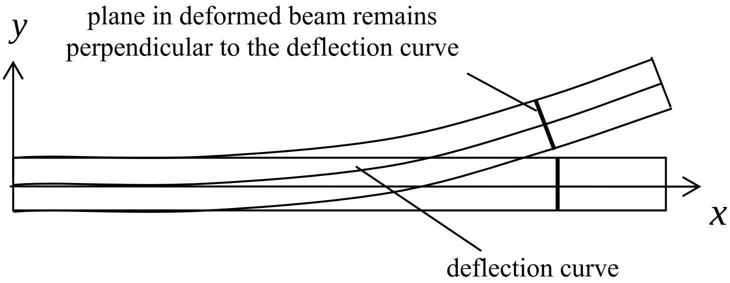
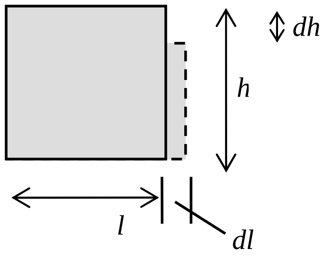


Figure 7.4.19: plane sections remain plane in the elementary beam theory



$$\varepsilon_{xx} = \frac{dl}{l}, \quad \varepsilon_{yy} = -\frac{dh}{h} \approx 0$$

Figure 7.4.20: transverse strain is neglected in the elementary beam theory

With these assumptions, consider now the element of beam shown in Fig. 7.4.21. Here, two material fibres ab and pq, of length Δx in the undeformed beam, deform to $a'b'$ and $p'q'$. The deflection curve has a radius of curvature *R*. The above two assumptions imply that, referring to the figure:

$$\begin{aligned} &\angle p'a'b' = \angle a'b'q' = \pi/2 && \text{(assumption 1)} \\ &|ap| = |a'p'|, \quad |bq| = |b'q'| && \text{(assumption 2)} \end{aligned} \qquad (7.4.15)$$

Since the fibre ab is on the neutral axis, by definition $|a'b'| = |ab|$. However the fibre pq, a distance *y* from the neutral axis, extends in length from Δx to length $\Delta x'$. The longitudinal strain for this fibre is

$$\varepsilon_{xx} = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(R-y)\Delta\theta - R\Delta\theta}{R\Delta\theta} = -\frac{y}{R} \qquad (7.4.16)$$

As one would expect, this relation implies that a small *R* (large curvature) is related to a large strain and a large *R* (small curvature) is related to a small strain. Further, for $y > 0$ (above the neutral axis), the strain is negative, whereas if $y < 0$ (below the neutral axis), the strain is positive[4], and the variation across the cross-section is linear.

[4] this is under the assumption that *R* is positive, which means that the beam is concave up; a negative *R* implies that the centre of curvature is below the beam

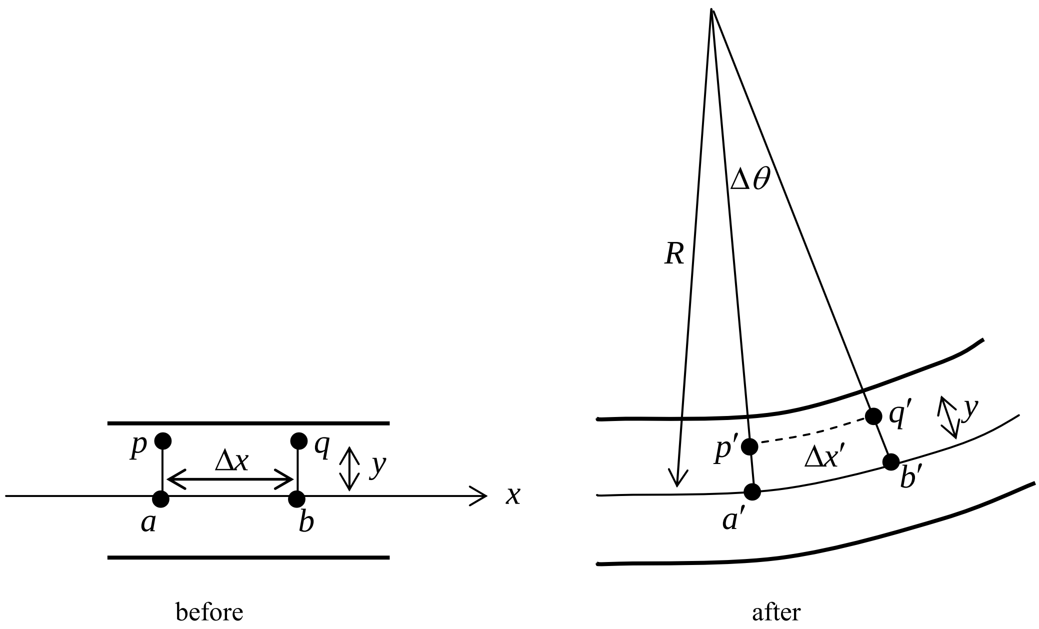


Figure 7.4.21: deformation of material fibres in an element of beam

To relate this deformation to the stresses arising in the beam, it is necessary to postulate the stress-strain law for the material out of which the beam is made. Here, it is assumed that the beam is isotropic linear elastic[5]. Since there are no forces acting in the z direction, the beam is in a state of plane stress, and the stress-strain equations are (see Eqns. 6.1.10)

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{zz} &= -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}, \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0
\end{aligned} \tag{7.4.17}$$

Yet another assumption is now made, that the transverse normal stresses, σ_{yy}, may be neglected in comparison with the flexural stresses σ_{xx}. This is similar to the above assumption #2 concerning the deformation, where the transverse normal strain was neglected in comparison with the longitudinal strain. It might seem strange at first that the transverse stress is neglected, since all loads are in the transverse direction. However, just as the tangential stresses are much larger than the radial stresses in the pressure vessel, it is found that the longitudinal stresses in a beam are very much greater than the transverse stresses. With this assumption, the first of Eqn. 7.4.17 reduces to a one-dimensional equation:

$$\varepsilon_{xx} = \sigma_{xx} / E \tag{7.4.18}$$

[5] the beam theory can be extended to incorporate more complex material models (constitutive equations)

and, from Eqn. 7.4.16, dropping the subscripts on σ,

$$\sigma = -\frac{E}{R} y \tag{7.4.19}$$

Finally, the resultant force of the normal stress distribution over the cross-section must be zero, and the resultant moment of the distribution is *M*, leading to the conditions

$$\begin{aligned} 0 &= \int_A \sigma dA = -\frac{E}{R}\int_A y dA \\ M &= -\int_A \sigma y dA = \frac{E}{R}\int_A y^2 dA = -\frac{\sigma}{y}\int_A y^2 dA \end{aligned} \tag{7.4.20}$$

and the integration is over the complete cross-sectional area *A*. The minus sign in the second of these equations arises because a positive moment and a positive *y* imply a compressive (negative) stress (see Fig. 7.4.4).

The quantity $\int_A y dA$ is the first moment of area about the neutral axis, and is equal to $\bar{y}A$, where $\bar{y}$ is the centroid of the section (see, for example, §3.2.1). Note that the horizontal component of the centroid will always be at the centre of the beam due to the symmetry of the beam about the plane of bending. Since the first moment of area is zero, it follows that $\bar{y} = 0$: *the neutral axis passes through the centroid of the cross-section*.

The quantity $\int_A y^2 dA$ is called the **second moment of area** or the **moment of inertia** about the neutral axis, and is denoted by the symbol *I*. It follows that the flexural stress is related to the moment through

$$\boxed{\sigma = -\frac{My}{I}} \qquad \textbf{Flexural stress in a beam} \tag{7.4.21}$$

This is one of the most famous and useful formulas in mechanics.

The Moment of Inertia

The moment of inertia depends on the shape of a beam's cross-section. Consider the important case of a rectangular cross section. Before determining the moment of inertia one must locate the centroid (neutral axis). Due to symmetry, the neutral axis runs through the centre of the cross-section. To evaluate *I* for a rectangle of height *h* and width *b*, consider a small strip of height dy at location *y*, Fig. 7.4.22. Then

$$I = \int_A y^2 dA = b\int_{-h/2}^{+h/2} y^2 dy = \frac{bh^3}{12} \tag{7.4.22}$$

This relation shows that the "taller" the cross-section, the larger the moment of inertia, something which holds generally for *I*. Further, the larger is *I*, the smaller is the flexural stress, which is always desirable.

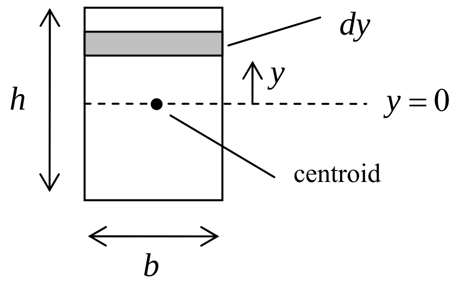


Figure 7.4.22: Evaluation of the moment of inertia for a rectangular cross-section

For a circular cross-section with radius *R*, consider Fig. 7.4.23. The moment of inertia is then

$$I = \int_A y^2 dA = \int_0^{2\pi}\int_0^R r^3 \sin^2\theta dr d\theta = \frac{\pi R^4}{4} \tag{7.4.23}$$

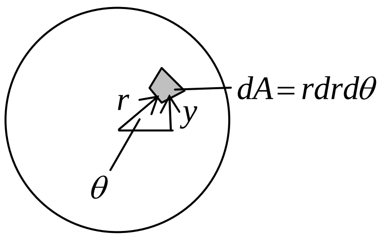


Figure 7.2.23: Moment of inertia for a circular cross-section

Example

Consider the beam shown in Fig. 7.4.24. It is loaded symmetrically by two concentrated forces and has a circular cross-section of radius 100mm. The reactions at the two supports are found to be 100N. Sectioning the beam to the left of the forces, and then to the right of the first force, one finds that

$$\begin{aligned} V &= 100, \qquad M = 100x \qquad (0 < x < 250) \\ V &= 0, \qquad M = 25000 \qquad (250 < x < l/2) \end{aligned} \tag{7.4.24}$$

where *l* is the length of the beam.

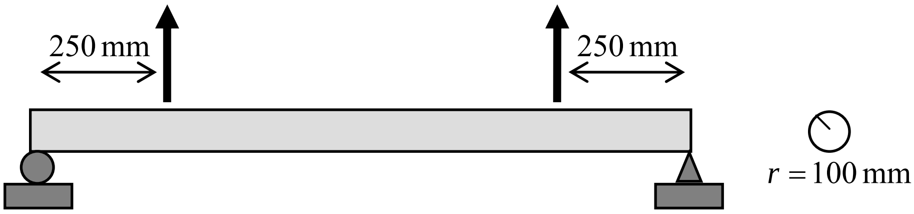


Figure 7.4.24: a loaded beam with circular cross-section

The maximum tensile stress is then

$$\sigma_{\max} = -\frac{M_{\max}(-y_{\max})}{I} = \frac{25000r}{\pi r^4/4} = 31.8\,\mathrm{MPa} \qquad (7.4.25)$$

and occurs at all sections between the two loads.

■

7.4.5 Shear Stresses in Beams

In the derivation of the flexural stress formula, Eqn. 7.4.21, it was assumed that plane sections remain plane. This implies that there is no shear strain and, for an isotropic elastic material, no shear stress, as indicated in Fig. 7.4.25.

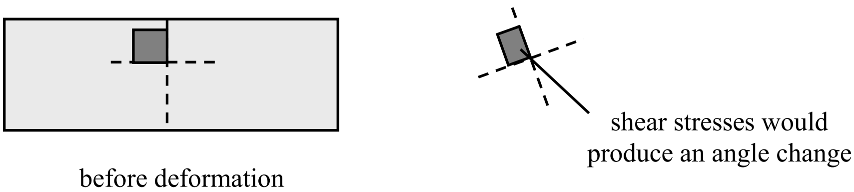


Figure 7.4.25: a section of beam before and after deformation

This fact will now be ignored, and an expression for the shear stress τ within a beam will be developed. It is implicitly assumed that this shear stress has little effect on the calculation of the flexural stress.

As in Fig. 7.4.18, consider the equilibrium of a thin section of beam, as shown in Fig. 7.4.26. The beam has *rectangular* cross-section (although the theory developed here is strictly for rectangular cross sections only, it can be used to give approximate shear stress values in any beam with a plane of symmetry). Consider the equilibrium of a section *of* this section, at the upper surface of the beam, shown hatched in Fig. 7.4.26. The stresses acting on this section are as shown. Again, the normal stress is compressive at the surface, consistent with the sign convention for a positive moment. Note that there are no shear stresses acting at the surface – there may be distributed normal loads or forces acting at the surface but, for clarity, these are not shown, and they are not necessary for the following calculation.

From equilibrium of forces in the horizontal direction of the surface section:

$$\left[-\int_A \sigma dA\right]_x + \left[\int_A \sigma dA\right]_{x+\Delta x} + \tau b \Delta x = 0 \tag{7.4.26}$$

The third term on the left here assumes that the shear stress is uniform over the section – this is similar to the calculations of §7.4.3 – for a very small section, the variation in stress is a small term and may be neglected. Using the bending stress formula, Eqn. 7.4.21,

$$-\int_A \frac{M(x+\Delta x) - M(x)}{\Delta x} \frac{y}{I} dA + \tau b = 0 \tag{7.4.27}$$

and, with Eqn. 7.4.13, as $\Delta x \to 0$,

$$\boxed{\tau = \frac{VQ}{Ib}} \qquad \textbf{Shear stress in a beam} \tag{7.4.28}$$

where Q is the first moment of area $\int_A y dA$ of the *surface section* of the cross-section.

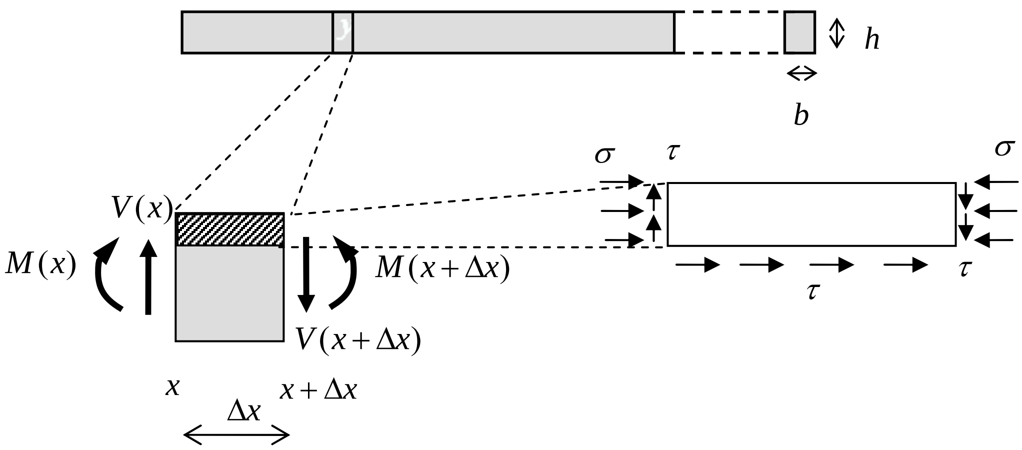


Figure 7.4.26: stresses and forces acting on a small section of material at the surface of a beam

As mentioned, this formula 7.4.28 can be used as an approximation of the shear stress in a beam of arbitrary cross-section, in which case b can be regarded as the depth of the beam at that section. For the rectangular beam, one has

$$Q = b \int_y^{h/2} y dy = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \tag{7.4.29}$$

so that

$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) \tag{7.4.30}$$

The maximum shear stress in the cross-section arises at the neutral surface:

$$\tau_{\max} = \frac{3V}{2bh} = \frac{3V}{2A} \tag{7.4.31}$$

and the shear stress dies away towards the upper and lower surfaces. Note that the average shear stress over the cross-section is V/A and the maximum shear stress is 150% of this value.

Finally, since the shear stress on a vertical cross-section has been evaluated, the shear stress on a longitudinal section has been evaluated, since the shear stresses on all four sides of an element are the same, as in Fig.7.4.6.

Example

Consider the simply supported beam loaded by a concentrated force shown in Fig. 7.4.27. The cross-section is rectangular with height $100\,\text{mm}$ and width $50\,\text{mm}$. The reactions at the supports are $5\,\text{kN}$ and $15\,\text{kN}$. To the left of the load, one has $V = 5\,\text{kN}$ and $M = 5x\,\text{kNm}$. To the right of the load, one has $V = -15\,\text{kN}$ and $M = 30 - 15x\,\text{kNm}$.

The maximum shear stress will occur along the neutral axis and will clearly occur where V is largest, so anywhere to the right of the load:

$$\tau_{\max} = \frac{3V_{\max}}{2A} = 4.5\,\text{MPa} \tag{7.4.32}$$

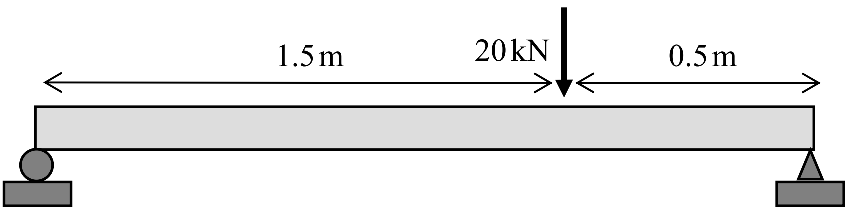


Figure 7.4.27: a simply supported beam

As an example of general shear stress evaluation, the shear stress at a point 25 mm below the top surface and 1 m in from the left-hand end is, from Eqn 7.4.30, $\tau = +1.125\,\text{MPa}$. The shear stresses acting on an element at this location are shown in Fig. 7.4.28.

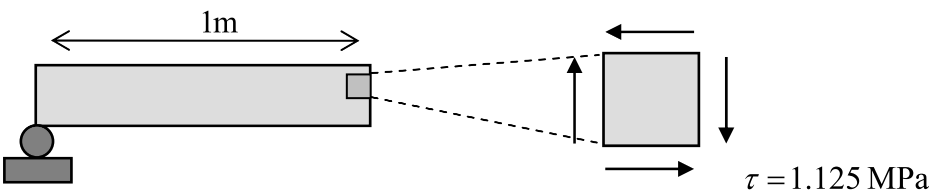


Figure 7.4.28: shear stresses acting at a point in the beam

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7.4.6 Approximate nature of the beam theory

The beam theory is only an approximate theory, with a number of simplifications made to the full equations of elasticity. The accuracy of the theory is briefly explored in this section.

When a beam is in **pure bending**, that is when the shear force is everywhere zero, the full elasticity solution shows that plane sections *do* actually remain plane and the beam theory is exact. For more complex loadings, plane sections *do* actually deform. For example, it will be shown in Book II that the initially plane sections of a cantilever subjected to an end force, Fig. 7.4.29, do not remain plane. Nevertheless, the beam theory prediction for normal and shear stress is exact in this simple case.

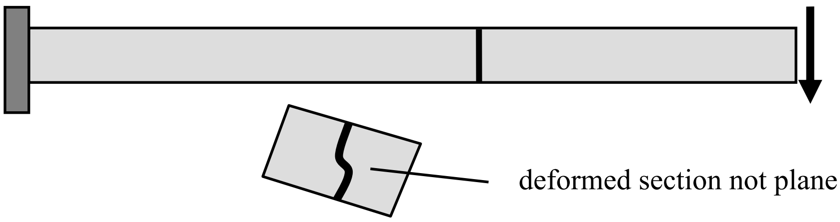


Figure 7.4.29: a cantilevered beam loaded by a force and moment

Consider next a cantilevered beam of length l and rectangular cross section, height h and width b, subjected to a uniformly distributed load p. With x measured from the cantilevered end, the shear force and moment are given by $V = p(l - x)$ and $M = (pl^2/2)\left(1 + 2x/l - 2(x/l)^2\right)$. The shear stress is

$$\tau = \frac{6p}{bh^3}\left(\frac{h^2}{4} - y^2\right)(l - x) \tag{7.4.33}$$

which turns out to be exact. The flexural stresses at the cantilevered end, at the upper surface, are

$$\frac{\sigma}{p} = \frac{3}{4}\left(\frac{l}{h}\right)^2 \tag{7.4.34}$$

The exact solution is, however (see Book II),

$$\frac{\sigma}{p} = \frac{3}{4}\left(\frac{l}{h}\right)^2 - \frac{1}{5} \tag{7.4.35}$$

It can be seen that the beam theory is a good approximation for the case when l/h is large, in which case the term 1/5 is negligible.

In summary, for most configurations, the elementary beam theory formulae for flexural stress and transverse shear stress are accurate to within about 3% for beams whose length-to-height ratio is greater than about 4.

7.4.7 Beam Deflection

Consider the deflection curve of a beam. The displacement of the neutral axis is denoted by v, positive upwards, as in Fig. 7.4.30. The slope at any point is then given by the first derivative, dv/dx.

For any type of material, provided the displacement is small, it can be shown that the radius of curvature R is related to the second derivative d^2v/dx^2 through (see the Appendix to this section, §7.4.10)

$$\frac{1}{R} = \frac{d^2 v}{dx^2} \tag{7.4.36}$$

and for this reason d^2v/dx^2 is called the **curvature** of the beam. Using Eqn. 7.4.19, $\sigma = -Ey/R$, and the flexural stress expression, Eqn. 7.4.21, $\sigma = -My/I$, one has the **moment-curvature equation**

$$\boxed{M(x) = EI\frac{d^2 v}{dx^2}} \qquad \textbf{moment-curvature equation} \tag{7.4.37}$$

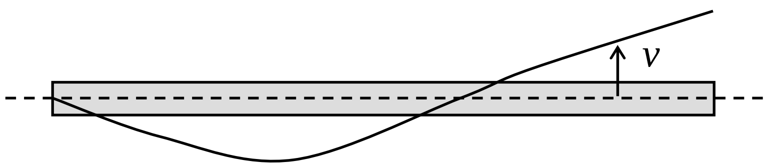


Figure 7.4.30: the deflection of a beam

With the moment known, this differential equation can be integrated twice to obtain the deflection. Boundary conditions must be supplied to obtain constants of integration.

Example

Consider the cantilevered beam of length L shown in Fig. 7.4.31, subjected to an end-force F and end-moment M_0. The moment is found to be $M(x) = F(L-x) + M_0$, with x measured from the clamped end. The moment-curvature equation is then

$$\begin{aligned}
&EI\frac{d^2 v}{dx^2} = (FL + M_0) - Fx \\
\rightarrow \quad &EI\frac{dv}{dx} = (FL + M_0)x - \frac{1}{2}Fx^2 + C_1 \\
\rightarrow \quad &EIv = \frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3 + C_1 x + C_2
\end{aligned} \tag{7.4.38}$$

The boundary conditions are that the displacement and slope are both zero at the clamped end, from which the two constant of integration can be obtained:

$$\begin{aligned} v(0) = 0 \quad &\to \quad C_2 = 0 \\ v'(0) = 0 \quad &\to \quad C_1 = 0 \end{aligned} \tag{7.4.39}$$

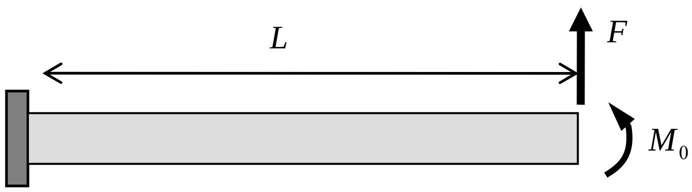


Figure 7.4.31: a cantilevered beam loaded by an end-force and moment

The slope and deflection are therefore

$$v = \frac{1}{EI}\left[\frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3\right], \qquad \frac{dv}{dx} = \frac{1}{EI}\left[(FL + M_0)x - \frac{1}{2}Fx^2\right] \tag{7.4.40}$$

The maximum deflection occurs at the end, where

$$v(L) = \frac{1}{EI}\left[\frac{1}{2}M_0 L^2 + \frac{1}{3}FL^3\right] \tag{7.4.41}$$

■

The term EI in Eqns. 7.4.40-41 is called the **flexural rigidity**, since it is a measure of the resistance of the beam to deflection.

Example

Consider the simply supported beam of length L shown in Fig. 7.4.32, subjected to a uniformly distributed load p over half its length. In this case, the moment is given by

$$M(x) = \begin{cases} \frac{3}{8}pLx - \frac{1}{2}px^2 & 0 < x < \frac{L}{2} \\ \frac{1}{8}pL(L - x) & \frac{L}{2} < x < L \end{cases} \tag{7.4.42}$$

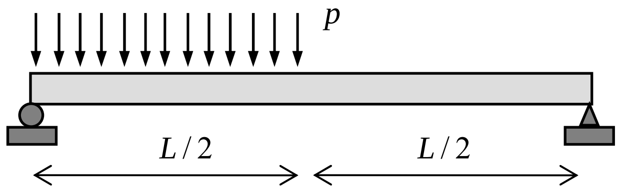


Figure 7.4.32: a simply supported beam subjected to a uniformly distributed load over half its length

It is necessary to apply the moment-curvature equation to each of the two regions $0 < x < L/2$ and $L/2 < x < L$ separately, since the expressions for the moment in these regions differ. Thus there will be four constants of integration:

$$
\begin{aligned}
&EI\frac{d^2v}{dx^2} = \frac{3}{8}pLx - \frac{1}{2}px^2 && EI\frac{d^2v}{dx^2} = \frac{1}{8}pL^2 - \frac{1}{8}pLx \\
&\to EI\frac{dv}{dx} = \frac{3}{16}pLx^2 - \frac{1}{6}px^3 + C_1 && \to EI\frac{dv}{dx} = \frac{1}{8}pL^2x - \frac{1}{16}pLx^2 + D_1 \\
&\to EIv = \frac{3}{48}pLx^3 - \frac{1}{24}px^4 + C_1x + C_2 && \to EIv = \frac{1}{16}pL^2x^2 - \frac{1}{48}pLx^3 + D_1x + D_2
\end{aligned}
\tag{7.4.43}
$$

The boundary conditions are: (i) no deflection at pin support, $v(0) = 0$ and (ii) no deflection at roller support, $v(L) = 0$, from which one finds that $C_2 = 0$ and $D_2 = -pL^4/24 - D_1L$. The other two necessary conditions are the **continuity conditions** where the two solutions meet. These are that (i) the deflection of both solutions agree at $x = L/2$ and (ii) the slope of both solutions agree at $x = L/2$. Using these conditions, one finds that

$$
C_1 = -\frac{9pL^3}{384}, \quad C_2 = -\frac{17pL^3}{384} \tag{7.4.44}
$$

so that

$$
v = \begin{cases} \dfrac{wL^4}{384EI}\left[-9\left(\dfrac{x}{L}\right) + 24\left(\dfrac{x}{L}\right)^3 - 16\left(\dfrac{x}{L}\right)^4\right] & 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{wL^4}{384EI}\left[1 - 17\left(\dfrac{x}{L}\right) + 24\left(\dfrac{x}{L}\right)^2 - 8\left(\dfrac{x}{L}\right)^3\right] & \dfrac{L}{2} < x < L \end{cases} \tag{7.4.45}
$$

The deflection is shown in Fig. 7.4.33. Note that the maximum deflection occurs in $0 < x < L/2$; it can be located by setting $dv/dx = 0$ there and solving.

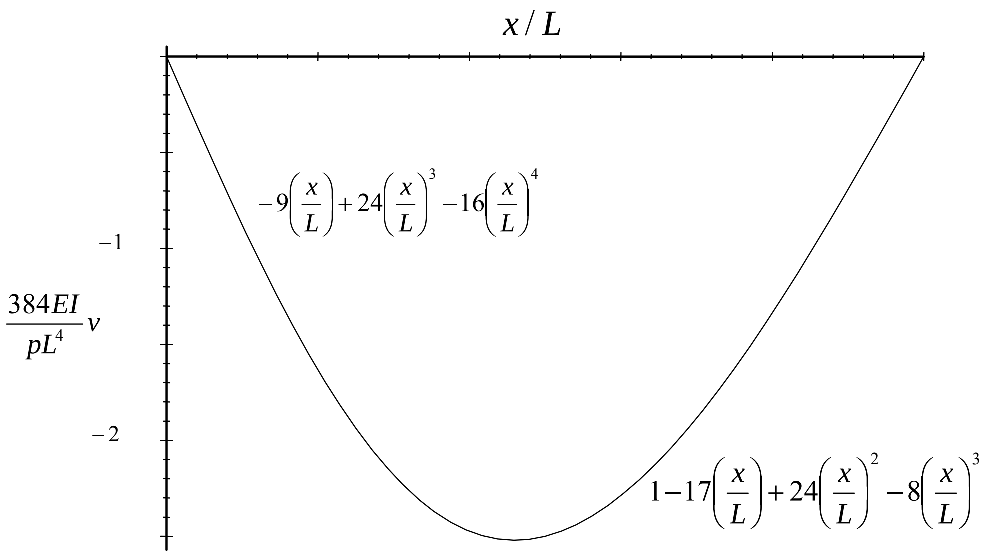


Figure 7.4.33: deflection of a beam

■

7.4.8 Statically Indeterminate Beams

Consider the beam shown in Fig. 7.4.34. It is cantilevered at one end and supported by a roller at its other end. A moment is applied at its centre. There are three unknown reactions in this problem, the reaction force at the roller and the reaction force and moment at the built-in end. There are only two equilibrium equations with which to determine these three unknowns and so it is not possible to solve the problem from equilibrium considerations alone. The beam is therefore statically indeterminate (see the end of section 2.3.3).

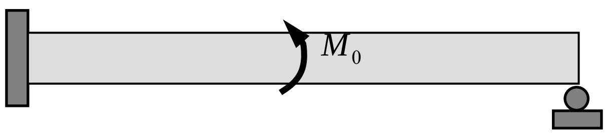


Figure 7.4.34: a cantilevered beam supported also by a roller

More examples of statically indeterminate beam problems are shown in Fig. 7.4.35. To solve such problems, one must consider the deformation of the beam. The following example illustrates how this can be achieved.

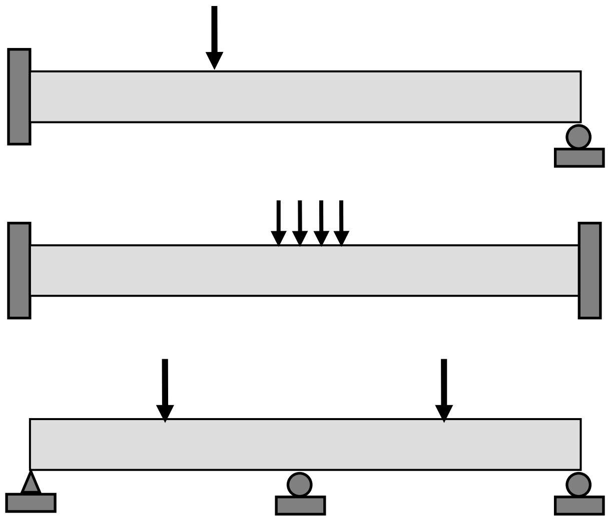


Figure 7.4.35: examples of statically indeterminate beams

Example

Consider the beam of length *L* shown in Fig. 7.4.36, cantilevered at end *A* and supported by a roller at end *B*. A moment M_0 is applied at *B*.

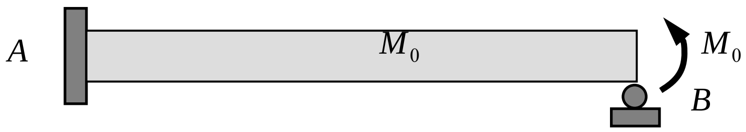


Figure 7.4.36: a statically indeterminate beam

The moment along the beam can be expressed in terms of the *unknown* reaction force at end *B*: $M(x) = R_B(L-x) + M_0$. As before, one can integrate the moment-curvature equation:

$$
\begin{aligned}
& EI\frac{d^2v}{dx^2} = R_B(L-x) + M_0 \\
& \rightarrow EI\frac{dv}{dx} = \left(R_B L + M_0\right)x - \frac{1}{2}R_B x^2 + C_1 \\
& \rightarrow EIv = \frac{1}{2}\left(R_B L + M_0\right)x^2 - \frac{1}{6}R_B x^3 + C_1 x + C_2
\end{aligned}
\tag{7.4.46}
$$

There are three boundary conditions, two to determine the constants of integration and one can be used to determine the unknown reaction R_B. The boundary conditions are (i) $v(0) = 0 \rightarrow C_2 = 0$, (ii) $dv/dx(0) = 0 \rightarrow C_1 = 0$ and (iii) $v(L) = 0$ from which one finds that $R_B = -3M_0/2L$. The slope and deflection are therefore

$$v = \frac{M_B L^2}{4EI}\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right]$$
$$\frac{dv}{dx} = \frac{M_B L}{4EI}\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)\right] \tag{7.4.47}$$

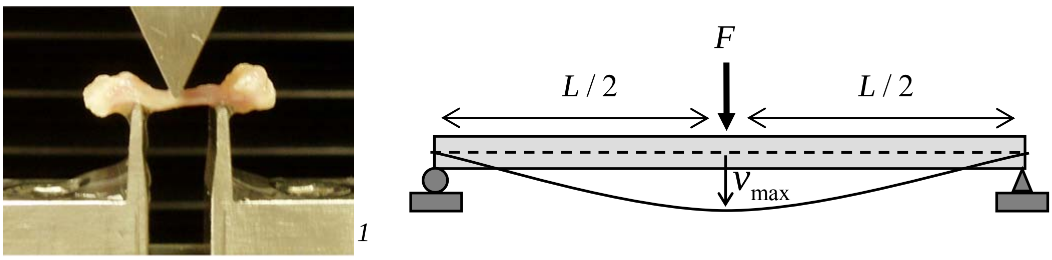
One can now return to the equilibrium equations to find the remaining reactions acting on the beam, which are $R_A = -R_B$ and $M_A = M_0 + LR_B$

■

7.4.9 The Three-point Bending Test

The 3-point bending test is a very useful experimental procedure. It is used to gather data on materials which are subjected to bending in service. It can also be used to get the Young's Modulus of a material for which it might be more difficult to get *via* a tension or other test.

A mouse bone is shown in the standard 3-point bend test apparatus in Fig. 7.4.37a. The idealised beam theory model of this test is shown in Fig. 7.4.37b. The central load is P, so the reactions at the supports are $P/2$. The moment is zero at the supports, varying linearly to a maximum $PL/4$ at the centre.



(a) (b)

Figure 7.4.37: the three-point bend test; (a) a mouse bone specimen, (b) idealised model

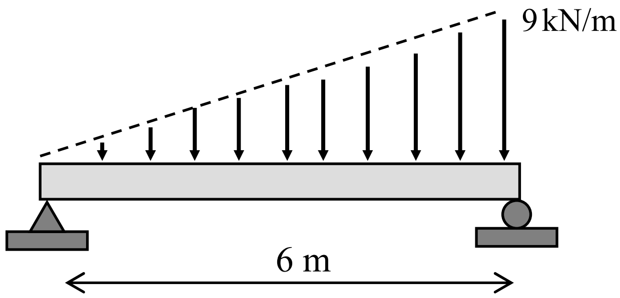
The maximum flexural stress then occurs at the outer fibres at the centre of the beam: for a circular cross-section, $\sigma_{max} = FL/\pi R^3$. Integrating the moment-curvature equation, and using the fact that the deflection is zero at the supports and, from symmetry, the slope is zero at the centre, the maximum deflection is seen to be $v_{max} = FL^3/12\pi R^4 E$. If one plots the load F against the deflection v_{max}, one will see a straight line (initially, before the elastic limit is reached); let the slope of this line be $\hat{E}$. The Young's modulus can then be evaluated through

$$E = \frac{L^3}{12\pi R^4}\hat{E} \tag{7.4.48}$$

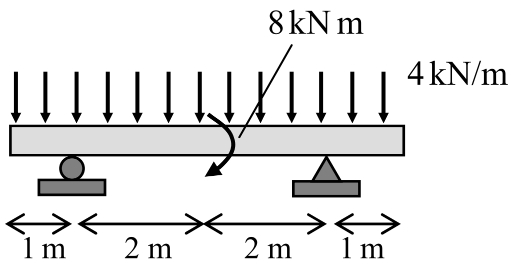
With $\sigma = E\varepsilon$, the maximum strain is $\varepsilon_{max} = FL/\pi ER^3 = 12Rv_{max}/L^2$. By carrying the test on beyond the elastic limit, the strength of the material at failure can be determined.

7.4.10 Problems

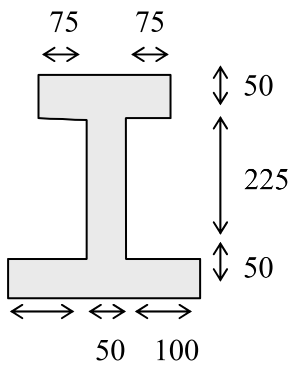
1. The simply supported beam shown below carries a vertical load that increases uniformly from zero at the left end to a maximum value of 9 kN/m at the right end. Draw the shearing force and bending moment diagrams



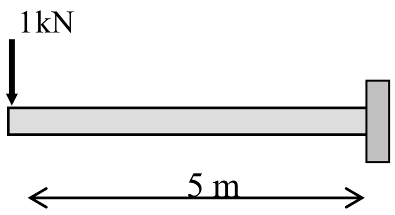
2. The beam shown below is imply supported at two points and overhangs the supports at each end. It is subjected to a uniformly distributed load of 4 kN/m as well as a couple of magnitude 8 kN m applied to the centre. Draw the shearing force and bending moment diagrams



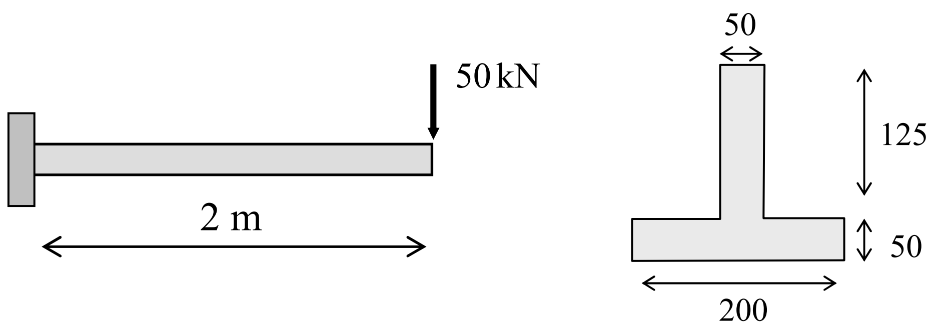
3. Evaluate the centroid of the beam cross-section shown below (all measurements in mm)



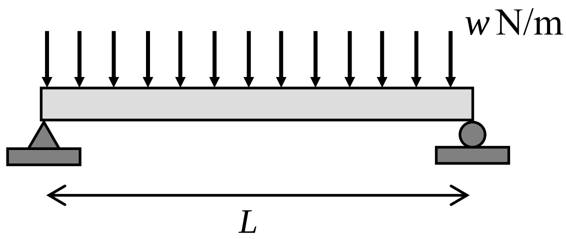
4. Determine the maximum tensile and compressive stresses in the following beam (it has a rectangular cross-section with height 75 mm and depth 50 mm)



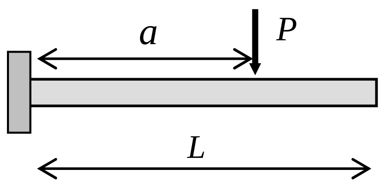
5. Consider the cantilever beam shown below. Determine the maximum shearing stress in the beam and determine the shearing stress 25 mm from the top surface of the beam at a section adjacent to the supporting wall. The cross-section is the "T" shape shown, for which $I = 40 \times 10^6 \text{ mm}^4$.
   [note: use the shear stress formula derived for rectangular cross-sections – as mentioned above, in this formula, *b* is the thickness of the beam *at the point where the shear stress is being evaluated*]



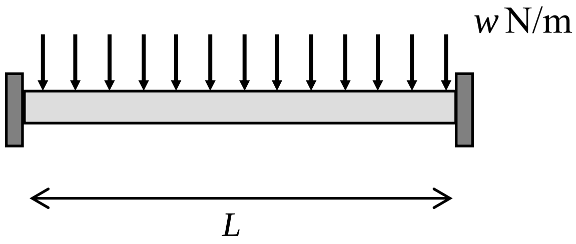
6. Obtain an expression for the maximum deflection of the simply supported beam shown here, subject to a uniformly distributed load of w N/m.



7. Determine the equation of the deflection curve for the cantilever beam loaded by a concentrated force *P* as shown below.



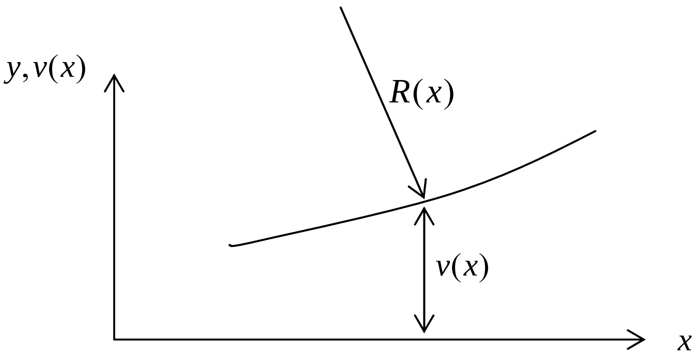
8. Determine the reactions for the following uniformly loaded beam clamped at both ends.



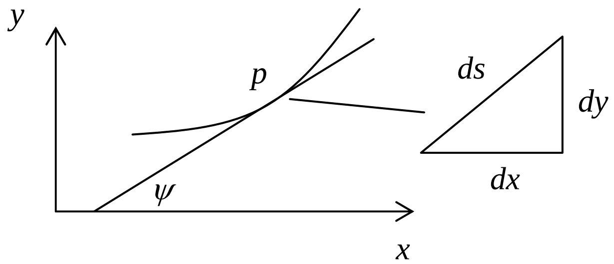
7.4.11 Appendix to §7.4

Curvature of the deflection curve

Consider a deflection curve with deflection $v(x)$ and radius of curvature $R(x)$, as shown in the figure below. Here, *deflection* is the transverse displacement (in the *y* direction) of the points that lie along the axis of the beam. A relationship between $v(x)$ and $R(x)$ is derived in what follows.



First, consider a curve (arc) *s*. The tangent to some point *p* makes an angle ψ with the *x* – axis, as shown below. As one move along the arc, ψ changes.



Define the **curvature** κ of the curve to be the rate at which ψ increases relative to *s*,

$$\kappa = \frac{d\psi}{ds}$$

Thus if the curve is very "curved", ψ is changing rapidly as one moves along the curve (as one increase *s*) and the curvature will be large.

From the above figure,

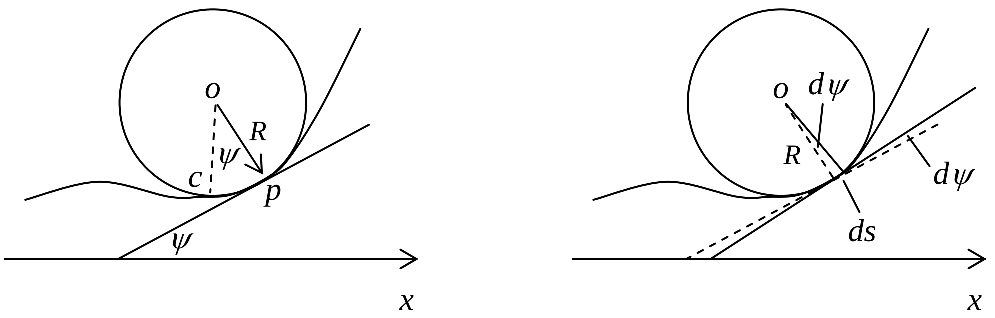
$$\tan\psi = \frac{dy}{dx}, \quad \frac{ds}{dx} = \frac{\sqrt{(dx)^2 + (dy)^2}}{dx} = \sqrt{1 + (dy/dx)^2}\,,$$

so that

$$\kappa = \frac{d\psi}{ds} = \frac{d\psi}{dx}\frac{dx}{ds} = \frac{d\left(\arctan(dy/dx)\right)}{dx}\frac{dx}{ds} = \frac{1}{1 + \left(dy/dx\right)^2}\frac{d^2y}{dx^2}\frac{dx}{ds}$$
$$= \frac{d^2y/dx^2}{\left[1 + \left(dy/dx\right)^2\right]^{3/2}}$$

Finally, it will be shown that the curvature is simply the reciprocal of the radius of curvature. Draw a circle to the point *p* with radius R. Arbitrarily measure the arc length *s* from the point *c*, which is a point on the circle such that $\angle cop = \psi$. Then arc length $s = R\psi$, so that

$$\kappa = \frac{d\psi}{ds} = \frac{1}{R}$$



Thus

$$\frac{1}{R} = \frac{\dfrac{d^2v}{dx^2}}{\left[1 + \left(\dfrac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$$

If one assumes now that the slopes of the deflection curve are small, then $dv/dx << 1$ and

$$\frac{1}{R} \approx \frac{d^2v}{dx^2}$$

Images used:

1. http://www.mc.vanderbilt.edu/root/vumc.php?site=CenterForBoneBiology&doc=20412

Answers to Selected Problems: Chapter 2

2.2

1. $F_{\text{cable}} = 31.4\,\text{kN}$, $F_{\text{rope}} = 56.4\,\text{kN}$
2. $150\,\text{N}$
3 $10g/\sqrt{3} \approx 56.6\text{N}$

2.3

3. $F = 600\text{N}$, $R_{xC} = 1000\text{N}$, $R_{yC} = F$
4. $M = -6.25\text{Nm}$, i.e. clockwise. Anywhere.

Answers to Selected Problems: Chapter 3

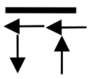
3.1

1. (a) $F = 0.8\text{N}$, (b) $M = 0.021\dot{3}\,\text{Nm}$, (c) $M = 0.005\dot{3}\,\text{Nm}$
2. $F = 0, \quad M = 0.005\dot{3}\,\text{Nm}$
3. $R_A = 2000\text{N/m}, \quad R_B = 1600\text{N/m}$
4. 2kPa

3.2

1. $x_c = \dfrac{1}{3}, \quad y_c = \dfrac{1}{3}$

3.3

2. $\sigma_N = S / l, \quad \sigma_s = 0$
3. at A:  No stress at B.
4. σ_{yz} is negative
5. σ_{yz} (positive)
6. bottom left: σ_{xz} (negative), top: σ_{zx} (negative), bottom right: σ_{yz} (positive)
7. bottom left: σ_{xx} (positive), top: σ_{zz} (positive), bottom right: σ_{yy} (positive)

3.4

2. $\sigma'_{xx} \approx 0.884, \sigma'_{22} \approx 2.116, \sigma'_{12} \approx -0.933$

3.5

2. (a) $\sigma_1 = \sigma_{xx}, \sigma_2 = 0$, (b) $\sigma'_{xx} = \sigma'_{yy} = \sigma_{xx} / 2, \quad \sigma'_{xy} = -\sigma_{xx} / 2$
3. (b) $\theta \approx -32^{\text{o}}$, (c) $\sigma_1 = +3.85, \sigma_2 = -2.85$
4. $\sigma_1 = 2\alpha, \sigma_2 = 0, \sigma_3 = 0$, Max shear is α, the original planes are planes of maximum shearing stress.
5. (b) $\sigma_1 = \tau$, (c) Max shear is $\frac{1}{2}\tau$, acts on planes oriented at 45^{o} to the principal planes acting in the 2-3 plane and the 1-3 plane
8. $\sigma_{xx}^{(2)} = \sigma_{xx}^{(w)}, \sigma_{xy}^{(2)} = \sigma_{xy}^{(w)}$
9. $$-t\int_{-b}^{+b} \sigma_{xy}(x,0)dx = 0, \quad -t\int_{-b}^{+b} \sigma_{yy}(x,0)dx - t\int_{-a}^{+a} p(x)dx = 0$$
$$-t\int_{-b}^{+b} (x+b)\sigma_{yy}(x,0)dx - t\int_{-a}^{+a} (x+b)\,p(x)dx = 0$$

Answers to Selected Problems: Chapter 4

4.1

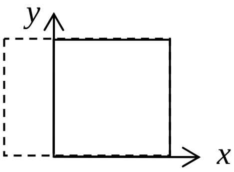
1.
(a) 12cm
(b) 0.6

2.
(a) $\varepsilon = 1.0$, $\varepsilon_t = 0.693$
(b) $\varepsilon = -0.25$, $\varepsilon_t = -0.288$
(c) $\varepsilon = 0.5$, $\varepsilon_t = 0.405$

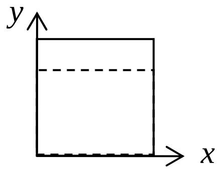
3.
$\varepsilon_{xy} = 0.0015$

4.
Small strains: $\varepsilon_{xx} = 0.00225$, $\varepsilon_{yy} = 0.002$, $\varepsilon_{xy} = -0.0008\dot{3}$
Actual strains: $\varepsilon_{xx} = 0.00225$, $\varepsilon_{yy} = 2.001386 \times 10^{-3}$, $\varepsilon_{xy} = -8.3166823 \times 10^{-4}$
Errors: 0%, 0.069%, 0.200%

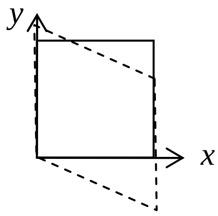
5.
(i)



(ii)



(iii)



6.
$\varepsilon_{xx} = 0$, $\varepsilon_{yy} \approx -3.8053 \times 10^{-3}$, $\varepsilon_{xy} \approx 1.6646 \times 10^{-4}$

4.2

1.

(a)

$\varepsilon_{xx} = 0.02,\ \varepsilon_{yy} = \varepsilon_{xy} = 0$

$\varepsilon'_{xx} = 0.015,\ \ \varepsilon'_{yy} = 0.005,\ \ \varepsilon'_{xy} = -8.66 \times 10^{-3}$

(b) $\varepsilon_1 = 0.02, \quad \varepsilon_2 = 0$

(c) $\max\left(\varepsilon_{xy}\right) = 0.01$

(d) 45 degrees.

2.

(a) $\varepsilon_{xx} = 0.01, \varepsilon_{yy} = -0.01, \varepsilon_{xy} = 0$

(b) $\varepsilon'_{xx} = 0, \varepsilon'_{yy} = 0, \varepsilon_{xy} = -0.01$

(c) the same as (b) using $\theta = 45$

(d) $\sqrt{1.0001} - 1$. Close to (b).

3.

(a) $\varepsilon_{xx} = 0.5, \varepsilon_{yy} = -0.5, \varepsilon_{xy} = 0$

(b)

$\varepsilon'_{xx} = 0, \varepsilon'_{yy} = 0, \varepsilon_{xy} = -0.5$

(c) the same as (b) using $\theta = 45$

(d) $\sqrt{2} - 1 \approx 0.414$. Not the same as (b).

Answers to Selected Problems: Chapter 5

5.2

1. Each component treated separately would be homogeneous and isotropic; the complete structure is not homogeneous; it is not isotropic along the interfaces between the separate components.

5.4

1. 5mm

Answers to Selected Problems: Chapter 6

6.1

Q.2
$\sigma_{xx} = 15.92308$ MPa, $\sigma_{yy} = 11.07692$ MPa, $\sigma_{xy} = 2.42308$ MPa
$\sigma_1 = 16.927, \sigma_2 = 10.073, \sigma_3 = 0$

5.
(a) $\varepsilon_{xx} = \sigma_0 / E,\ \varepsilon_{yy} = \varepsilon_{zz} = -\nu\sigma_0 / E,\ \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$
(b) $\varepsilon_1 = \sigma_0 / E,\ \varepsilon_2 = \varepsilon_3 = -\nu\sigma_0 / E$

6. 3.1×10^{-6} rads

6.2

1.
normal strains: $0, -\sigma_o(1-\nu^2)/E, \sigma_o\nu(1-\nu)/E$
normal stresses: $-\nu\sigma_0, -\sigma_0, 0$, (b) $\sigma_o / 2$

2
normal strains: $0, -\sigma_o(1-2\nu)(1+\nu)/(1-\nu)E, 0$
normal stresses: $-\nu\sigma_0/(1-\nu), -\sigma_0, -\nu\sigma_0/(1-\nu)$]

6.3

2. $\sigma_2 = -0.6969$ MPa, $\sigma_3 = -0.5360$ MPa, $\varepsilon_1 = -8.463\times10^{-5}$

3. $$\sigma_2 = -p\frac{\nu_t(1+\nu_f)}{1-\nu_t\nu_f}, \quad \sigma_3 = -p\frac{\nu_f(1+\nu_t)}{1-\nu_t\nu_f}$$

4.
(a) $\sigma_1 = 48.25, \sigma_2 = 36.75, \sigma_6 = -4.82$
(b) $\varepsilon_1 = 0.0317, \varepsilon_2 = 0.1623, \varepsilon_6 = -3.4486\times10^{-5}$
(c) $G = 69.9\text{kPa}$, $E_2 = 200\text{Pa}$, $\nu_{21} = 0.0904$
(d) principal stresses ($\theta = 0$): 50, 35, 0
principal strains $\theta = 20.02$): $\varepsilon_{p1} = 0.1623, \varepsilon_{p2} = 0.0317$
(e) No.

5. $\theta \approx 48.2^{\circ}$.

Answers to Selected Problems: Chapter 7

7.1

1. $L/\left(1+A_1L_2/A_2L_1\right)$

7.2

1. $\tau_{\max}=20T/\pi d^3,\ \phi_B=-12LT/\pi Gd^4$

7.3

1. 0.25 MPa
2. $\sigma_t/2$
3. (a) 15MPa, 30MPa, (b) 15MPa, (c) 750N
4. 0.13, 0.03, -0.07 ($\times 10^{-3}$), 0.015mm

7.4

1. $V=9-0.75x^2$ kN, $M=9x-0.25x^3$ kN m
2. $M=-2x^2;-2x^2+10x-10;-2x^2+10x-2;-2x^2+24x-72$ kN m
3. $\bar{y}=152.3$ mm
4. 107 MPa
5. 4.78 MPa; -0.43 MPa
6. $\delta_{\max}=-\left(5wL^4\right)/\left(384EI\right)$
7. $EIv=-(Pa^3/6)(x/a)^2\left(3-x/a\right);\ EIv=-(Pa^3/6)\left(3x/a-1\right)$
8. $R=wL/2;\ M=-wL^2/12$

1 Differential Equations for Solid Mechanics

Simple problems involving homogeneous stress states have been considered so far, wherein the stress is the same throughout the component under study. An exception to this was the varying stress field in the loaded beam, but there a simplified set of elasticity equations was used. Here the question of varying stress and strain fields in materials is considered. In order to solve such problems, a differential formulation is required. In this Chapter, a number of differential equations will be derived, relating the stresses and body forces (**equations of motion**), the strains and displacements (**strain-displacement relations**) and the strains with each other (**compatibility relations**). These equations are derived from physical principles and so apply to any type of material, although the latter two are derived under the assumption of small strain.

1.1 The Equations of Motion

In Part I, balance of forces and moments acting on any component was enforced in order to ensure that the component was in equilibrium. Here, allowance is made for stresses which vary continuously throughout a material, and force equilibrium of any portion of material is enforced.

One-Dimensional Equation

Consider a one-dimensional differential element of length Δx and cross sectional area A, Fig. 1.1.1. Let the *average* body force per unit volume acting on the element be b and the *average* acceleration and density of the element be a and ρ. Stresses σ act on the element.

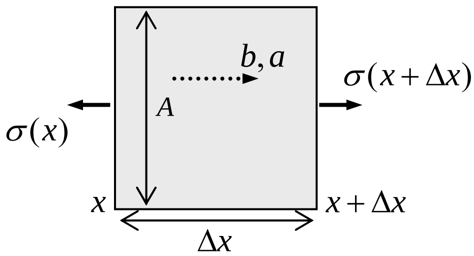


Figure 1.1.1: a differential element under the action of surface and body forces

The net surface force acting is $\sigma(x+\Delta x)A - \sigma(x)A$. If the element is small, then the body force and velocity can be assumed to vary linearly over the element and the average will act at the centre of the element. Then the body force acting on the element is $Ab\Delta x$ and the inertial force is $\rho A\Delta x a$. Applying Newton's second law leads to

$$\begin{aligned} &\sigma(x+\Delta x)A - \sigma(x)A + b\Delta x A = \rho a \Delta x A \\ &\rightarrow \frac{\sigma(x+\Delta x) - \sigma(x)}{\Delta x} + b = \rho a \end{aligned} \tag{1.1.1}$$

so that, by the definition of the derivative, in the limit as $\Delta x \to 0$,

$$\boxed{\frac{d\sigma}{dx} + b = \rho a} \quad \textbf{1-d Equation of Motion} \tag{1.1.2}$$

which is the one-dimensional **equation of motion**. Note that this equation was derived on the basis of a physical law and must therefore be satisfied for all materials, whatever they be composed of.

The derivative $d\sigma / dx$ is the **stress gradient** – physically, it is a measure of how rapidly the stresses are changing.

Example

Consider a bar of length l which hangs from a ceiling, as shown in Fig. 1.1.2.

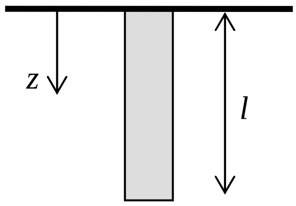


Figure 1.1.2: a hanging bar

The gravitational force is $F = mg$ downward and the body force per unit volume is thus $b = \rho g$. There are no accelerating material particles. Taking the *z* axis positive down, an integration of the equation of motion gives

$$\frac{d\sigma}{dz} + \rho g = 0 \quad \to \quad \sigma = -\rho g z + c \tag{1.1.3}$$

where *c* is an arbitrary constant. The lower end of the bar is free and so the stress there is zero, and so

$$\sigma = \rho g (l - z) \tag{1.1.4}$$

■

Two-Dimensional Equations

Consider now a two dimensional infinitesimal element of width and height Δx and Δy and unit depth (into the page).

Looking at the normal stress components acting in the x−direction, and allowing for variations in stress over the element surfaces, the stresses are as shown in Fig. 1.1.3.

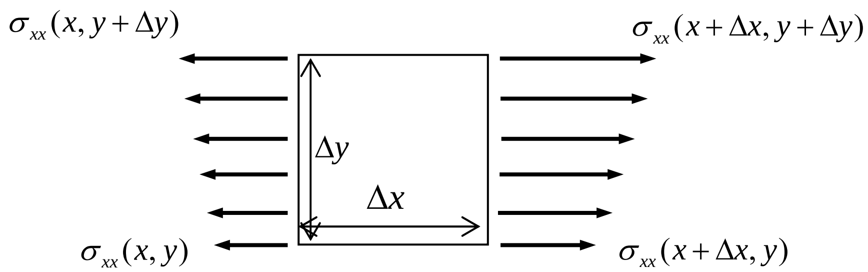


Figure 1.1.3: varying stresses acting on a differential element

Using a (two dimensional) Taylor series and dropping higher order terms then leads to the linearly varying stresses illustrated in Fig. 1.1.4. (where $\sigma_{xx} \equiv \sigma_{xx}(x, y)$ and the partial derivatives are evaluated at (x, y)), which is a reasonable approximation when the element is small.

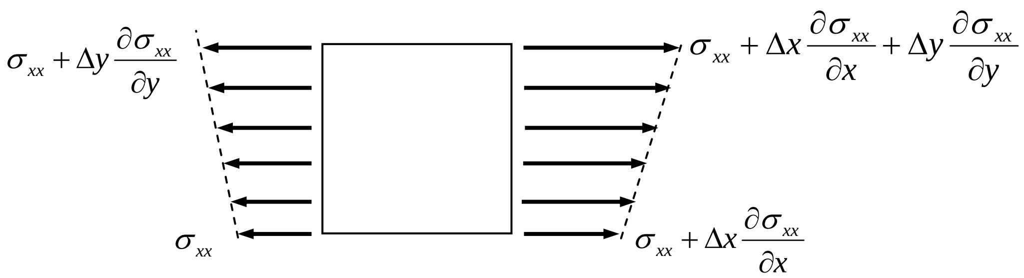


Figure 1.1.4: linearly varying stresses acting on a differential element

The effect (resultant force) of this linear variation of stress on the plane can be replicated by a *constant* stress acting over the whole plane, the size of which is the *average* stress. For the left and right sides, one has, respectively,

$$\sigma_{xx} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y}, \quad \sigma_{xx} + \Delta x \frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y} \tag{1.1.5}$$

One can take away the stress $(1/2)\Delta y \partial \sigma_{xx} / \partial y$ from both sides without affecting the net force acting on the element so one finally has the representation shown in Fig. 1.1.5.

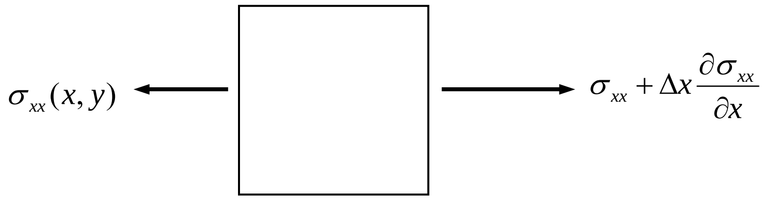


Figure 1.1.5: net stresses acting on a differential element

Carrying out the same procedure for the shear stresses contributing to a force in the x –direction leads to the stresses shown in Fig. 1.1.6.

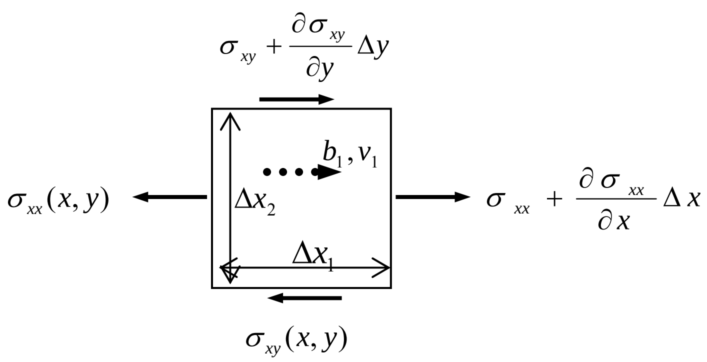


Figure 1.1.6: normal and shear stresses acting on a differential element

Take a_x, b_x to be the average acceleration and body force, and ρ to be the average density. Newton's law then yields

$$-\sigma_{xx}\Delta y+\left(\sigma_{xx}+\Delta x\frac{\partial\sigma_{xx}}{\partial x}\right)\Delta y-\sigma_{xy}\Delta x+\left(\sigma_{xy}+\Delta y\frac{\partial\sigma_{xy}}{\partial y}\right)\Delta y+b_x\Delta x\Delta y=\rho a_x\Delta x\Delta y \tag{1.1.6}$$

which, dividing through by $\Delta x\Delta y$ and taking the limit, gives

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x=\rho a_x \tag{1.1.7}$$

A similar analysis for force components in the $y-$direction yields another equation and one then has the two-dimensional equations of motion:

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x&=\rho a_x\\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+b_y&=\rho a_y\end{aligned}} \qquad \textbf{2-D Equations of Motion} \tag{1.1.8}$$

Three-Dimensional Equations

Similarly, one can consider a three-dimensional element, and one finds that

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x&=\rho a_x\\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y&=\rho a_y\\ \frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z&=\rho a_z\end{aligned}} \qquad \textbf{3-D Equations of Motion} \tag{1.1.9}$$

These three equations express force-balance in, respectively, the x, y, z directions.

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signant par $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ les projections algébriques de la force accélératrice qui serait capable de produire à elle seule le mouvement effectif d'une particule, et prenant x, y, z, t pour variables indépendantes, on obtiendra, à la place des équations (1), celles qui suivent

$$(2)\quad \left\{ \begin{aligned} &\frac{dA}{dx} + \frac{dF}{dy} + \frac{dE}{dz} + \rho X = \rho\mathfrak{X}, \\ &\frac{dF}{dx} + \frac{dB}{dy} + \frac{dD}{dz} + \rho Y = \rho\mathfrak{Y}, \\ &\frac{dE}{dx} + \frac{dD}{dy} + \frac{dC}{dz} + \rho Z = \rho\mathfrak{Z}. \end{aligned} \right.$$

Enfin, si l'on nomme ξ, η, ζ les déplacements de la particule qui, au bout d'un temps t, coïncide avec le point (x, y, z), mesurés parallèlement aux axes coordonnés, on trouvera, en supposant ces déplacements très-petits,

Figure 1.1.7: from Cauchy's Exercices de Mathematiques (1829)

The Equations of Equlibrium

If the material is not moving (or is moving at constant velocity) and is in static equilibrium, then the equations of motion reduce to the **equations of equilibrium**,

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= 0 \end{aligned}} \qquad \textbf{3-D Equations of Equilibrium} \qquad (1.1.10)$$

These equations express the force balance between surface forces and body forces in a material. The equations of equilibrium may also be used as a good approximation in the analysis of materials which have relatively small accelerations.

1.1.2 Problems

1. What does the one-dimensional equation of motion say about the stresses in a bar in the absence of any body force or acceleration?

2. Does equilibrium exist for the following two dimensional stress distribution in the absence of body forces?

$$\begin{aligned} &\sigma_{xx} = 3x^2 + 4xy - 8y^2 \\ &\sigma_{xy} = \sigma_{yx} = x^2/2 - 6xy - 2y^2 \\ &\sigma_{yy} = 2x^2 + xy + 3y^2 \\ &\sigma_{zz} = \sigma_{zx} = \sigma_{xz} = \sigma_{zy} = \sigma_{yz} = 0 \end{aligned}$$

3. The elementary beam theory predicts that the stresses in a circular beam due to bending are

$$\sigma_{xx} = My/I, \quad \sigma_{xy} = \sigma_{yx} = V(R^2 - y^2)/3I \qquad (I = \pi R^4/4)$$

and all the other stress components are zero. Do these equations satisfy the equations of equilibrium?

4. With respect to axes $0xyz$ the stress state is given in terms of the coordinates by the matrix

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} xy & y^2 & 0 \\ y^2 & yz & z^2 \\ 0 & z^2 & xz \end{bmatrix}$$

Determine the body force acting on the material if it is at rest.

5. What is the acceleration of a material particle of density $\rho = 0.3\text{kgm}^{-3}$, subjected to the stress

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 2x^2 - x^4 & 2xy & 2xz \\ 2xy & 2y^2 - y^4 & 2yz \\ 2xz & 2yz & 2z^2 - z^4 \end{bmatrix}$$

and gravity (the z axis is directed vertically upwards from the ground).

6. A fluid at rest is subjected to a hydrostatic pressure p and the force of gravity only.
   (a) Write out the equations of motion for this case.
   (b) A very basic formula of hydrostatics, to be found in any elementary book on fluid mechanics, is that giving the pressure variation in a static fluid,

$$\Delta p = \rho g h$$

   where ρ is the density of the fluid, g is the acceleration due to gravity, and h is the vertical distance between the two points in the fluid (the relative depth). Show that this formula is but a special case of the equations of motion.

1.2 The Strain-Displacement Relations

The strain was introduced in Book I: §4. The concepts examined there are now extended to the case of strains which vary continuously throughout a material.

1.2.1 The Strain-Displacement Relations

Normal Strain

Consider a line element of length Δx emanating from position (x, y) and lying in the x-direction, denoted by AB in Fig. 1.2.1. After deformation the line element occupies $A'B'$, having undergone a translation, extension and rotation.

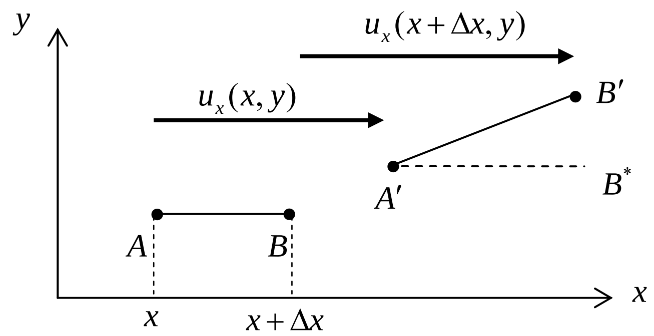


Figure 1.2.1: deformation of a line element

The particle that was originally at x has undergone a displacement $u_x(x, y)$ and the other end of the line element has undergone a displacement $u_x(x+\Delta x, y)$. By the definition of (small) normal strain,

$$\varepsilon_{xx} = \frac{A'B^* - AB}{AB} = \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x} \tag{1.2.1}$$

In the limit $\Delta x \to 0$ one has

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{1.2.2}$$

This partial derivative is a **displacement gradient**, a measure of how rapid the displacement changes through the material, and is the strain *at* (x, y). Physically, it represents the (approximate) unit change in length of a line element, as indicated in Fig. 1.2.2.

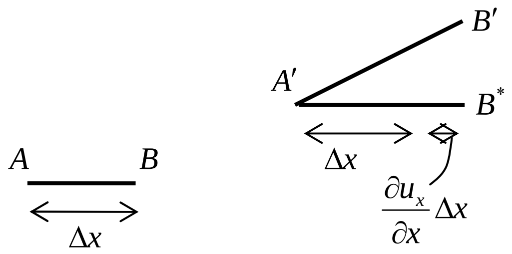


Figure 1.2.2: unit change in length of a line element

Similarly, by considering a line element initially lying in the *y* direction, the strain in the *y* direction can be expressed as

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{1.2.3}$$

Shear Strain

The particles *A* and *B* in Fig. 1.2.1 also undergo displacements in the *y* direction and this is shown in Fig. 1.2.3. In this case, one has

$$B^* B' = \frac{\partial u_y}{\partial x}\Delta x \tag{1.2.4}$$

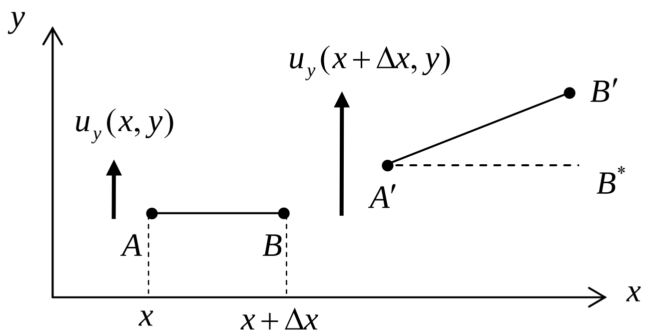


Figure 1.2.3: deformation of a line element

A similar relation can be derived by considering a line element initially lying in the *y* direction. A summary is given in Fig. 1.2.4. From the figure,

$$\theta \approx \tan\theta = \frac{\partial u_y / \partial x}{1 + \partial u_x / \partial x} \approx \frac{\partial u_y}{\partial x}$$

provided that (i) θ is small and (ii) the displacement gradient $\partial u_x / \partial x$ is small. A similar expression for the angle λ can be derived, and hence the shear strain can be written in terms of displacement gradients.

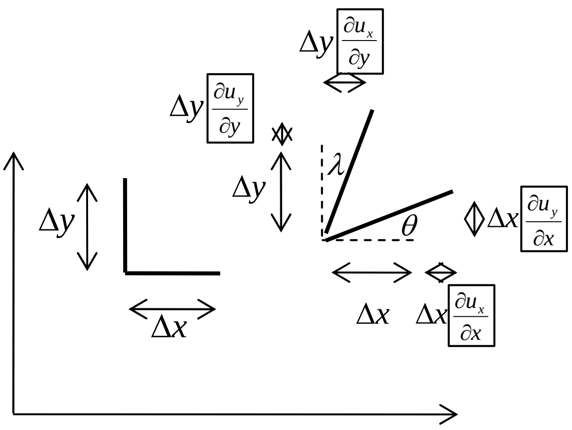


Figure 1.2.4: strains in terms of displacement gradients

The Small-Strain Stress-Strain Relations

In summary, one has

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{aligned}} \quad \textbf{2-D Strain-Displacement relations} \qquad (1.2.5)$$

1.2.2 Geometrical Interpretation of Small Strain

A geometric interpretation of the strain was given in Book I: §4.1.4. This interpretation is repeated here, only now in terms of displacement gradients.

Positive Normal Strain

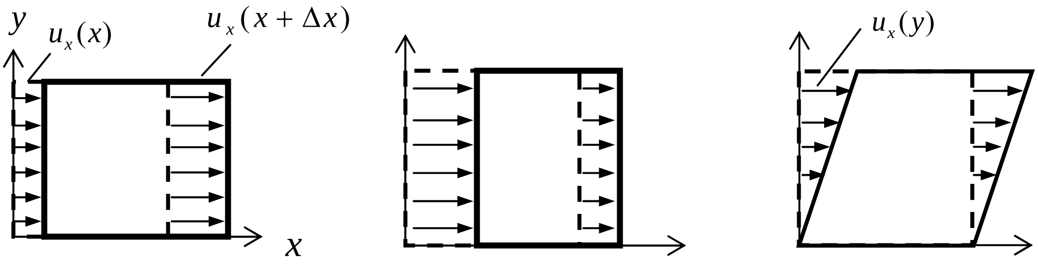
Fig. 1.2.5a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} > 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial y}{\partial x}\right) = 0 \qquad (1.2.6)$$

Negative Normal Strain

Fig 1.2.5b,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} < 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \qquad (1.2.7)$$



(a) (b) (c)

Figure 1.2.5: some simple deformations; (a) positive normal strain, (b) negative normal strain, (c) simple shear

Simple Shear

Fig. 1.2.5c,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = \frac{1}{2}\frac{\partial u_x}{\partial y} \tag{1.2.8}$$

Pure Shear

Fig 1.2.6a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} \tag{1.2.9}$$

1.2.3 The Rotation

Consider an arbitrary deformation (omitting normal strains for ease of description), as shown in Fig. 1.2.6. As usual, the angles θ and λ are small, equal to their tangents, and $\theta = \partial u_y / \partial x$, $\lambda = \partial u_x / \partial y$.

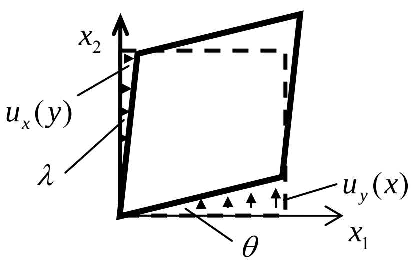
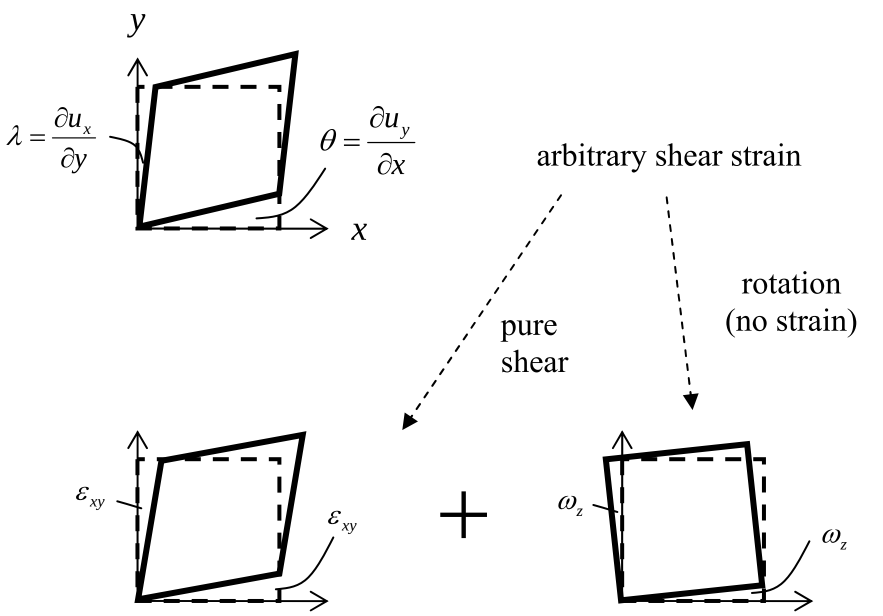


Figure 1.2.6: arbitrary deformation (shear and rotation)

Now this arbitrary deformation can be decomposed into a pure shear and a rigid rotation as depicted in Fig. 1.2.7. In the pure shear, $\theta = \lambda = \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda)$. In the rotation, the angle of rotation is then $\frac{1}{2}(\theta + \lambda)$.



$$\varepsilon_{xy} = \frac{1}{2}(\lambda + \theta) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \omega_z = \frac{1}{2}(\theta - \lambda) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

Figure 1.2.7: decomposition of a strain into a pure shear and a rotation

This leads one to define the **rotation** of a material particle, ω_z, the "z" signifying the axis about which the element is rotating:

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{1.2.10}$$

The rotation will in general vary throughout a material. When the rotation is everywhere zero, the material is said to be **irrotational**.

For a pure rotation, note that

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0, \qquad \frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} \tag{1.2.11}$$

1.2.4 Fixing Displacements

The strains give information about the deformation of material particles but, since they do not encompass translations and rotations, they do not give information about the precise location in space of particles. To determine this, one must specify *three* displacement components (in two-dimensional problems). Mathematically, this is equivalent to saying that one cannot uniquely determine the displacements from the strain-displacement relations 1.2.5.

Example

Consider the strain field $\varepsilon_{xx} = 0.01, \varepsilon_{yy} = \varepsilon_{xy} = 0$. The displacements can be obtained by integrating the strain-displacement relations:

$$\begin{aligned} u_x &= \int \varepsilon_{xx} dx = 0.01x + f(y) \\ u_y &= \int \varepsilon_{yy} dy = g(x) \end{aligned} \tag{1.2.12}$$

where *f* and *g* are unknown functions of *y* and *x* respectively. Substituting the displacement expressions into the shear strain relation gives

$$f'(y) = -g'(x). \tag{1.2.13}$$

Any expression of the form $F(x) = G(y)$ which holds for all *x* and *y* implies that *F* and *G* are constant[1]. Since f', g' are constant, one can integrate to get $f(y) = A + Dy, \; g(x) = B + Cx$. From 1.2.13, $C = -D$, and

$$\begin{aligned} u_x &= 0.01x + A - Cy \\ u_y &= B + Cx \end{aligned} \tag{1.2.14}$$

There are three arbitrary constants of integration, which can be determined by specifying three displacement components. For example, suppose that it is known that

$$u_x(0,0) = 0, u_y(0,0) = 0, u_x(0,a) = b. \tag{1.2.15}$$

In that case, $A = 0, B = 0, C = -b/a$, and, finally,

$$\begin{aligned} u_x &= 0.01x + (b/a)y \\ u_y &= -(b/a)x \end{aligned} \tag{1.2.16}$$

which corresponds to Fig. 1.2.8, with (b/a) being the (tan of the small) angle by which the element has rotated.

[1] since, if this was not so, a change in *x* would change the left hand side of this expression but would not change the right hand side and so the equality cannot hold

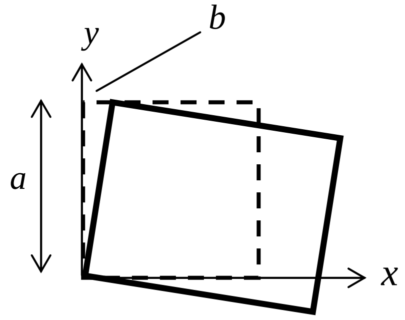


Figure 1.2.8: an element undergoing a normal strain and a rotation

■

In general, the displacement field will be of the form

$$\begin{aligned} u_x &= \quad \cdots\cdots \quad + A - Cy \\ u_y &= \quad \cdots\cdots \quad + B + Cx \end{aligned} \tag{1.2.17}$$

and indeed Eqn. 1.2.16 is of this form. Physically, *A*, *B* and *C* represent the possible rigid body motions *of the material as a whole*, since they are the same for all material particles. *A* corresponds to a translation in the x direction, *B* corresponds to a translation in the x direction, and *C* corresponds to a positive (counterclockwise) rotation.

1.2.5 Three Dimensional Strain

The three-dimensional stress-strain relations analogous to Eqns. 1.2.5 are

$$\boxed{\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\ &\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned}}$$

3-D Stress-Strain relations (1.2.18)

The rotations are

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \tag{1.2.19}$$

1.2.6 Problems

1. The displacement field in a material is given by

$$u_x = A(3x - y), \quad u_y = Axy^2$$

where *A* is a small constant.

(a) Evaluate the strains. What is the rotation ω_z? Sketch the deformation and any rigid body motions of a differential element at the point $(1,1)$
(b) Sketch the deformation and rigid body motions at the point $(0,2)$, by using a pure shear strain superimposed on the rotation.

2. The strains in a material are given by
$$\varepsilon_{xx} = \alpha x, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = \alpha$$
Evaluate the displacements in terms of three arbitrary constants of integration, in the form of Eqn. 1.2.17,
$$\begin{aligned} u_x &= \quad \cdots\cdots \quad + A - Cy \\ u_y &= \quad \cdots\cdots \quad + B + Cx \end{aligned}$$
What is the rotation?

3. The strains in a material are given by
$$\varepsilon_{xx} = Axy, \quad \varepsilon_{yy} = Ay^2, \quad \varepsilon_{xy} = Ax$$
where A is a small constant. Evaluate the displacements in terms of three arbitrary constants of integration. What is the rotation?

4. Show that, in a state of plane strain ($\varepsilon_{zz} = 0$) with zero body force,
$$\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}$$
where e is the volumetric strain (dilatation), the sum of the normal strains: $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ (see Book I, §4.3).

1.3 Compatibility of Strain

As seen in the previous section, the displacements can be determined from the strains through integration, to within a rigid body motion. In the two-dimensional case, there are three strain-displacement relations but only two displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**.

1.3.1 The Compatibility Relations

Differentiating the first of 1.2.5 twice with respect to y, the second twice with respect to x and the third once each with respect to x and y yields

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y} \right)$$

It follows that

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}} \qquad \textbf{2-D Compatibility Equation} \qquad (1.3.1)$$

This compatibility condition is an equation which must be satisfied by the strains at all material particles.

Physical Meaning of the Compatibility Condition

When all material particles in a component deform, translate and rotate, they need to meet up again very much like the pieces of a jigsaw puzzle must fit together. Fig. 1.3.1 illustrates possible deformations and rigid body motions for three line elements in a material. Compatibility ensures that they stay together after the deformation.

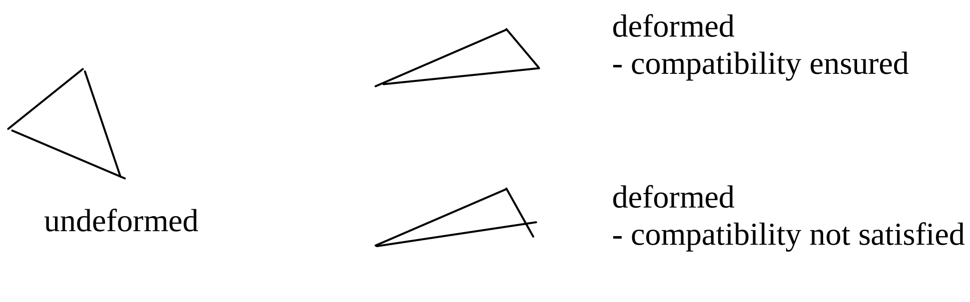


Figure 1.3.1: Deformation and Compatibility

The Three Dimensional Case

There are six compatibility relations to be satisfied in the three dimensional case :

$$
\begin{aligned}
&\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\
&\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\
&\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right)
\end{aligned}
\tag{1.3.2}
$$

By inspection, it will be seen that these are satisfied by Eqns. 1.2.19.

1.3.2 Problems

1. The displacement field in a material is given by

   $$u_x = Axy, \quad u_y = Ay^2,$$

   where A is a small constant. Determine
   (a) the components of small strain
   (b) the rotation
   (c) the principal strains
   (d) whether the compatibility condition is satisfied

7 3D Elasticity

7.1 Vectors, Tensors and the Index Notation

The equations governing three dimensional mechanics problems can be quite lengthy. For this reason, it is essential to use a short-hand notation called the **index notation**[1]. Consider first the notation used for vectors.

7.1.1 Vectors

Vectors are used to describe physical quantities which have both a magnitude and a direction associated with them. Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow. Analytically, in what follows, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **b**.

The **dot product** of two vectors **a** and **b** is denoted by $\mathbf{a}\cdot\mathbf{b}$ and is a scalar defined by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta . \tag{7.1.1}$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$.

Cartesian Coordinate System

So far the short discussion has been in **symbolic notation**[2], that is, no reference to 'axes' or 'components' or 'coordinates' is made, implied or required. Vectors exist independently of any coordinate system. The symbolic notation is very useful, but there are many circumstances in which use of the component forms of vectors is more helpful – or essential. To this end, introduce the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ having the properties

$$\mathbf{e}_1\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_3 = \mathbf{e}_3\cdot\mathbf{e}_1 = 0, \tag{7.1.2}$$

so that they are mutually perpendicular, and

$$\mathbf{e}_1\cdot\mathbf{e}_1 = \mathbf{e}_2\cdot\mathbf{e}_2 = \mathbf{e}_3\cdot\mathbf{e}_3 = 1, \tag{7.1.3}$$

so that they are unit vectors. Such a set of orthogonal unit vectors is called an **orthonormal** set, Fig. 7.1.1. This set of vectors forms a **basis,** by which is meant that any other vector can be written as a **linear combination** of these vectors, i.e. in the form

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{7.1.4}$$

where a_1, a_2 and a_3 are scalars, called the **Cartesian components** or **coordinates** of **a** along the given three directions. The unit vectors are called **base vectors** when used for

[1] or **indicial** or **subscript** or **suffix** notation
[2] or **absolute** or **invariant** or **direct** or **vector** notation

this purpose. The components a_1, a_2 and a_3 are measured along lines called the x_1, x_2 and x_3 **axes**, drawn through the base vectors.

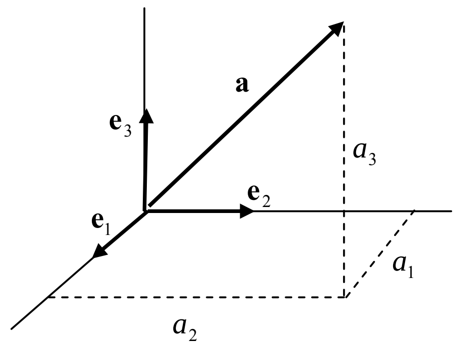


Figure 7.1.1: an orthonormal set of base vectors and Cartesian coordinates

Note further that this orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is **right-handed**, by which is meant $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (or $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ or $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$).

In the index notation, the expression for the vector **a** in terms of the components a_1, a_2, a_3 and the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \tag{7.1.5}$$

This can be simplified further by using Einstein's **summation convention**, whereby the summation sign is dropped and it is understood that for a repeated index (*i* in this case) a summation over the range of the index (3 in this case[3]) is implied. Thus one writes $\mathbf{a} = a_i\mathbf{e}_i$. This can be further shortened to, simply, a_i.

The dot product of two vectors **u** and **v**, referred to this coordinate system, is

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1v_1(\mathbf{e}_1\cdot\mathbf{e}_1) + u_1v_2(\mathbf{e}_1\cdot\mathbf{e}_2) + u_1v_3(\mathbf{e}_1\cdot\mathbf{e}_3) \\
&\quad + u_2v_1(\mathbf{e}_2\cdot\mathbf{e}_1) + u_2v_2(\mathbf{e}_2\cdot\mathbf{e}_2) + u_2v_3(\mathbf{e}_2\cdot\mathbf{e}_3) \\
&\quad\quad + u_3v_1(\mathbf{e}_3\cdot\mathbf{e}_1) + u_3v_2(\mathbf{e}_3\cdot\mathbf{e}_2) + u_3v_3(\mathbf{e}_3\cdot\mathbf{e}_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned} \tag{7.1.6}$$

The dot product of two vectors written in the index notation reads

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_iv_i} \quad \textbf{Dot Product} \tag{7.1.7}$$

[3] 2 in the case of a two-dimensional space/analysis

The repeated index *i* is called a **dummy index**, because it can be replaced with any other letter and the sum is the same; for example, this could equally well be written as $\mathbf{u} \cdot \mathbf{v} = u_j v_j$ or $u_k v_k$.

Introduce next the **Kronecker delta symbol** δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{7.1.8}$$

Note that $\delta_{11} = 1$ but, using the index notation, $\delta_{ii} = 3$. The Kronecker delta allows one to write the expressions defining the orthonormal basis vectors (7.1.2, 7.1.3) in the compact form

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \qquad \textbf{Orthonormal Basis Rule} \tag{7.1.9}$$

Example

Recall the equations of motion, Eqns. 1.1.9, which in full read

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 &= \rho a_1 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 &= \rho a_2 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= \rho a_3 \end{aligned} \tag{7.1.10}$$

The index notation for these equations is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.1.11}$$

Note the dummy index *j*. The index *i* is called a **free index**; if one term has a fee index *i*, then, to be consistent, all terms must have it. One free index, as here, indicates three separate equations.

7.1.2 Matrix Notation

The symbolic notation $\mathbf{v}$ and index notation $v_i \mathbf{e}_i$ (or simply v_i) can be used to denote a vector. Another notation is the **matrix notation**: the vector **v** can be represented by a 3×1 matrix (a **column vector**):

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Matrices will be denoted by square brackets, so a shorthand notation for this matrix/vector would be $[\mathbf{v}]$. The elements of the matrix $[\mathbf{v}]$ can be written in the index notation v_i.

Note the distinction between a vector and a 3×1 matrix: the former is a mathematical object independent of any coordinate system, the latter is a representation of the vector in a particular coordinate system – matrix notation, as with the index notation, relies on a particular coordinate system.

As an example, the dot product can be written in the matrix notation as

$$\underbrace{[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]}_{\text{“short” matrix notation}} = \underbrace{\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\text{“full” matrix notation}}$$

Here, the notation $[\mathbf{u}^{\mathrm{T}}]$ denotes the 1×3 matrix (the **row vector**). The result is a 1×1 matrix, $u_i v_i$.

The matrix notation for the Kronecker delta δ_{ij} is the identity matrix

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, for example, in both index and matrix notation:

$$\delta_{ij} u_j = u_i \qquad [\mathbf{I}][\mathbf{u}] = [\mathbf{u}] \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad (7.1.12)$$

Matrix – Matrix Multiplication

When discussing vector transformation equations further below, it will be necessary to multiply various matrices with each other (of sizes 3×1, 1×3 and 3×3). It will be helpful to write these matrix multiplications in the short-hand notation.

First, it has been seen that the dot product of two vectors can be represented by $[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]$ or $u_i v_i$. Similarly, the matrix multiplication $[\mathbf{u}][\mathbf{v}^{\mathrm{T}}]$ gives a 3×3 matrix with element form $u_i v_j$ or, in full,

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

This operation is called the **tensor product** of two vectors, written in symbolic notation as $\mathbf{u}\otimes\mathbf{v}$ (or simply **uv**).

Next, the matrix multiplication

$$[\mathbf{Q}][\mathbf{u}] \equiv \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

is a 3×1 matrix with elements $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij}u_j$. The elements of $[\mathbf{Q}][\mathbf{u}]$ are the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$, which can be expressed as $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}])_i \equiv u_j Q_{ij}$.

The expression $[\mathbf{u}][\mathbf{Q}]$ is meaningless, but $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ {▲Problem 4} is a 1×3 matrix with elements $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}])_i \equiv u_j Q_{ji}$.

This leads to the following rule:

1. if a vector pre-multiplies a matrix $[\mathbf{Q}]$ $\rightarrow$ the vector is the transpose $[\mathbf{u}^{\mathrm{T}}]$
2. if a matrix $[\mathbf{Q}]$ pre-multiplies the vector $\rightarrow$ the vector is $[\mathbf{u}]$
3. if summed indices are "beside each other", as the *j* in $u_j Q_{ji}$ or $Q_{ij}u_j$
   $\rightarrow$ the matrix is $[\mathbf{Q}]$
4. if summed indices are not beside each other, as the *j* in $u_j Q_{ij}$
   $\rightarrow$ the matrix is the transpose, $[\mathbf{Q}^{\mathrm{T}}]$

Finally, consider the multiplication of 3×3 matrices. Again, this follows the "beside each other" rule for the summed index. For example, $[\mathbf{A}][\mathbf{B}]$ gives the 3×3 matrix {▲Problem 8} $([\mathbf{A}][\mathbf{B}])_{ij} = A_{ik}B_{kj}$, and the multiplication $[\mathbf{A}^{\mathrm{T}}][\mathbf{B}]$ is written as $([\mathbf{A}^{\mathrm{T}}][\mathbf{B}])_{ij} = A_{ki}B_{kj}$. There is also the important identity

$$([\mathbf{A}][\mathbf{B}])^{\mathrm{T}} = [\mathbf{B}^{\mathrm{T}}][\mathbf{A}^{\mathrm{T}}] \tag{7.1.13}$$

Note also the following:

(i) if there is no free index, as in $u_i v_i$, there is one element
(ii) if there is one free index, as in $u_j Q_{ji}$, it is a 3×1 (or 1×3) matrix
(iii) if there are two free indices, as in $A_{ki} B_{kj}$, it is a 3×3 matrix

7.1.3 Vector Transformation Rule

Introduce two Cartesian coordinate systems with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ and common origin *o*, Fig. 7.1.2. The vector $\mathbf{u}$ can then be expressed in two ways:

$$\mathbf{u} = u_i \mathbf{e}_i = u'_i \mathbf{e}'_i \tag{7.1.14}$$

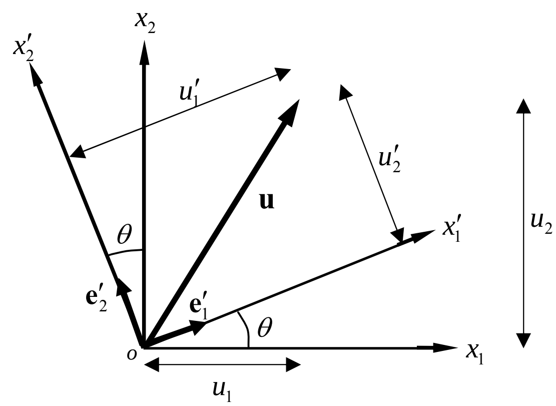


Figure 7.1.2: a vector represented using two different coordinate systems

Note that the x'_i coordinate system is obtained from the x_i system by a *rotation* of the base vectors. Fig. 7.1.2 shows a rotation θ about the x_3 axis (the sign convention for rotations is positive counterclockwise).

Concentrating for the moment on the two dimensions $x_1 - x_2$, from trigonometry (refer to Fig. 7.1.3),

$$\begin{aligned} \mathbf{u} &= u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 \\ &= \left[|OB| - |AB|\right]\mathbf{e}_1 + \left[|BD| + |CP|\right]\mathbf{e}_2 \\ &= \left[\cos\theta\, u'_1 - \sin\theta\, u'_2\right]\mathbf{e}_1 + \left[\sin\theta\, u'_1 + \cos\theta\, u'_2\right]\mathbf{e}_2 \end{aligned} \tag{7.1.15}$$

and so

$$\begin{aligned} u_1 &= \cos\theta\, u'_1 - \sin\theta\, u'_2 \\ u_2 &= \sin\theta\, u'_1 + \cos\theta\, u'_2 \end{aligned} \tag{7.1.16}$$

vector components in first coordinate system

vector components in second coordinate system

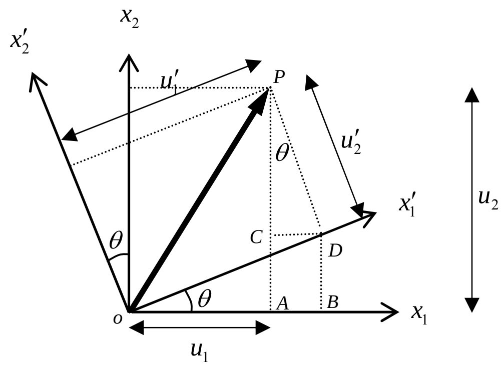


Figure 7.1.3: geometry of the 2D coordinate transformation

In matrix form, these transformation equations can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \tag{7.1.17}$$

The 2×2 matrix is called the **transformation matrix** or **rotation matrix** $[\mathbf{Q}]$. By pre-multiplying both sides of these equations by the inverse of $[\mathbf{Q}]$, $[\mathbf{Q}^{-1}]$, one obtains the transformation equations transforming from $[u_1 \quad u_2]^{\mathrm{T}}$ to $[u_1' \quad u_2']^{\mathrm{T}}$:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.1.18}$$

It can be seen that the components of $[\mathbf{Q}]$ are the **directions cosines**, i.e. the cosines of the angles between the coordinate directions:

$$Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j' \tag{7.1.19}$$

It is straight forward to show that, in the full three dimensions, Fig. 7.1.4, the components in the two coordinate systems are also related through

$$\boxed{\begin{array}{ll} u_i = Q_{ij} u_j' & \ldots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \\ u_i' = Q_{ji} u_j & \ldots \quad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{array}} \quad \textbf{Vector Transformation Rule} \tag{7.1.20}$$

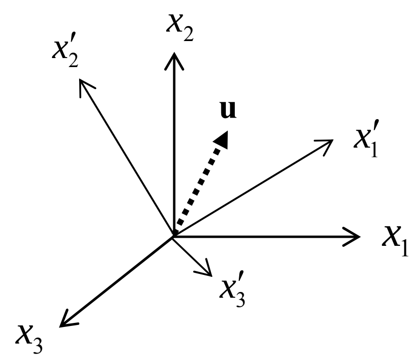


Figure 7.1.4: two different coordinate systems in a 3D space

Orthogonality of the Transformation Matrix $[\mathbf{Q}]$

From 7.1.20, it follows that

$$\begin{aligned} u_i &= Q_{ij} u'_j & \quad \ldots \quad [\mathbf{u}] &= [\mathbf{Q}][\mathbf{u}'] \\ &= Q_{ij} Q_{kj} u_k & \quad \ldots \quad &= [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{aligned} \tag{7.1.21}$$

and so

$$Q_{ij} Q_{kj} = \delta_{ik} \quad \ldots \quad [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}] = [\mathbf{I}] \tag{7.1.22}$$

A matrix such as this for which $[\mathbf{Q}^{\mathrm{T}}] = [\mathbf{Q}^{-1}]$ is called an **orthogonal matrix**.

Example

Consider a Cartesian coordinate system with base vectors $\mathbf{e}_i$. A coordinate transformation is carried out with the new basis given by

$$\begin{aligned} \mathbf{e}'_1 &= a_1^{(1)} \mathbf{e}_1 + a_2^{(1)} \mathbf{e}_2 + a_3^{(1)} \mathbf{e}_3 \\ \mathbf{e}'_2 &= a_1^{(2)} \mathbf{e}_1 + a_2^{(2)} \mathbf{e}_2 + a_3^{(2)} \mathbf{e}_3 \\ \mathbf{e}'_3 &= a_1^{(3)} \mathbf{e}_1 + a_2^{(3)} \mathbf{e}_2 + a_3^{(3)} \mathbf{e}_3 \end{aligned}$$

What is the transformation matrix?

<u>Solution</u>

The transformation matrix consists of the direction cosines $Q_{ij} = \cos(x_i, x'_j) = \mathbf{e}_i \cdot \mathbf{e}'_j$, so

$$[\mathbf{Q}] = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & a_1^{(3)} \\ a_2^{(1)} & a_2^{(2)} & a_2^{(3)} \\ a_3^{(1)} & a_3^{(2)} & a_3^{(3)} \end{bmatrix}$$

■

7.1.4 Tensors

The concept of the **tensor** is discussed in detail in Part III, where it is indispensable for the description of large-strain deformations. For small deformations, it is not so necessary; the main purpose for introducing the tensor here (in a rather non-rigorous way) is that it helps to deepen one's understanding of the concept of stress.

A **second-order tensor**[4] **A** may be *defined* as an operator that acts on a vector **u** generating another vector **v**, so that $\mathbf{T}(\mathbf{u}) = \mathbf{v}$, or

$$\boxed{\mathbf{Tu} = \mathbf{v}} \qquad \textbf{Second-order Tensor} \qquad (7.1.23)$$

The second-order tensor **T** is a **linear operator**, by which is meant

$$\begin{aligned} \mathbf{T}(\mathbf{a}+\mathbf{b}) &= \mathbf{Ta}+\mathbf{Tb} && \ldots \text{ distributive} \\ \mathbf{T}(\alpha\mathbf{a}) &= \alpha(\mathbf{Ta}) && \ldots \text{ associative} \end{aligned}$$

for scalar α. In a Cartesian coordinate system, the tensor **T** has nine components and can be represented in the matrix form

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

The rule 7.1.23, which is expressed in symbolic notation, can be expressed in the index and matrix notation when **T** is referred to particular axes:

$$u_i = T_{ij} v_j \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad (7.1.24)$$

Again, one should be careful to distinguish between a tensor such as **T** and particular matrix representations of that tensor. The relation 7.1.23 is a **tensor relation**, relating vectors and a tensor and is valid in all coordinate systems; the matrix representation of this tensor relation, Eqn. 7.1.24, is to be sure valid in all coordinate systems, but the entries in the matrices of 7.1.24 depend on the coordinate system chosen.

---

[4] to be called simply a tensor in what follows

Note also that the transformation formulae for vectors, Eqn. 7.1.20, is not a tensor relation; although 7.1.20 looks similar to the tensor relation 7.1.24, the former relates the components of a vector to the components of the *same* vector in different coordinate systems, whereas (by definition of a tensor) the relation 7.1.24 relates the components of a vector to those of a different vector in the same coordinate system.

For these reasons, the notation $u_i = Q_{ij}u'$ in Eqn. 7.1.20 is more formally called **element form**, the Q_{ij} being elements of a matrix rather than components of a tensor. This distinction between element form and index notation should be noted, but the term "index notation" is used for both tensor and matrix-specific manipulations in these notes.

Example

Recall the strain-displacement relations, Eqns. 1.2.19, which in full read

$$\begin{gathered}\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \quad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)\end{gathered} \tag{7.1.25}$$

The index notation for these equations is

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.1.26}$$

This expression has two free indices and as such indicates nine separate equations. Further, with its two subscripts, ε_{ij}, the strain, is a tensor. It can be expressed in the matrix notation

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \partial u_1/\partial x_1 & \frac{1}{2}(\partial u_1/\partial x_2 + \partial u_2/\partial x_1) & \frac{1}{2}(\partial u_1/\partial x_3 + \partial u_3/\partial x_1) \\ \frac{1}{2}(\partial u_2/\partial x_1 + \partial u_1/\partial x_2) & \partial u_2/\partial x_2 & \frac{1}{2}(\partial u_2/\partial x_3 + \partial u_3/\partial x_2) \\ \frac{1}{2}(\partial u_3/\partial x_1 + \partial u_1/\partial x_3) & \frac{1}{2}(\partial u_3/\partial x_2 + \partial u_2/\partial x_3) & \partial u_3/\partial x_3 \end{bmatrix}$$

7.1.5 Tensor Transformation Rule

Consider now the tensor definition 7.1.23 expressed in two different coordinate systems:

$$\begin{array}{lll} u_i = T_{ij}v_j & [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] & \text{in} \quad \{x_i\} \\ u'_i = T'_{ij}v'_j & [\mathbf{u}'] = [\mathbf{T}'][\mathbf{v}'] & \text{in} \quad \{x'_i\} \end{array} \tag{7.1.27}$$

From the vector transformation rule 7.1.20,

$$\begin{aligned} u_i' &= Q_{ji} u_j \qquad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \\ v_i' &= Q_{ji} v_j \qquad [\mathbf{v}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \end{aligned} \tag{7.1.28}$$

Combining 7.1.27-28,

$$Q_{ji} u_j = T_{ij}' Q_{kj} v_k \qquad [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] = [\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.29}$$

and so

$$Q_{mi} Q_{ji} u_j = Q_{mi} T_{ij}' Q_{kj} v_k \qquad [\mathbf{u}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.30}$$

(Note that $Q_{mi} Q_{ji} u_j = \delta_{mj} u_j = u_m$.) Comparing with 7.1.24, it follows that

$$\boxed{\begin{aligned} T_{ij} &= Q_{ip} Q_{jq} T_{pq}' \qquad \ldots \quad [\mathbf{T}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}] \\ T_{ij}' &= Q_{pi} Q_{qj} T_{pq} \qquad \ldots \quad [\mathbf{T}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{T}][\mathbf{Q}] \end{aligned}} \quad \textbf{Tensor Transformation Rule} \tag{7.1.31}$$

7.1.6 Problems

1. Write the following in index notation: $|\mathbf{v}|$, $\mathbf{v} \cdot \mathbf{e}_1$, $\mathbf{v} \cdot \mathbf{e}_k$.
2. Show that $\delta_{ij} a_i b_j$ is equivalent to $\mathbf{a} \cdot \mathbf{b}$.
3. Evaluate or simplify the following expressions:
   (a) δ_{kk} (b) $\delta_{ij}\delta_{ij}$ (c) $\delta_{ij}\delta_{jk}$
4. Show that $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ is a 1×3 matrix with elements $u_j Q_{ji}$ (write the matrices out in full)
5. Show that $([\mathbf{Q}][\mathbf{u}])^{\mathrm{T}} = [\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$
6. Are the three elements of $[\mathbf{Q}][\mathbf{u}]$ the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$?
7. What is the index notation for $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$?
8. Write out the 3×3 matrices $[\mathbf{A}]$ and $[\mathbf{B}]$ in full, i.e. in terms of A_{11}, A_{12}, etc. and verify that $[\mathbf{AB}]_{ij} = A_{ik} B_{kj}$ for $i = 2,\ j = 1$.
9. What is the index notation for
   (a) $[\mathbf{A}][\mathbf{B}^{\mathrm{T}}]$
   (b) $[\mathbf{v}^{\mathrm{T}}][\mathbf{A}][\mathbf{v}]$ (there is no ambiguity here, since $([\mathbf{v}^{\mathrm{T}}][\mathbf{A}])[\mathbf{v}] = [\mathbf{v}^{\mathrm{T}}]([\mathbf{A}][\mathbf{v}])$)
   (c) $[\mathbf{B}^{\mathrm{T}}][\mathbf{A}][\mathbf{B}]$
10. The angles between the axes in two coordinate systems are given in the table below.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

Construct the corresponding transformation matrix $[\mathbf{Q}]$ and verify that it is orthogonal.

11. Consider a two-dimensional problem. If the components of a vector **u** in one coordinate system are

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

what are they in a second coordinate system, obtained from the first by a positive rotation of 30°? Sketch the two coordinate systems and the vector to see if your answer makes sense.

12. Consider again a two-dimensional problem with the same change in coordinates as in Problem 11. The components of a 2D tensor in the first system are

$$\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

What are they in the second coordinate system?

7.2 Analysis of Three Dimensional Stress and Strain

The concept of traction and stress was introduced and discussed in Part I, §3.1-3.5. For the most part, the discussion was confined to two-dimensional states of stress. Here, the fully three dimensional stress state is examined. There will be some repetition of the earlier analyses.

7.2.1 The Traction Vector and Stress Components

Consider a traction vector **t** acting on a surface element, Fig. 7.2.1. Introduce a Cartesian coordinate system with base vectors $\mathbf{e}_i$ so that one of the base vectors is a normal to the surface and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 7.1.1, the $\mathbf{e}_3$ direction is taken to be normal to the plane, and a superscript on **t** denotes this normal:

$$\mathbf{t}^{(\mathbf{e}_3)} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3 \tag{7.2.1}$$

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the direction of the normal and the second denotes the direction of the component to the plane. Thus the three components of the traction vector shown in Fig. 7.2.1 are $\sigma_{31}, \sigma_{32}, \sigma_{33}$:

$$\mathbf{t}^{(\mathbf{e}_3)} = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \tag{7.2.2}$$

The first two stresses, the components acting tangential to the surface, are shear stresses whereas σ_{33}, acting normal to the plane, is a normal stress.

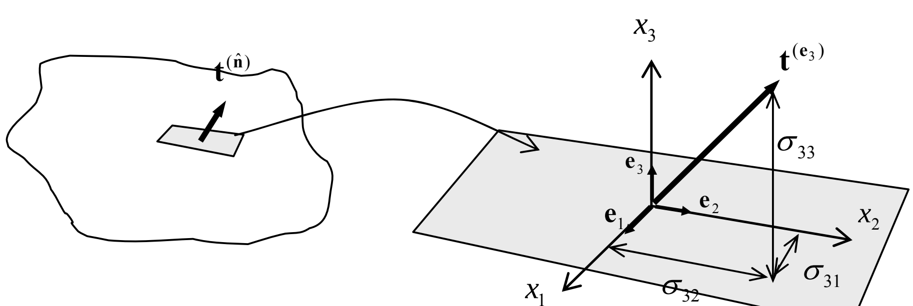


Figure 7.2.1: components of the traction vector

Consider the three traction vectors $\mathbf{t}^{(\mathbf{e}_1)}, \mathbf{t}^{(\mathbf{e}_2)}, \mathbf{t}^{(\mathbf{e}_3)}$ acting on the surface elements whose outward normals are aligned with the three base vectors $\mathbf{e}_j$, Fig. 7.2.2a. The three (or six) surfaces can be amalgamated into one diagram as in Fig. 7.2.2b.

In terms of stresses, the traction vectors are

$$\begin{aligned}\mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3\end{aligned} \quad \text{or} \quad \mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ij}\mathbf{e}_j \tag{7.2.3}$$

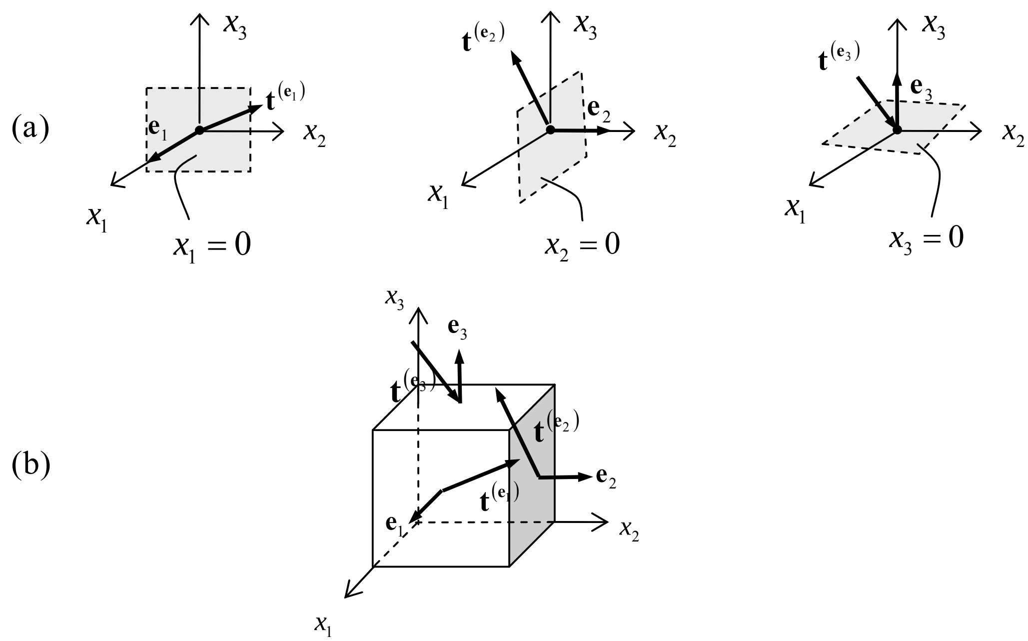


Figure 7.2.2: the three traction vectors acting at a point; (a) on mutually orthogonal planes, (b) the traction vectors illustrated on a box element

The components of the three traction vectors, i.e. the stress components, can now be displayed on a box element as in Fig. 7.2.3. Note that the stress components will vary slightly over the surfaces of an elemental box of finite size. However, it is assumed that the element in Fig. 7.2.3 is small enough that the stresses can be treated as constant, so that they are the stresses acting *at* the origin.

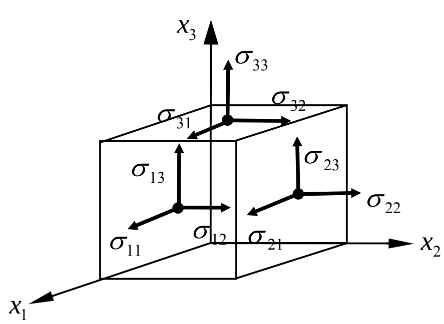


Figure 7.2.3: the nine stress components with respect to a Cartesian coordinate system

The nine stresses can be conveniently displayed in 3×3 matrix form:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{7.2.4}$$

It is important to realise that, if one were to take an element at some different orientation to the element in Fig. 7.2.3, but at the *same material particle*, for example aligned with the axes x_1', x_2', x_3' shown in Fig. 7.2.4, one would then have different tractions acting and the nine stresses would be different also. The stresses acting in this new orientation can be represented by a new matrix:

$$\left[\sigma_{ij}'\right] = \begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} \tag{7.2.5}$$

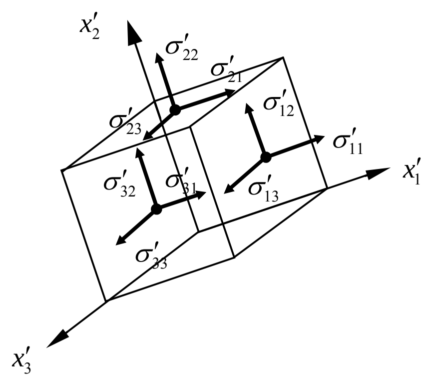


Figure 7.2.4: the stress components with respect to a Cartesian coordinate system different to that in Fig. 7.2.3

7.2.2 Cauchy's Law

Cauchy's Law, which will be proved below, states that the normal to a surface, $\mathbf{n} = n_i \mathbf{e}_i$, is related to the traction vector $\mathbf{t}^{(\mathbf{n})} = t_i \mathbf{e}_i$ acting on that surface, according to

$$t_i = \sigma_{ji} n_j \tag{7.2.6}$$

Writing the traction and normal in vector form and the stress in 3×3 matrix form,

$$\left[t_i^{(\mathbf{n})}\right] = \begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix}, \quad \left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad \left[n_i\right] = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.7}$$

and Cauchy's law in matrix notation reads

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.8}$$

Note that it is the transpose stress matrix which is used in Cauchy's law. Since the stress matrix is symmetric, one can express Cauchy's law in the form

$$\boxed{t_i = \sigma_{ij} n_j} \qquad \textbf{Cauchy's Law} \tag{7.2.9}$$

Cauchy's law is illustrated in Fig. 7.2.5; in this figure, positive stresses σ_{ij} are shown.

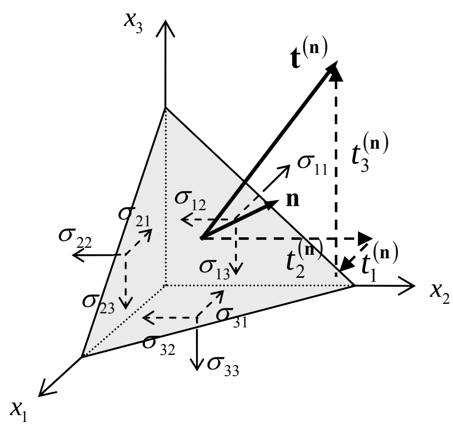


Figure 7.2.5: Cauchy's Law; given the stresses and the normal to a plane, the traction vector acting on the plane can be determined

Normal and Shear Stress

It is useful to be able to evaluate the normal stress σ_N and shear stress σ_S acting on any plane, Fig. 7.2.6. For this purpose, note that the stress acting normal to a plane is the projection of $\mathbf{t}^{(\mathbf{n})}$ in the direction of $\mathbf{n}$,

$$\sigma_N = \mathbf{n} \cdot \mathbf{t}^{(\mathbf{n})} \tag{7.2.10}$$

The magnitude of the shear stress acting on the surface can then be obtained from

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} \tag{7.2.11}$$

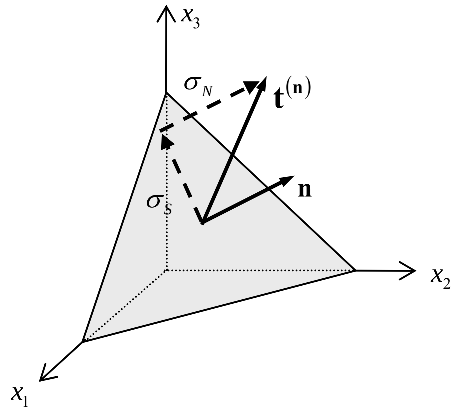


Figure 7.2.6: the normal and shear stress acting on an arbitrary plane through a point

Example

The state of stress at a point with respect to a Cartesian coordinates system $0x_1x_2x_3$ is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine:

(a) the traction vector acting on a plane through the point whose unit normal is $\mathbf{n} = (1/3)\mathbf{e}_1 + (2/3)\mathbf{e}_2 - (2/3)\mathbf{e}_3$

(b) the component of this traction acting perpendicular to the plane

(c) the shear component of traction on the plane

<u>Solution</u>

(a) From Cauchy's law,

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (-2/3)\mathbf{e}_1 + 3\mathbf{e}_2 - \hat{\mathbf{e}}_3$.

(b) The component normal to the plane is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

(c) The shearing component of traction is

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \left\{\left[\left(-\tfrac{2}{3}\right)^2 + (3)^2 + (-1)^2\right] - \left[\left(\tfrac{22}{9}\right)^2\right]\right\}^{1/2} \approx 2.1$$

■

Proof of Cauchy's Law

Cauchy's law can be proved using force equilibrium of material elements. First, consider a tetrahedral free-body, with vertex at the origin, Fig. 7.2.7. It is required to determine the traction **t** in terms of the nine stress components (which are all shown positive in the diagram).

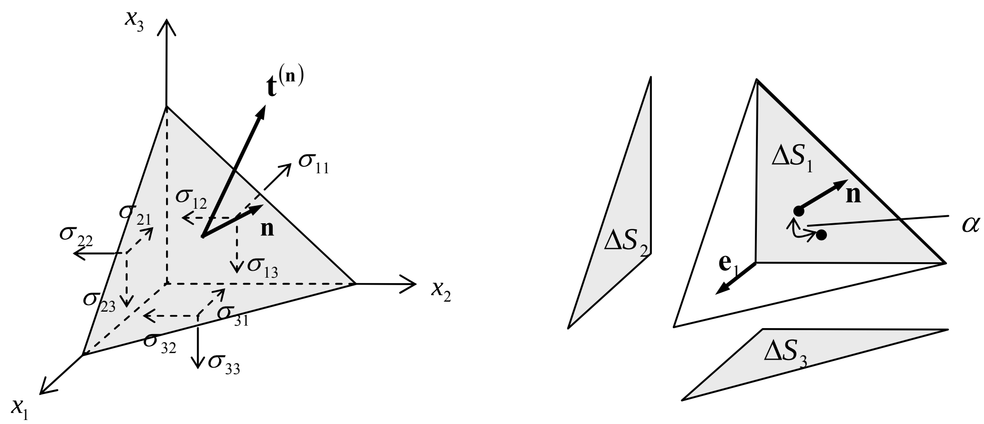


Figure 7.2.7: proof of Cauchy's Law

The components of the unit normal, n_i, are the direction cosines of the normal vector, i.e. the cosines of the angles between the normal and each of the coordinate directions:

$$\cos(\mathbf{n}, \mathbf{e}_i) = \mathbf{n} \cdot \mathbf{e}_i = n_i \tag{7.2.12}$$

Let the area of the base of the tetrahedran, with normal **n**, be ΔS. The area ΔS_1 is then $\Delta S \cos\alpha$, where α is the angle between the planes, as shown to the right of Fig. 7.2.7; this angle is the same as that between the vectors **n** and $\mathbf{e}_1$, so $\Delta S_1 = n_1 \Delta S$, and similarly for the other surfaces:

$$\Delta S_i = n_i \Delta S \tag{7.2.13}$$

The resultant surface force on the body, acting in the x_i direction, is then

$$\sum F_i = t_i \Delta S - \sigma_{ji} \Delta S_j = t_i \Delta S - \sigma_{ji} n_j \Delta S \tag{7.2.14}$$

For equilibrium, this expression must be zero, and one arrives at Cauchy's law.

<u>Note:</u>
As proved in Part III, this result holds also in the general case of accelerating material elements in the presences of body forces.

7.2.3 The Stress Tensor

Cauchy's law 7.2.9 is of the same form as 7.1.24 and so by definition the stress is a tensor. Denote the stress tensor in symbolic notation by $\boldsymbol{\sigma}$. Cauchy's law in symbolic form then reads

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \tag{7.2.15}$$

Further, the transformation rule for stress follows the general tensor transformation rule 7.1.31:

$$\boxed{\begin{array}{ll} \sigma_{ij} = Q_{ip}Q_{jq}\sigma'_{pq} & \ldots \quad [\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}] \\ \sigma'_{ij} = Q_{pi}Q_{qj}\sigma_{pq} & \ldots \quad [\boldsymbol{\sigma}'] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}][\mathbf{Q}] \end{array}} \quad \textbf{Stress Transformation Rule} \tag{7.2.16}$$

As with the normal and traction vectors, the components and hence matrix representation of the stress changes with coordinate system, as with the two different matrix representations 7.2.4 and 7.2.5. However, there is only one stress tensor $\boldsymbol{\sigma}$ at a point. Another way of looking at this is to note that an infinite number of planes pass through a point, and on each of these planes acts a traction vector, and each of these traction vectors has three (stress) components. *All* of these traction vectors taken together define the complete **state of stress** at a point.

Example

The state of stress at a point with respect to an $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) What are the stress components with respect to axes $0x'_1x'_2x'_3$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_2 axis, Fig. 7.2.8?

(b) Use Cauchy's law to evaluate the normal and shear stress on a plane with normal $\mathbf{n} = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_3$ and relate your result with that from (a)

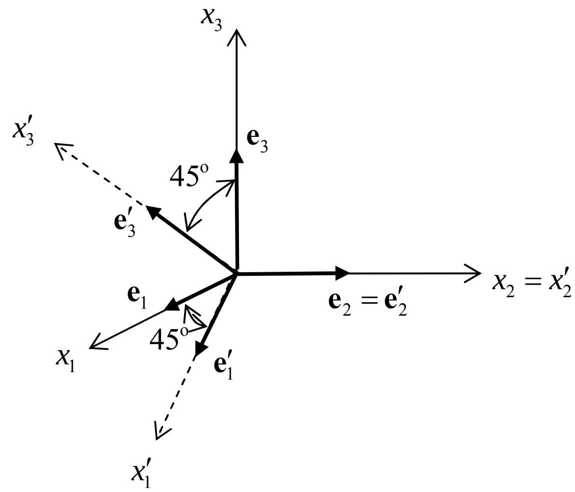


Figure 7.2.8: two different coordinate systems at a point

<u>Solution</u>

(a) The transformation matrix is

$$[Q_{ij}] = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ as expected. The rotated stress components are therefore

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{\sqrt{2}} & \frac{1}{2} \\ \frac{3}{\sqrt{2}} & 3 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{3}{2} \end{bmatrix}$$

and the new stress matrix is symmetric as expected.

(b) From Cauchy's law, the traction vector is

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (\sqrt{2})\mathbf{e}_1 - (1/\sqrt{2})\mathbf{e}_2 + (1/\sqrt{2})\mathbf{e}_3$. The normal and shear stress on the plane are

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = 3/2$$

and

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \sqrt{3 - (3/2)^2} = \sqrt{3}/2$$

The normal to the plane is equal to $\mathbf{e}_3'$ and so σ_N should be the same as σ_{33}' and it is. The stress σ_S should be equal to $\sqrt{(\sigma_{31}')^2 + (\sigma_{32}')^2}$ and it is. The results are displayed in Fig. 7.2.9, in which the traction is represented in different ways, with components $\left(t_1^{(\mathbf{n})}, t_2^{(\mathbf{n})}, t_3^{(\mathbf{n})}\right)$ and $\left(\sigma'_{31}, \sigma'_{32}, \sigma'_{33}\right)$.

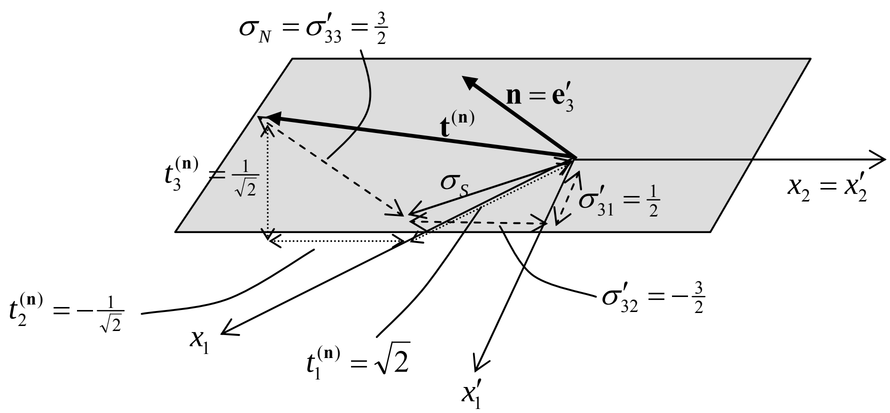


Figure 7.2.9: traction and stresses acting on a plane

Isotropic State of Stress

Suppose the state of stress in a body is

$$\sigma_{ij} = \sigma_0 \delta_{ij} \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \tag{7.2.17}$$

One finds that the application of the stress tensor transformation rule yields the very same components no matter what the new coordinate system{▲Problem 3}. In other words, no shear stresses act, no matter what the orientation of the plane through the point. This is termed an **isotropic state of stress**, or a **spherical state of stress**. One example of isotropic stress is the stress arising in a fluid at rest, which cannot support shear stress, in which case

$$[\boldsymbol{\sigma}] = -p[\mathbf{I}] \tag{7.2.18}$$

where the scalar *p* is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic state of stress**.

7.2.4 Principal Stresses

For certain planes through a material particle, there are traction vectors which act normal to the plane, as in Fig. 7.2.10. In this case the traction can be expressed as a scalar multiple of the normal vector, $\mathbf{t}^{(\mathbf{n})} = \sigma \mathbf{n}$.

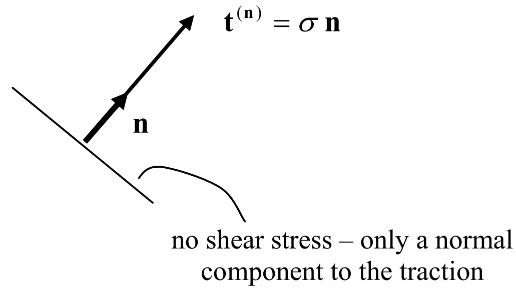


Figure 7.2.10: a purely normal traction vector

From Cauchy's law then, for these planes,

$$\boldsymbol{\sigma}\mathbf{n} = \sigma\,\mathbf{n}, \quad \sigma_{ij}n_j = \sigma n_i, \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad (7.2.19)$$

This is a standard **eigenvalue problem** from Linear Algebra: given a matrix $\lfloor \sigma_{ij} \rfloor$, find the **eigenvalues** σ and associated **eigenvectors n** such that Eqn. 7.2.19 holds.
To solve the problem, first re-write the equation in the form

$$(\boldsymbol{\sigma} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}, \quad (\sigma_{ij} - \sigma\delta_{ij})n_j = 0, \quad \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (7.2.20)$$

or

$$\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7.2.21)$$

This is a set of three homogeneous equations in three unknowns (if one treats σ as known). From basic linear algebra, this system has a solution (apart from $n_i = 0$) if and only if the determinant of the coefficient matrix is zero, i.e. if

$$\det(\boldsymbol{\sigma} - \sigma\mathbf{I}) = \det \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} = 0 \qquad (7.2.22)$$

Evaluating the determinant, one has the following cubic **characteristic equation** of the stress tensor $\boldsymbol{\sigma}$,

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \qquad \textbf{Characteristic Equation} \qquad (7.2.23)$$

and the **principal scalar invariants** of the stress tensor are

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{31}
\end{aligned}
\qquad (7.2.24)
$$

(I_3 is the determinant of the stress matrix.) The characteristic equation 7.2.23 can now be solved for the eigenvalues σ and then Eqn. 7.2.21 can be used to solve for the eigenvectors **n**.

Now another theorem of linear algebra states that the eigenvalues of a real (that is, the components are real), symmetric matrix (such as the stress matrix) are all real and further that the associated eigenvectors are mutually orthogonal. This means that the three roots of the characteristic equation are real and that the three associated eigenvectors form a mutually orthogonal system. This is illustrated in Fig. 7.2.11; the eigenvalues are called **principal stresses** and are labelled $\sigma_1, \sigma_2, \sigma_3$ and the three corresponding eigenvectors are called **principal directions**, the directions in which the principal stresses act. The planes on which the principal stresses act (to which the principal directions are normal) are called the **principal planes**.

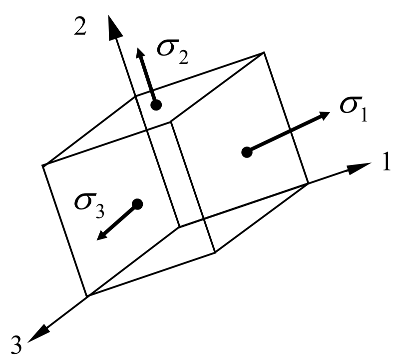


Figure 7.2.11: the three principal stresses acting at a point and the three associated principal directions 1, 2 and 3

Once the principal stresses are found, as mentioned, the principal directions can be found by solving Eqn. 7.2.21, which can be expressed as

$$
\begin{aligned}
(\sigma_{11} - \sigma)n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 &= 0 \\
\sigma_{21}n_1 + (\sigma_{22} - \sigma)n_2 + \sigma_{23}n_3 &= 0 \\
\sigma_{31}n_1 + \sigma_{32}n_2 + (\sigma_{33} - \sigma)n_3 &= 0
\end{aligned}
\qquad (7.2.25)
$$

Each principal stress value in this equation gives rise to the three components of the associated principal direction vector, n_1, n_2, n_3. The solution also requires that the magnitude of the normal be specified: for a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$. The directions of the normals are also chosen so that they form a right-handed set.

Example

The stress at a point is given with respect to the axes $Ox_1x_2x_3$ by the values

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}.$$

Determine (a) the principal values, (b) the principal directions (and sketch them).

<u>Solution:</u>
(a)
The principal values are the solution to the characteristic equation

$$\begin{vmatrix} 5-\sigma & 0 & 0 \\ 0 & -6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{vmatrix} = (-10+\sigma)(5-\sigma)(15+\sigma) = 0$$

which yields the three principal values $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$.
(b)
The eigenvectors are now obtained from Eqn. 7.2.25. First, for $\sigma_1 = 10$,

$$\begin{aligned} -5n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 16n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 9n_3 &= 0 \end{aligned}$$

and using also the equation $n_1^2 + n_2^2 + n_3^2 = 1$ leads to $\mathbf{n}_1 = -(3/5)\mathbf{e}_2 + (4/5)\mathbf{e}_3$. Similarly, for $\sigma_2 = 5$ and $\sigma_3 = -15$, one has, respectively,

$$\begin{aligned} 0n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 11n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 4n_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 20n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 + 9n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 + 16n_3 &= 0 \end{aligned}$$

which yield $\mathbf{n}_2 = \mathbf{e}_1$ and $\mathbf{n}_3 = (4/5)\mathbf{e}_2 + (3/5)\mathbf{e}_3$. The principal directions are sketched in Fig. 7.2.12. Note that the three components of each principal direction, n_1, n_2, n_3, are the direction cosines: the cosines of the angles between that principal direction and the three coordinate axes. For example, for σ_1 with $n_1 = 0, n_2 = -3/5, n_3 = 4/5$, the angles made with the coordinate axes x_1, x_2, x_3 are, respectively, 0, 127° and 37°.

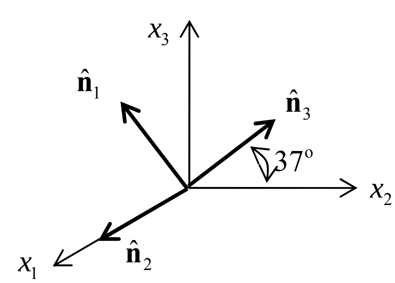


Figure 7.2.12: principal directions

■

Invariants

The principal stresses $\sigma_1, \sigma_2, \sigma_3$ are independent of any coordinate system; the $0x_1x_2x_3$ axes to which the stress matrix in Eqn. 7.2.19 is referred can have any orientation – the same principal stresses will be found from the eigenvalue analysis. This is expressed by using the symbolic notation for the problem: $\boldsymbol{\sigma}\,\mathbf{n} = \sigma\,\mathbf{n}$, which is independent of any coordinate system. Thus the principal stresses are intrinsic properties of the stress state at a point. It follows that the functions I_1, I_2, I_3 in the characteristic equation Eqn. 7.2.23 are also independent of any coordinate system, and hence the name principal scalar invariants (or simply **invariants**) of the stress.

The stress invariants can also be written neatly in terms of the principal stresses:

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{7.2.26}$$

Also, if one chooses a coordinate system to coincide with the principal directions, Fig. 7.2.12, the stress matrix takes the simple form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{7.2.27}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions – the stress matrix has the form 7.2.27 no matter what orientation the planes through the point.

Example

The two stress matrices from the Example of §7.2.3, describing the stress state at a point with respect to different coordinate systems, are

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \qquad [\sigma'_{ij}] = \begin{bmatrix} 3/2 & 3/\sqrt{2} & 1/2 \\ 3/\sqrt{2} & 3 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 3/2 \end{bmatrix}$$

The first invariant is the sum of the normal stresses, the diagonal terms, and is the same for both as expected:

$$I_1 = 2 + 3 + 1 = \tfrac{3}{2} + 3 + \tfrac{3}{2} = 6$$

The other invariants can also be obtained from either matrix, and are

$$I_2 = 6, \;\; I_3 = -3$$

■

7.2.5 Maximum and Minimum Stress Values

Normal Stresses

The three principal stresses include the maximum and minimum normal stress components acting at a point. To prove this, first let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions*. Consider next an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. From Cauchy's law (see Fig. 7.2.13 – the stress matrix in Cauchy's law is now with respect to the principal directions 1, 2 and 3), the normal stress acting on the plane with normal $\mathbf{n}$ is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}, \qquad \sigma_N = \sigma_{ij} n_j n_i \tag{7.2.28}$$

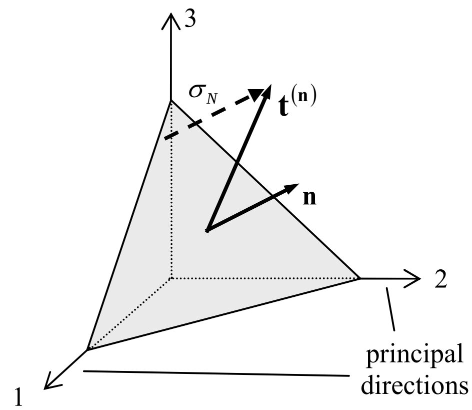


Figure 7.2.13: normal stress acting on a plane defined by the unit normal n

Thus

$$\sigma_N = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.29}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \tag{7.2.30}$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_3 \tag{7.2.31}$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45^{o} to the principal planes and that they have magnitude equal to half the difference between the principal stresses. First, again, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors in the principal directions and consider an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. The normal stress is given by Eqn. 7.2.29,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.32}$$

Cauchy's law gives the components of the traction vector as

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{bmatrix} \tag{7.2.33}$$

and so the shear stress on the plane is, from Eqn. 7.2.11,

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \tag{7.2.34}$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right) n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right) n_2^2 + \sigma_3^2 - \left[\left(\sigma_1 - \sigma_3\right) n_1^2 + \left(\sigma_2 - \sigma_3\right) n_2^2 + \sigma_3\right]^2 \tag{7.2.35}$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned} \frac{\partial\left(\sigma_S^2\right)}{\partial n_1} &= n_1\left(\sigma_1 - \sigma_3\right)\left\{\sigma_1 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right) n_1^2 + \left(\sigma_2 - \sigma_3\right) n_2^2\right]\right\} = 0 \\ \frac{\partial\left(\sigma_S^2\right)}{\partial n_2} &= n_2\left(\sigma_2 - \sigma_3\right)\left\{\sigma_2 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right) n_1^2 + \left(\sigma_2 - \sigma_3\right) n_2^2\right]\right\} = 0 \end{aligned} \tag{7.2.36}$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 7.2.36 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}\left(\sigma_2 - \sigma_3\right)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned}
\mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|
\end{aligned} \tag{7.2.37}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{7.2.38}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 7.2.14.

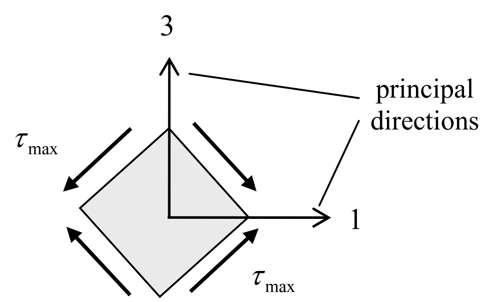


Figure 7.2.14: maximum shear stress at a point

Example

Consider the stress state examined in the Example of §7.2.4:

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

The principal stresses were found to be $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{25}{2}$$

One of the planes upon which they act is shown in Fig. 7.2.15 (see Fig. 7.2.12)

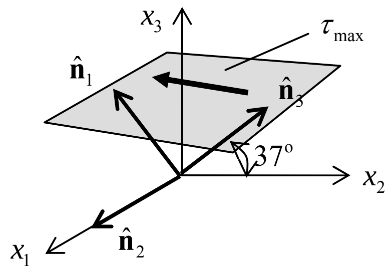


Figure 7.2.15: maximum shear stress

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7.2.6 Mohr's Circles of Stress

The Mohr's circle for 2D stress states was discussed in Part I, §3.5.4. For the 3D case, following on from section 7.2.5, one has the conditions

$$\begin{aligned} \sigma_N &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \\ \sigma_S^2 + \sigma_N^2 &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned} \tag{7.2.39}$$

Solving these equations gives

$$\begin{aligned} n_1^2 &= \frac{(\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\ n_2^2 &= \frac{(\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\ n_3^2 &= \frac{(\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \end{aligned} \tag{7.2.40}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and noting that the squares of the normal components must be positive, one has that

$$\begin{aligned} (\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2 &\geq 0 \\ (\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2 &\leq 0 \\ (\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2 &\geq 0 \end{aligned} \tag{7.2.41}$$

and these can be re-written as

$$\begin{aligned} \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_2 + \sigma_3)\right]^2 &\geq \left[\tfrac{1}{2}(\sigma_2 - \sigma_3)\right]^2 \\ \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right]^2 &\leq \left[\tfrac{1}{2}(\sigma_1 - \sigma_3)\right]^2 \\ \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right]^2 &\geq \left[\tfrac{1}{2}(\sigma_1 - \sigma_2)\right]^2 \end{aligned} \tag{7.2.42}$$

If one takes coordinates (σ_N, σ_S), the equality signs here represent circles in (σ_N, σ_S) stress space, Fig. 7.2.16. Each point (σ_N, σ_S) in this stress space represents the stress on a particular plane through the material particle in question. Admissible (σ_N, σ_S) pairs are given by the conditions Eqns. 7.2.42; they must lie inside a circle of centre $\left(\frac{1}{2}(\sigma_1 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_3)$. This is the large circle in Fig. 7.2.16. The points must lie outside the circle with centre $\left(\frac{1}{2}(\sigma_2 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_2 - \sigma_3)$ and also outside the circle with centre $\left(\frac{1}{2}(\sigma_1 + \sigma_2), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_2)$; these are the two smaller circles in the figure. Thus the admissible points in stress space lie in the shaded region of Fig. 7.2.16.

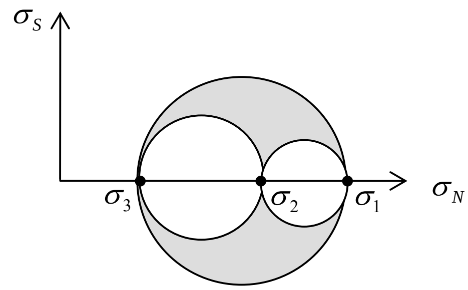


Figure 7.2.16: admissible points in stress space

7.2.7 Three Dimensional Strain

The strain ε_{ij}, in symbolic form $\boldsymbol{\varepsilon}$, is a tensor and as such it follows the same rules as for the stress tensor. In particular, it follows the general tensor transformation rule 7.2.16; it has principal values ε which satisfy the characteristic equation 7.2.23 and these include the maximum and minimum normal strain at a point. There are three principal strain invariants given by 7.2.24 or 7.2.26 and the maximum shear strain occurs on planes oriented at 45° to the principal directions.

7.2.8 Problems

1. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -2 \end{bmatrix}$$

Use Cauchy's law to determine the traction vector acting on a plane trough this point whose unit normal is $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$. What is the normal stress acting on the plane? What is the shear stress acting on the plane?

2. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

What are the stress components with respect to axes $0x_1'x_2'x_3'$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_3 axis

3. Show, in both the index and matrix notation, that the components of an isotropic stress state remain unchanged under a coordinate transformation.

4. Consider a two-dimensional problem. The stress transformation formulae are then, in full,
$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' \\ \sigma_{21}' & \sigma_{22}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
Multiply the right hand side out and use the fact that the stress tensor is symmetric ($\sigma_{12} = \sigma_{21}$ - not true for all tensors). What do you get? Look familiar?

5. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by
$$[\sigma_{ij}] = \begin{bmatrix} 5/2 & -1/2 & 0 \\ -1/2 & 5/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Evaluate the principal stresses and the principal directions. What is the maximum shear stress acting at the point?

7.3 Governing Equations of Three Dimensional Elasticity

7.3.1 Hooke's Law and Lamé's Constants

Linear elasticity was introduced in Part I, §4.2. The three-dimensional Hooke's law for isotropic linear elastic solids (Part I, Eqns. 4.2.9) can be expressed in index notation as

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \tag{7.3.1}$$

where (see also Part I, Eqns. 6.2.21)

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{7.3.2}$$

are the Lamé constants (μ is the Shear Modulus). Eqns. 7.3.1 can be inverted to obtain {▲Problem 1}

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} \tag{7.3.3}$$

7.3.2 Navier's Equations

The governing equations of elasticity are Hooke's law (Eqn. 7.3.1), the equations of motion, Eqn. 1.1.9 (see Eqns. 7.1.10-11),

$$\frac{\partial\sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.3.4}$$

and the strain-displacement relations, Eqn. 1.2.19 (see Eqns. 7.1.25-26),

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.3.5}$$

Substituting 7.3.5 into 7.3.1 and then into 7.3.4 leads to the 3D Navier's equations {▲Problem 2}

$$\boxed{(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i = \rho a_i} \qquad \textbf{Navier's Equations} \tag{7.3.6}$$

These reduce to the 2D plane strain Navier's equations, Eqns. 3.1.4, by setting $u_3 = 0$ and $\partial/\partial x_3 = 0$. They do not reduce to the plane stress equations since the latter are only an approximate solution to the equations of elasticity which are valid only in the limit as the thickness of the thin plate of plane stress tends to zero.

7.3.3 Problems

1. Invert Eqns. 7.3.1 to get 7.3.3.

2. Derive the 3D Navier's equations from 7.3.6 from 7.3.1, 7.3.4 and 7.3.5

Answers to Selected Problems: Part II, Chapter 1

1.1

2. Yes
3. One of the equations of equilibrium is not satisfied.
4. $\mathbf{b} = \begin{bmatrix} -3x_2 \\ -x_3 \\ -x_1 \end{bmatrix}$
5. $\mathbf{a} = \begin{bmatrix} \frac{40}{3} x_1 (2 - x_1^2) \\ \frac{40}{3} x_2 (2 - x_2^2) \\ \frac{40}{3} x_3 (2 - x_3^2) - 9.81 \end{bmatrix}$

1.2

1. $\varepsilon_{xx} = 3A, \quad \varepsilon_{yy} = 2Axy, \quad \varepsilon_{xy} = \frac{1}{2} A\left(-1 + y^2\right), \quad \omega_z = \frac{1}{2} A\left(y^2 + 1\right)$
2. $u_x = \frac{1}{2}\alpha x^2 + A - Cy, \quad u_y = 2\alpha x + B + Cx, \quad \omega_z = \alpha + C$
3. $u_x = \frac{1}{2} Ax^2 y + C_3 - C_4 y, \quad u_y = \frac{1}{3} Ay^3 + Ax^2 - \frac{1}{6} Ax^3 + C_2 + C_4 x,$
   $\omega_z = Ax - \frac{1}{8} Ax^2 + C_4$

1.3

1. $\varepsilon_{xx} = Ay, \quad \varepsilon_{yy} = 2Ay, \quad \varepsilon_{xy} = \frac{1}{2} Ax, \quad \omega_z = \frac{1}{2} Ax, \quad \frac{A}{2}\left[3y \pm \sqrt{x^2 + y^2}\right]$

Answers to Selected Problems: Part II, Chapter 7

7.1

1. $\sqrt{v_i v_i}$, v_1 , v_k
3. 3, 3, δ_{ik}
6. No
7. $a_i b_i c_j$
9. $A_{ij} B_{kj}$, $v_i A_{ij} v_j$, $B_{ji} A_{jk} B_{kl}$
10. $\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{bmatrix}$
11. $\begin{bmatrix} \sqrt{3}+3/2 \\ -1+3\sqrt{3}/2 \end{bmatrix}$
12. $\dfrac{1}{4}\begin{bmatrix} 5+2\sqrt{3} & \sqrt{3}-6 \\ \sqrt{3}+10 & 7-2\sqrt{3} \end{bmatrix}$

7.2

1. $\sqrt{v_i v_i}$, v_1 , v_k
3.

Section 7.2

1. $\sigma_N = 4/3$, $\sigma_S \approx 2.62$
2. $\begin{bmatrix} 4 & 0 & \sqrt{2} \\ 0 & -2 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & -2 \end{bmatrix}$
4. The 2D stress transformation equations
5. $\sigma_i = 1,2,3$

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}\mathbf{e}_1 - \frac{1}{\sqrt{2}}\mathbf{e}_2,\ \mathbf{n}_2 = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2,\ \mathbf{n}_3 = \mathbf{e}_3$$

$$\tau_{\max} = 1$$

1 Vectors & Tensors

The mathematical modeling of the physical world requires knowledge of quite a few different mathematics subjects, such as Calculus, Differential Equations and Linear Algebra. These topics are usually encountered in fundamental mathematics courses. However, in a more thorough and in-depth treatment of mechanics, it is essential to describe the physical world using the concept of the **tensor**, and so we begin this book with a comprehensive chapter on the tensor.

The chapter is divided into three parts. The first part covers vectors (§1.1-1.7). The second part is concerned with second, and higher-order, tensors (§1.8-1.15). The second part covers much of the same ground as done in the first part, mainly generalizing the vector concepts and expressions to tensors. The final part (§1.16-1.19) (not required in the vast majority of applications) is concerned with generalizing the earlier work to curvilinear coordinate systems.

The first part comprises basic vector algebra, such as the dot product and the cross product; the mathematics of how the components of a vector transform between different coordinate systems; the symbolic, index and matrix notations for vectors; the differentiation of vectors, including the gradient, the divergence and the curl; the integration of vectors, including line, double, surface and volume integrals, and the integral theorems.

The second part comprises the definition of the tensor (and a re-definition of the vector); dyads and dyadics; the manipulation of tensors; properties of tensors, such as the trace, transpose, norm, determinant and principal values; special tensors, such as the spherical, identity and orthogonal tensors; the transformation of tensor components between different coordinate systems; the calculus of tensors, including the gradient of vectors and higher order tensors and the divergence of higher order tensors and special fourth order tensors.

In the first two parts, attention is restricted to rectangular Cartesian coordinates (except for brief forays into cylindrical and spherical coordinates). In the third part, curvilinear coordinates are introduced, including covariant and contravariant vectors and tensors, the metric coefficients, the physical components of vectors and tensors, the metric, coordinate transformation rules, tensor calculus, including the Christoffel symbols and covariant differentiation, and curvilinear coordinates for curved surfaces.

1.1 Vector Algebra

1.1.1 Scalars

A physical quantity which is completely described by a single real number is called a **scalar**. Physically, it is something which has a magnitude, and is completely described by this magnitude. Examples are **temperature, density** and **mass**. In the following, lowercase (usually Greek) letters, e.g. α, β, γ, will be used to represent scalars.

1.1.2 Vectors

The concept of the **vector** is used to describe physical quantities which have both a magnitude and a direction associated with them. Examples are **force**, **velocity**, **displacement** and **acceleration**.

Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow, Fig. 1.1.1a.

Analytically, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **r**, **q**.

The **magnitude** (or **length**) of a vector is denoted by $|\mathbf{a}|$ or a. It is a scalar and must be non-negative. Any vector whose length is 1 is called a **unit vector**; unit vectors will usually be denoted by **e**.

(a) (b)

Figure 1.1.1: (a) a vector; (b) addition of vectors

1.1.3 Vector Algebra

The operations of addition, subtraction and multiplication familiar in the algebra of numbers (or scalars) can be extended to an algebra of vectors.

The following definitions and properties fundamentally *define* the vector:

1. Sum of Vectors:
   The addition of vectors **a** and **b** is a vector **c** formed by placing the initial point of **b** on the terminal point of **a** and then joining the initial point of **a** to the terminal point of **b**. The sum is written $\mathbf{c} = \mathbf{a} + \mathbf{b}$. This definition is called the parallelogram law for vector addition because, in a geometrical interpretation of vector addition, **c** is the diagonal of a parallelogram formed by the two vectors **a** and **b**, Fig. 1.1.1b. The following properties hold for vector addition:
   $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ … commutative law
   $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ … associative law

2. The Negative Vector:
   For each vector **a** there exists a **negative vector**. This vector has direction opposite to that of vector **a** but has the same magnitude; it is denoted by $-\mathbf{a}$. A geometrical interpretation of the negative vector is shown in Fig. 1.1.2a.

3. Subtraction of Vectors and the Zero Vector:
   The **subtraction** of two vectors **a** and **b** is defined by $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$, Fig. 1.1.2b. If $\mathbf{a} = \mathbf{b}$ then $\mathbf{a} - \mathbf{b}$ is defined as the **zero vector** (or **null vector**) and is represented by the symbol **o**. It has zero magnitude and unspecified direction. A **proper vector** is any vector other than the null vector. Thus the following properties hold:
   $$\mathbf{a} + \mathbf{o} = \mathbf{a}$$
   $$\mathbf{a} + (-\mathbf{a}) = \mathbf{o}$$

4. Scalar Multiplication:
   The product of a vector **a** by a scalar α is a vector $\alpha\mathbf{a}$ with magnitude $|\alpha|$ times the magnitude of **a** and with direction the same as or opposite to that of **a**, according as α is positive or negative. If $\alpha = 0$, $\alpha\mathbf{a}$ is the null vector. The following properties hold for scalar multiplication:
   $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$ … distributive law, over addition of scalars
   $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$ … distributive law, over addition of vectors
   $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$ … associative law for scalar multiplication

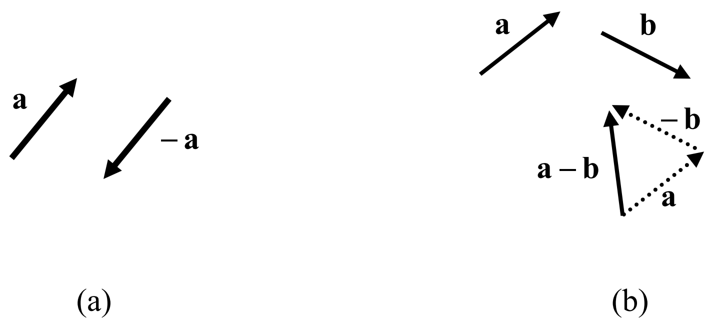


Figure 1.1.2: (a) negative of a vector; (b) subtraction of vectors

Note that when two vectors **a** and **b** are equal, they have the same direction and magnitude, regardless of the position of their initial points. Thus $\mathbf{a} = \mathbf{b}$ in Fig. 1.1.3. A particular position in space is not assigned here to a vector – it just has a magnitude and a direction. Such vectors are called **free**, to distinguish them from certain special vectors to which a particular position in space is actually assigned.

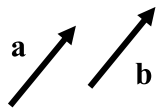


Figure 1.1.3: equal vectors

The vector as something with "magnitude and direction" and defined by the above rules is an element of one case of the mathematical structure, the **vector space**. The vector space is discussed in the next section, §1.2.

1.1.4 The Dot Product

The **dot product** of two vectors **a** and **b** (also called the **scalar product**) is denoted by $\mathbf{a} \cdot \mathbf{b}$. It is a scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta . \tag{1.1.1}$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$, Fig. 1.1.4.

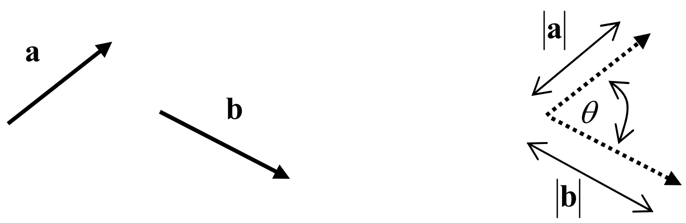


Figure 1.1.4: the dot product

An important property of the dot product is that if for two (proper) vectors **a** and **b**, the relation $\mathbf{a} \cdot \mathbf{b} = 0$, then **a** and **b** are perpendicular. The two vectors are said to be **orthogonal**. Also, $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos(0)$, so that the length of a vector is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

Another important property is that the **projection** of a vector **u** along the direction of a unit vector **e** is given by $\mathbf{u} \cdot \mathbf{e}$. This can be interpreted geometrically as in Fig. 1.1.5.

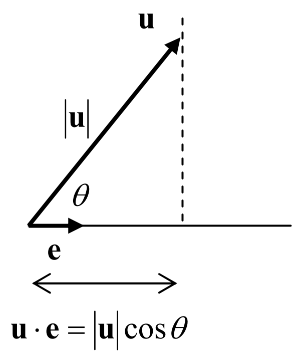


Figure 1.1.5: the projection of a vector along the direction of a unit vector

It follows that any vector **u** can be decomposed into a component parallel to a (unit) vector **e** and another component perpendicular to **e**, according to

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{e})\mathbf{e} + [\mathbf{u} - (\mathbf{u}\cdot\mathbf{e})\mathbf{e}] \qquad (1.1.2)$$

The dot product possesses the following properties (which can be proved using the above definition) {▲Problem 6}:

(1) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (commutative)
(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(\alpha\mathbf{b})$
(4) $\mathbf{a}\cdot\mathbf{a} \geq 0$; and $\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{o}$

1.1.5 The Cross Product

The **cross product** of two vectors **a** and **b** (also called the **vector product**) is denoted by $\mathbf{a}\times\mathbf{b}$. It is a vector with magnitude

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta \qquad (1.1.3)$$

with θ defined as for the dot product. It can be seen from the figure that the magnitude of $\mathbf{a}\times\mathbf{b}$ is equivalent to the area of the parallelogram determined by the two vectors **a** and **b**.

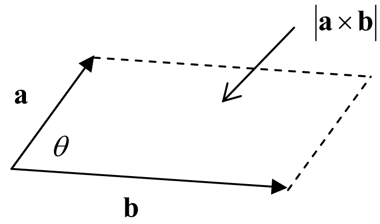


Figure 1.1.6: the magnitude of the cross product

The direction of this new vector is perpendicular to both **a** and **b**. Whether $\mathbf{a}\times\mathbf{b}$ points "up" or "down" is determined from the fact that the three vectors **a**, **b** and $\mathbf{a}\times\mathbf{b}$ form a **right handed system**. This means that if the thumb of the right hand is pointed in the direction of $\mathbf{a}\times\mathbf{b}$, and the open hand is directed in the direction of **a**, then the curling of the fingers of the right hand so that it closes should move the fingers through the angle θ, $0\le\theta\le\pi$, bringing them to **b**. Some examples are shown in Fig. 1.1.7.

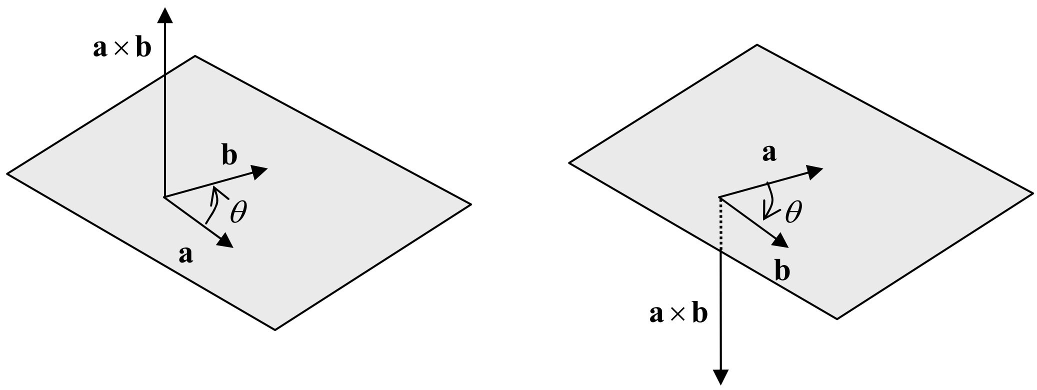


Figure 1.1.7: examples of the cross product

The cross product possesses the following properties (which can be proved using the above definition):

(1) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ (*not* commutative)
(2) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(\alpha\mathbf{b})$
(4) $\mathbf{a}\times\mathbf{b}=\mathbf{o}$ if and only if **a** and **b** $(\neq\mathbf{o})$ are parallel ("linearly dependent")

The Triple Scalar Product

The **triple scalar product**, or **box product**, of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is defined by

$$\boxed{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}} \quad \textbf{Triple Scalar Product} \qquad (1.1.4)$$

Its importance lies in the fact that, if the three vectors form a right-handed triad, then the volume V of a parallelepiped spanned by the three vectors is equal to the box product.

To see this, let **e** be a unit vector in the direction of $\mathbf{u}\times\mathbf{v}$, Fig. 1.1.8. Then the projection of **w** on $\mathbf{u}\times\mathbf{v}$ is $h=\mathbf{w}\cdot\mathbf{e}$, and

$$\begin{aligned}\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})&=\mathbf{w}\cdot(|\mathbf{u}\times\mathbf{v}|\mathbf{e})\\&=|\mathbf{u}\times\mathbf{v}|h\\&=V\end{aligned} \qquad (1.1.5)$$

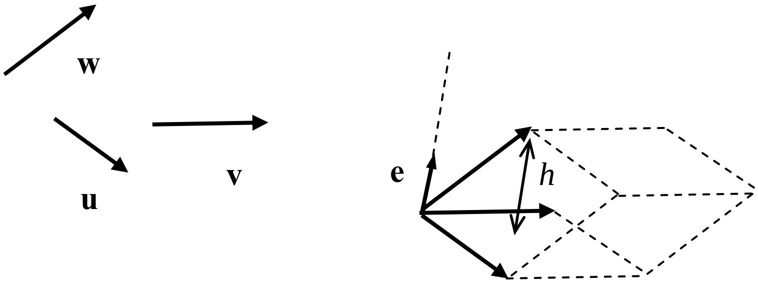


Figure 1.1.8: the triple scalar product

Note: if the three vectors do not form a right handed triad, then the triple scalar product yields the negative of the volume. For example, using the vectors above, $(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u} = -V$.

1.1.6 Vectors and Points

Vectors are objects which have magnitude and direction, but they do not have any specific location in space. On the other hand, a **point** has a certain position in space, and the only characteristic that distinguishes one point from another is its position. Points cannot be "added" together like vectors. On the other hand, a vector **v** can be added to a point **p** to give a new point **q**, $\mathbf{q} = \mathbf{v} + \mathbf{p}$, Fig. 1.1.9. Similarly, the "difference" between two points gives a vector, $\mathbf{q} - \mathbf{p} = \mathbf{v}$. Note that the notion of point as defined here is slightly different to the familiar point in space with axes and origin – the concept of origin is not necessary for these points and their simple operations with vectors.

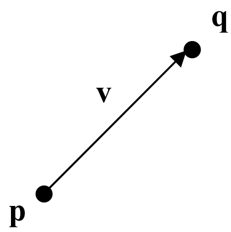
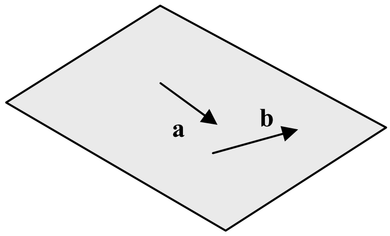


Figure 1.1.9: adding vectors to points

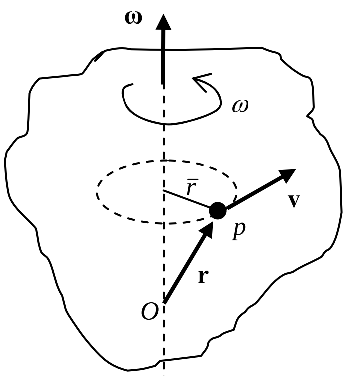
1.1.7 Problems

1. Which of the following are scalars and which are vectors?
   (i) weight
   (ii) specific heat
   (iii) momentum
   (iv) energy
   (v) volume
2. Find the magnitude of the sum of three unit vectors drawn from a common vertex of a cube along three of its sides.

3. Consider two **non-collinear** (not parallel) vectors **a** and **b**. Show that a vector **r** lying in the same plane as these vectors can be written in the form $\mathbf{r} = p\mathbf{a} + q\mathbf{b}$, where *p* and *q* are scalars. [Note: one says that all the vectors **r** in the plane are specified by the **base** vectors **a** and **b**.]
4. Show that the dot product of two vectors **u** and **v** can be interpreted as the magnitude of **u** times the component of **v** in the direction of **u**.
5. The work done by a force, represented by a vector **F**, in moving an object a given distance is the product of the component of force in the given direction times the distance moved. If the vector **s** represents the direction and magnitude (distance) the object is moved, show that the work done is equivalent to $\mathbf{F} \cdot \mathbf{s}$.
6. Prove that the dot product is commutative, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. [Note: this is equivalent to saying, for example, that the work done in problem 5 is also equal to the component of **s** in the direction of the force, times the magnitude of the force.]
7. Sketch $\mathbf{b} \times \mathbf{a}$ if **a** and **b** are as shown below.



8. Show that $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$.
9. Suppose that a rigid body rotates about an axis *O* with angular speed *w*, as shown below. Consider a point *p* in the body with position vector **r**. Show that the velocity **v** of *p* is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is the vector with magnitude ω and whose direction is that in which a right-handed screw would advance under the rotation. [Note: let *s* be the arc-length traced out by the particle as it rotates through an angle θ on a circle of radius $\bar{r}$, then $v = |\mathbf{v}| = \bar{r}\omega$ (since $s = \bar{r}\theta,\ ds/dt = \bar{r}(d\theta/dt)$).]



10. Show, geometrically, that the dot and cross in the triple scalar product can be interchanged: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
11. Show that the **triple vector product** $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ lies in the plane spanned by the vectors **a** and **b**.

1.2 Vector Spaces

The notion of the vector presented in the previous section is here re-cast in a more formal and abstract way, using some basic concepts of Linear Algebra and Topology. This might seem at first to be unnecessarily complicating matters, but this approach turns out to be helpful in unifying and bringing clarity to much of the theory which follows.

Some background theory which complements this material is given in Appendix A to this Chapter, §1.A.

1.2.1 The Vector Space

The vectors introduced in the previous section obey certain rules, those listed in §1.1.3. It turns out that many other mathematical objects obey the same list of rules. For that reason, the mathematical structure defined by these rules is given a special name, the **linear space** or **vector space**.

First, a **set** is any well-defined list, collection, or class of objects, which could be finite or infinite. An example of a set might be

$$B = \{x \mid x \le 3\} \tag{1.2.1}$$

which reads "*B* is the set of objects *x* such that *x* satisfies the property $x \le 3$". Members of a set are referred to as **elements**.

Consider now the **field**[1] of real numbers *R*. The elements of *R* are referred to as **scalars**. Let *V* be a non-empty set of elements $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ with rules of **addition** and **scalar multiplication**, that is there is a **sum** $\mathbf{a} + \mathbf{b} \in V$ for any $\mathbf{a}, \mathbf{b} \in V$ and a **product** $\alpha\mathbf{a} \in V$ for any $\mathbf{a} \in V$, $\alpha \in R$. Then *V* is called a **(real)**[2] **vector space** over *R* if the following eight axioms hold:

1. *associative law for addition*: for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$, one has $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
2. *zero element*: there exists an element $\mathbf{o} \in V$, called the zero element, such that $\mathbf{a} + \mathbf{o} = \mathbf{o} + \mathbf{a} = \mathbf{a}$ for every $\mathbf{a} \in V$
3. *negative* (or *inverse*): for each $\mathbf{a} \in V$ there exists an element $-\mathbf{a} \in V$, called the negative of **a**, such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = 0$
4. *commutative law for addition*: for any $\mathbf{a}, \mathbf{b} \in V$, one has $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
5. *distributive law, over addition of elements of V*: for any $\mathbf{a}, \mathbf{b} \in V$ and scalar $\alpha \in R$, $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$
6. *distributive law, over addition of scalars*: for any $\mathbf{a} \in V$ and scalars $\alpha, \beta \in R$, $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$

---

[1] A **field** is another mathematical structure (see Appendix A to this Chapter, §1.A). For example, the set of complex numbers is a field. In what follows, the only field which will be used is the familiar set of real numbers with the usual operations of addition and multiplication.

[2] "real", since the associated field is the reals. The word *real* will usually be omitted in what follows for brevity.

7. *associative law for multiplication*: for any $\mathbf{a} \in V$ and scalars $\alpha,\ \beta \in R$,
   $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$
8. *unit multiplication*: for the unit scalar $1 \in R$, $1\mathbf{a} = \mathbf{a}$ for any $\mathbf{a} \in V$.

The set of vectors as objects with "magnitude and direction" discussed in the previous section satisfy these rules and therefore form a vector space over *R*. However, despite the name "vector" space, other objects, which are *not* the familiar geometric vectors, can also form a vector space over *R*, as will be seen in a later section.

1.2.2 Inner Product Space

Just as the vector of the previous section is an element of a vector space, next is introduced the notion that the vector dot product is one example of the more general **inner product**.

First, a **function** (or **mapping**) is an assignment which assigns to *each* element of a set *A* a *unique* element of a set B, and is denoted by

$$f : A \to B \tag{1.2.2}$$

An **ordered pair** (a, b) consists of two elements *a* and *b* in which one of them is designated the first element and the other is designated the second element The **product set** (or **Cartesian product**) $A \times B$ consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\} \tag{1.2.3}$$

Now let *V* be a real vector space. An **inner product** (or **scalar product**) on *V* is a mapping that associates to each ordered pair of elements **x**, **y**, a scalar, denoted by $\langle x, y \rangle$,

$$\langle \cdot , \cdot \rangle : V \times V \to R \tag{1.2.4}$$

that satisfies the following properties, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\alpha \in R$:

1. *additivity*: $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
2. *homogeneity*: $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$
3. *symmetry*: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
4. *positive definiteness*: $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ when $\mathbf{x} \neq \mathbf{o}$

From these properties, it follows that, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in V$, then $\mathbf{x} = 0$

A vector space with an associated inner product is called an **inner product space**.

Two elements of an inner product space are said to be **orthogonal** if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \tag{1.2.5}$$

and a set of elements of *V*, $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots\}$, are said to form an **orthogonal set** if every element in the set is orthogonal to every other element:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0, \quad \langle \mathbf{x}, \mathbf{z} \rangle = 0, \quad \langle \mathbf{y}, \mathbf{z} \rangle = 0, \quad \text{etc.} \tag{1.2.6}$$

The above properties are those listed in §1.1.4, and so the set of vectors with the associated dot product forms an inner product space.

Euclidean Vector Space

The set of real triplets (x_1, x_2, x_3) under the usual rules of addition and multiplication forms a vector space R^3. With the inner product defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

one has the inner product space known as (three dimensional) **Euclidean vector space**, and denoted by *E*. This inner product allows one to take distances (and angles) between elements of *E* through the norm (length) and metric (distance) concepts discussed next.

1.2.3 Normed Space

Let *V* be a real vector space. A **norm** on *V* is a real-valued function,

$$\| \ \| : V \to R \tag{1.2.7}$$

that satisfies the following properties, for $\mathbf{x}, \mathbf{y} \in V$, $\alpha \in R$:

1. *positivity*: $\|\mathbf{x}\| \ge 0$
2. *triangle inequality*: $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$
3. *homogeneity*: $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$
4. *positive definiteness*: $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{o}$

A vector space with an associated norm is called a **normed vector space**. Many different norms can be defined on a given vector space, each one giving a different normed linear space. A natural norm for the inner product space is

$$\|\mathbf{x}\| \equiv \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \tag{1.2.8}$$

It can be seen that this norm indeed satisfies the defining properties. When the inner product is the vector dot product, the norm defined by 1.2.8 is the familiar vector "length".

One important consequence of the definitions of inner product and norm is the **Schwarz inequality**, which states that

$$\left|\langle \mathbf{x}, \mathbf{y} \rangle\right| \le \|\mathbf{x}\|\|\mathbf{y}\| \tag{1.2.9}$$

One can now define the **angle** between two elements of *V* to be

$$\theta : V \times V \to R, \qquad \theta(\mathbf{x}, \mathbf{y}) \equiv \cos^{-1}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \tag{1.2.10}$$

The quantity inside the curved brackets here is necessarily between -1 and $+1$, by the Schwarz inequality, and hence the angle θ is indeed a real number.

1.2.4 Metric Spaces

Metric spaces are built on the concept of "distance" between objects. This is a generalization of the familiar distance between two points on the real line.

Consider a set *X*. A **metric** is a real valued function,

$$d(\cdot,\cdot): X \times X \to R \tag{1.2.11}$$

that satisfies the following properties, for $\mathbf{x}, \mathbf{y} \in X$:

1. positive: $d(\mathbf{x}, \mathbf{y}) \ge 0$ and $d(\mathbf{x}, \mathbf{x}) = 0$, for all $\mathbf{x}, \mathbf{y} \in X$
2. strictly positive: if $d(\mathbf{x}, \mathbf{y}) = 0$ then $\mathbf{x} = \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in X$
3. symmetry: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$, for all $\mathbf{x}, \mathbf{y} \in X$
4. triangle inequality: $d(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$

A set *X* with an associated metric is called a **metric space**. The set *X* can have more than one metric defined on it, with different metrics producing different metric spaces.

Consider now a normed vector space. This space naturally has a metric defined on it:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \tag{1.2.12}$$

and thus the normed vector space *is* a metric space. For the set of vectors with the dot product, this gives the "distance" between two vectors $\mathbf{x}, \mathbf{y}$.

1.2.5 The Affine Space

Consider a set *P*, the elements of which are called **points**. Consider also an associated vector space *V*. An **affine space** consists of the set *P*, the set *V*, and two operations which connect *P* and *V*:

(i) given two points $\mathbf{p}, \mathbf{q} \in P$, one can define a **difference**, $\mathbf{q} - \mathbf{p}$ which is a unique element **v** of *V*, i.e. $\mathbf{v} = \mathbf{q} - \mathbf{p} \in V$

(ii) given a point $\mathbf{p} \in P$ and $\mathbf{v} \in V$, one can define the **sum** $\mathbf{v} + \mathbf{p}$ which is a unique point **q** of *P*, i.e. $\mathbf{q} = \mathbf{v} + \mathbf{p} \in P$

and for which the following property holds, for $\mathbf{p}, \mathbf{q}, \mathbf{r} \in P$: $(\mathbf{q} - \mathbf{r}) + (\mathbf{r} - \mathbf{p}) = (\mathbf{q} - \mathbf{p})$.

From the above, one has for the affine space that $\mathbf{p} - \mathbf{p} = \mathbf{o}$ and $\mathbf{q} - \mathbf{p} = -(\mathbf{p} - \mathbf{q})$, for all $\mathbf{p}, \mathbf{q} \in P$.

Note that one can take the sum of vectors, according to the structure of the vector space, but one cannot take the sum of points, only the difference between two points. Further, there is no notion of **origin** in the affine space. One can choose some fixed $\mathbf{o} \in P$ to be an origin. In that case, $\mathbf{v} = \mathbf{p} - \mathbf{o}$ is called the **position vector** of **p** relative to **o**.

Suppose now that the associated vector space is a Euclidean vector space, i.e. an inner product space. Define the **distance** between two points through the inner product associated with *V*,

$$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\| = \sqrt{\langle \mathbf{q} - \mathbf{p}, \mathbf{q} - \mathbf{p} \rangle} \qquad (1.2.13)$$

It can be shown that this mapping $d : P \times P \to R$ is a metric, i.e. it satisfies the metric properties, and thus *P* is a metric space (although it is not a vector space). In this case, *P* is referred to as **Euclidean point space**, **Euclidean affine space** or, simply, **Euclidean space**. Whereas in Euclidean vector space there is a zero element, the origin $(0,0,0)$, in Euclidean point space there is none – apart from that, the two spaces are the same and, apart from certain special cases, one does not need to distinguish between them.

1.3 Cartesian Vectors

So far the discussion has been in **symbolic notation**[1], that is, no reference to 'axes' or 'components' or 'coordinates' is made, implied or required. The vectors exist independently of any coordinate system. It turns out that much of vector (tensor) mathematics is more concise and easier to manipulate in such notation than in terms of corresponding component notations. However, there are many circumstances in which use of the component forms of vectors (and tensors) is more helpful – or essential. In this section, vectors are discussed in terms of components – **component form**.

1.3.1 The Cartesian Basis

Consider three dimensional (Euclidean) space. In this space, consider the three unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ having the properties

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \tag{1.3.1}$$

so that they are mutually perpendicular (mutually **orthogonal**), and

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \tag{1.3.2}$$

so that they are unit vectors. Such a set of orthogonal unit vectors is called an **orthonormal** set, Fig. 1.3.1. Note further that this orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is **right-handed**, by which is meant $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (or $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ or $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$).

This set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms a **basis,** by which is meant that any other vector can be written as a **linear combination** of these vectors, i.e. in the form

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{1.3.3}$$

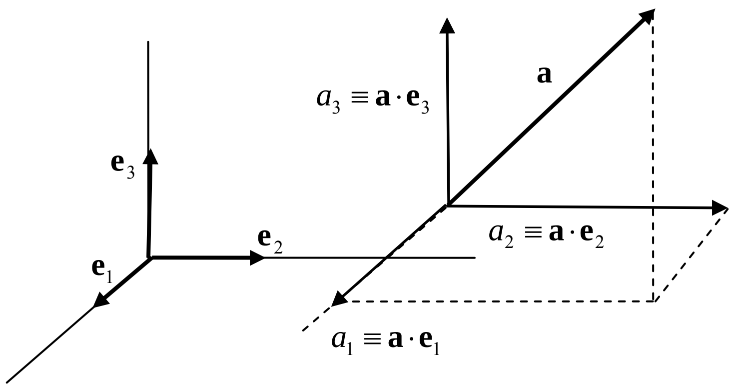


Figure 1.3.1: an orthonormal set of base vectors and Cartesian components

[1] or **absolute** or **invariant** or **direct** or **vector** notation

By repeated application of Eqn. 1.1.2 to a vector **a**, and using 1.3.2, the scalars in 1.3.3 can be expressed as (see Fig. 1.3.1)

$$a_1 = \mathbf{a} \cdot \mathbf{e}_1, \quad a_2 = \mathbf{a} \cdot \mathbf{e}_2, \quad a_3 = \mathbf{a} \cdot \mathbf{e}_3 \tag{1.3.4}$$

The scalars a_1, a_2 and a_3 are called the **Cartesian components** of **a** in the given basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The unit vectors are called **base vectors** when used for this purpose.

Note that it is not necessary to have three mutually orthogonal vectors, or vectors of unit size, or a right-handed system, to form a basis – only that the three vectors are not co-planar. The right-handed orthonormal set is often the easiest basis to use in practice, but this is not always the case – for example, when one wants to describe a body with curved boundaries (e.g., see §1.6.10).

The dot product of two vectors **u** and **v**, referred to the above basis, can be written as

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \cdot (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1v_1(\mathbf{e}_1 \cdot \mathbf{e}_1) + u_1v_2(\mathbf{e}_1 \cdot \mathbf{e}_2) + u_1v_3(\mathbf{e}_1 \cdot \mathbf{e}_3) \\
&\quad + u_2v_1(\mathbf{e}_2 \cdot \mathbf{e}_1) + u_2v_2(\mathbf{e}_2 \cdot \mathbf{e}_2) + u_2v_3(\mathbf{e}_2 \cdot \mathbf{e}_3) \\
&\quad\quad + u_3v_1(\mathbf{e}_3 \cdot \mathbf{e}_1) + u_3v_2(\mathbf{e}_3 \cdot \mathbf{e}_2) + u_3v_3(\mathbf{e}_3 \cdot \mathbf{e}_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned} \tag{1.3.5}$$

Similarly, the cross product is

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \times (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1v_1(\mathbf{e}_1 \times \mathbf{e}_1) + u_1v_2(\mathbf{e}_1 \times \mathbf{e}_2) + u_1v_3(\mathbf{e}_1 \times \mathbf{e}_3) \\
&\quad + u_2v_1(\mathbf{e}_2 \times \mathbf{e}_1) + u_2v_2(\mathbf{e}_2 \times \mathbf{e}_2) + u_2v_3(\mathbf{e}_2 \times \mathbf{e}_3) \\
&\quad\quad + u_3v_1(\mathbf{e}_3 \times \mathbf{e}_1) + u_3v_2(\mathbf{e}_3 \times \mathbf{e}_2) + u_3v_3(\mathbf{e}_3 \times \mathbf{e}_3) \\
&= (u_2v_3 - u_3v_2)\mathbf{e}_1 - (u_1v_3 - u_3v_1)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3
\end{aligned} \tag{1.3.6}$$

This is often written in the form

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{1.3.7}$$

that is, the cross product is equal to the determinant of the 3×3 matrix

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

1.3.2 The Index Notation

The expression for the cross product in terms of components, Eqn. 1.3.6, is quite lengthy – for more complicated quantities things get unmanageably long. Thus a short-hand notation is used for these component equations, and this **index notation**[2] is described here.

In the index notation, the expression for the vector **a** in terms of the components a_1, a_2, a_3 and the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \tag{1.3.8}$$

This can be simplified further by using Einstein's **summation convention**, whereby the summation sign is dropped and it is understood that for a repeated index (i in this case) a summation over the range of the index (3 in this case[3]) is implied. Thus one writes $\mathbf{a} = a_i\mathbf{e}_i$. This can be further shortened to, simply, a_i.

The dot product of two vectors written in the index notation reads

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_i v_i} \qquad \textbf{Dot Product} \tag{1.3.9}$$

The repeated index i is called a **dummy index**, because it can be replaced with any other letter and the sum is the same; for example, this could equally well be written as $\mathbf{u}\cdot\mathbf{v} = u_j v_j$ or $u_k v_k$.

For the purpose of writing the vector cross product in index notation, the **permutation symbol** (or **alternating symbol**) ε_{ijk} can be introduced:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if two or more indices are equal} \end{cases} \tag{1.3.10}$$

For example (see Fig. 1.3.2),

$$\begin{aligned} \varepsilon_{123} &= +1 \\ \varepsilon_{132} &= -1 \\ \varepsilon_{122} &= 0 \end{aligned}$$

[2] or **indicial** or **subscript** or **suffix** notation
[3] 2 in the case of a two-dimensional space/analysis

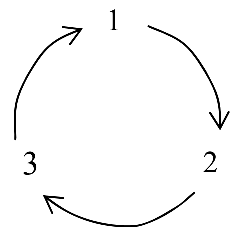


Figure 1.3.2: schematic for the permutation symbol (clockwise gives +1)

Note that

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj} \tag{1.3.11}$$

and that, in terms of the base vectors {▲Problem 7},

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k \tag{1.3.12}$$

and {▲Problem 7}

$$\varepsilon_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j)\cdot \mathbf{e}_k \,. \tag{1.3.13}$$

The cross product can now be written concisely as {▲Problem 8}

$$\boxed{\mathbf{u} \times \mathbf{v} = \varepsilon_{ijk} u_i v_j \mathbf{e}_k} \qquad \textbf{Cross Product} \tag{1.3.14}$$

Introduce next the **Kronecker delta symbol** δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{1.3.15}$$

Note that $\delta_{11} = 1$ but, using the index notation, $\delta_{ii} = 3$. The Kronecker delta allows one to write the expressions defining the orthonormal basis vectors (1.3.1, 1.3.2) in the compact form

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \qquad \textbf{Orthonormal Basis Rule} \tag{1.3.16}$$

The triple scalar product (1.1.4) can now be written as

$$\begin{aligned}(\mathbf{u} \times \mathbf{v})\cdot \mathbf{w} &= (\varepsilon_{ijk} u_i v_j \mathbf{e}_k)\cdot w_m \mathbf{e}_m \\ &= \varepsilon_{ijk} u_i v_j w_m \delta_{km} \\ &= \varepsilon_{ijk} u_i v_j w_k \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\end{aligned} \tag{1.3.17}$$

Note that, since the determinant of a matrix is equal to the determinant of the transpose of a matrix, this is equivalent to

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \tag{1.3.18}$$

Here follow some useful formulae involving the permutation and Kronecker delta symbol {▲Problem 13}:

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{kpq} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \\ \varepsilon_{ijk}\varepsilon_{ijp} &= 2\delta_{pk} \end{aligned} \tag{1.3.19}$$

Finally, here are some other important identities involving vectors; the third of these is called **Lagrange's identity**[4] {▲Problem 15}:

$$\begin{aligned} (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix} \\ (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{d})]\mathbf{c} - [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{d} \\ [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{d} &= [\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a} + [\mathbf{a}\cdot(\mathbf{d}\times\mathbf{c})]\mathbf{b} + [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{d})]\mathbf{c} \end{aligned} \tag{1.3.20}$$

1.3.3 Matrix Notation for Vectors

The symbolic notation $\mathbf{v}$ and index notation $v_i\mathbf{e}_i$ (or simply v_i) can be used to denote a vector. Another notation is the **matrix notation**: the vector **v** can be represented by a 3×1 matrix (a **column vector**):

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Matrices will be denoted by square brackets, so a shorthand notation for this matrix/vector would be $[\mathbf{v}]$. The elements of the matrix $[\mathbf{v}]$ can be written in the **element form** v_i. The element form for a matrix is essentially the same as the index notation for the vector it represents.

[4] to be precise, the special case of 1.3.20c, 1.3.20a, is Lagrange's identity

Formally, a vector can be represented by the ordered triplet of real numbers, (v_1, v_2, v_3). The set of all vectors can be represented by R^3, the set of all ordered triplets of real numbers:

$$R^3 = \{(v_1, v_2, v_3) \mid v_1, v_2, v_3 \in R\} \tag{1.3.21}$$

It is important to *note the distinction between a vector and a matrix*: the former is a mathematical object independent of any basis, the latter is a representation of the vector with respect to a particular basis – use a different set of basis vectors and the elements of the matrix will change, but the matrix is still describing the same vector. Said another way, there is a difference between an element (vector) **v** of Euclidean vector space and an ordered triplet $v_i \in R^3$. This notion will be discussed more fully in the next section.

As an example, the dot product can be written in the matrix notation as

$$[\mathbf{u}^{\mathrm{T}}][\mathbf{v}] = [u_1 \quad u_2 \quad u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

"short"
matrix notation

"full"
matrix notation

Here, the notation $[\mathbf{u}^{\mathrm{T}}]$ denotes the 1×3 matrix (the **row vector**). The result is a 1×1 matrix, i.e. a scalar, in element form $u_i v_i$.

1.3.4 Cartesian Coordinates

Thus far, the notion of an origin has not been used. Choose a point **o** in Euclidean (point) space, to be called the **origin**. An origin together with a right-handed orthonormal basis $\{\mathbf{e}_i\}$ constitutes a (**rectangular**) **Cartesian coordinate system**, Fig. 1.3.3.

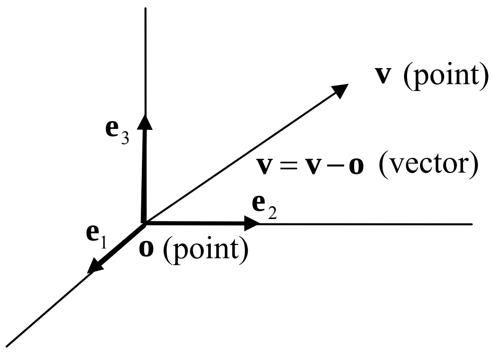


Figure 1.3.3: a Cartesian coordinate system

A second point **v** then defines a **position vector** $\mathbf{v}-\mathbf{o}$, Fig. 1.3.3. The components of the vector $\mathbf{v}-\mathbf{o}$ are called the (**rectangular**) **Cartesian coordinates** of the point **v** [5]. For brevity, the vector $\mathbf{v}-\mathbf{o}$ is simply labelled **v**, that is, one uses the same symbol for both the position vector and associated point.

1.3.5 Problems

1. Evaluate $\mathbf{u}\cdot\mathbf{v}$ where $\mathbf{u}=\mathbf{e}_1+3\mathbf{e}_2-2\mathbf{e}_3$, $\mathbf{v}=4\mathbf{e}_1-2\mathbf{e}_2+4\mathbf{e}_3$.
2. Prove that for any vector **u**, $\mathbf{u}=(\mathbf{u}\cdot\mathbf{e}_1)\mathbf{e}_1+(\mathbf{u}\cdot\mathbf{e}_2)\mathbf{e}_2+(\mathbf{u}\cdot\mathbf{e}_3)\mathbf{e}_3$. [Hint: write **u** in component form.]
3. Find the projection of the vector $\mathbf{u}=\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3$ on the vector $\mathbf{v}=4\mathbf{e}_1-4\mathbf{e}_2+7\mathbf{e}_3$.
4. Find the angle between $\mathbf{u}=3\mathbf{e}_1+2\mathbf{e}_2-6\mathbf{e}_3$ and $\mathbf{v}=4\mathbf{e}_1-3\mathbf{e}_2+\mathbf{e}_3$.
5. Write down an expression for a unit vector parallel to the resultant of two vectors **u** and **v** (in symbolic notation). Find this vector when $\mathbf{u}=2\mathbf{e}_1+4\mathbf{e}_2-5\mathbf{e}_3$, $\mathbf{v}=\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$ (in component form). Check that your final vector is indeed a unit vector.
6. Evaluate $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u}=-\mathbf{e}_1-2\mathbf{e}_2+2\mathbf{e}_3$, $\mathbf{v}=2\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3$.
7. Verify that $\mathbf{e}_i\times\mathbf{e}_j=\varepsilon_{ijm}\mathbf{e}_m$. Hence, by dotting each side with $\mathbf{e}_k$, show that $\varepsilon_{ijk}=\left(\mathbf{e}_i\times\mathbf{e}_j\right)\cdot\mathbf{e}_k$.
8. Show that $\mathbf{u}\times\mathbf{v}=\varepsilon_{ijk}u_i v_j\mathbf{e}_k$.
9. The triple scalar product is given by $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\varepsilon_{ijk}u_i v_j w_k$. Expand this equation and simplify, so as to express the triple scalar product in full (non-index) component form.
10. Write the following in index notation: $|\mathbf{v}|$, $\mathbf{v}\cdot\mathbf{e}_1$, $\mathbf{v}\cdot\mathbf{e}_k$.
11. Show that $\delta_{ij}a_i b_j$ is equivalent to $\mathbf{a}\cdot\mathbf{b}$.
12. Verify that $\varepsilon_{ijk}\varepsilon_{ijk}=6$.
13. Verify that $\varepsilon_{ijk}\varepsilon_{kpq}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}$ and hence show that $\varepsilon_{ijk}\varepsilon_{ijp}=2\delta_{pk}$.
14. Evaluate or simplify the following expressions:
    (a) δ_{kk} (b) $\delta_{ij}\delta_{ij}$ (c) $\delta_{ij}\delta_{jk}$ (d) $\varepsilon_{1jk}\delta_{3j}v_k$
15. Prove Lagrange's identity 1.3.20c.
16. If **e** is a unit vector and **a** an arbitrary vector, show that
$$\mathbf{a}=(\mathbf{a}\cdot\mathbf{e})\mathbf{e}+\mathbf{e}\times(\mathbf{a}\times\mathbf{e})$$
which is another representation of Eqn. 1.1.2, where **a** can be resolved into components parallel and perpendicular to **e**.

[5] That is, "components" are used for vectors and "coordinates" are used for points

1.4 Matrices and Element Form

1.4.1 Matrix – Matrix Multiplication

In the next section, §1.5, regarding vector transformation equations, it will be necessary to multiply various matrices with each other (of sizes 3×1, 1×3 and 3×3). It will be helpful to write these matrix multiplications in a short-hand element form and to develop some short "rules" which will be beneficial right through this chapter.

First, it has been seen that the dot product of two vectors can be represented by $[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]$, or $u_i v_i$. Similarly, the matrix multiplication $[\mathbf{u}][\mathbf{v}^{\mathrm{T}}]$ gives a 3×3 matrix with element form $u_i v_j$ or, in full,

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

This type of matrix represents the **tensor product** of two vectors, written in symbolic notation as $\mathbf{u}\otimes\mathbf{v}$ (or simply **uv**). Tensor products will be discussed in detail in §1.8 and §1.9.

Next, the matrix multiplication

$$[\mathbf{Q}][\mathbf{u}] \equiv \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1.4.1}$$

is a 3×1 matrix with elements $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij}u_j$ {▲Problem 1}. The elements of $[\mathbf{Q}][\mathbf{u}]$ are the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$, which in element form reads $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}])_i \equiv u_j Q_{ij}$.

The expression $[\mathbf{u}][\mathbf{Q}]$ is meaningless, but $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ {▲Problem 2} is a 1×3 matrix with elements $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}])_i \equiv u_j Q_{ji}$.

This leads to the following rule:

1. if a vector pre-multiplies a matrix $[\mathbf{Q}]$ $\rightarrow$ it is the transpose $[\mathbf{u}^{\mathrm{T}}]$
2. if a matrix $[\mathbf{Q}]$ pre-multiplies the vector $\rightarrow$ it is $[\mathbf{u}]$
3. if summed indices are "beside each other", as the *j* in $u_j Q_{ji}$ or $Q_{ij}u_j$
   $\rightarrow$ the matrix is $[\mathbf{Q}]$
4. if summed indices are not beside each other, as the *j* in $u_j Q_{ij}$
   $\rightarrow$ the matrix is the transpose, $[\mathbf{Q}^{\mathrm{T}}]$

Finally, consider the multiplication of 3×3 matrices. Again, this follows the "beside each other" rule for the summed index. For example, $[\mathbf{A}][\mathbf{B}]$ gives the 3×3 matrix {▲Problem 6} $([\mathbf{A}][\mathbf{B}])_{ij} = A_{ik}B_{kj}$, and the multiplication $[\mathbf{A}^{\mathrm{T}}][\mathbf{B}]$ is written as $([\mathbf{A}^{\mathrm{T}}][\mathbf{B}])_{ij} = A_{ki}B_{kj}$. There is also the important identity

$$([\mathbf{A}][\mathbf{B}])^{\mathrm{T}} = [\mathbf{B}^{\mathrm{T}}][\mathbf{A}^{\mathrm{T}}] \tag{1.4.2}$$

Note also the following (which applies to both the index notation and element form):

(i) if there is no free index, as in $u_i v_i$, there is one element (representing a scalar)

(ii) if there is one free index, as in $u_j Q_{ji}$, it is a 3×1 (or 1×3) matrix (representing a vector)

(iii) if there are two free indices, as in $A_{ki}B_{kj}$, it is a 3×3 matrix (representing, as will be seen later, a second-order tensor)

1.4.2 The Trace of a Matrix

Another important notation involving matrices is the **trace** of a matrix, defined to be the sum of the diagonal terms, and denoted by

$$\boxed{\mathrm{tr}[\mathbf{A}] = A_{11} + A_{22} + A_{33} \equiv A_{ii}} \qquad \textbf{The Trace} \tag{1.4.3}$$

1.4.3 Problems

1. Show that $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij}u_j$. To do this, multiply the matrix and the vector in Eqn. 1.4.1 and write out the resulting vector in full; Show that the three elements of the vector are $Q_{1j}u_j$, $Q_{2j}u_j$ and $Q_{3j}u_j$.
2. Show that $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ is a 1×3 matrix with elements $u_j Q_{ji}$ (write the matrices out in full).
3. Show that $([\mathbf{Q}][\mathbf{u}])^{\mathrm{T}} = [\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$.
4. Are the three elements of $[\mathbf{Q}][\mathbf{u}]$ the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$?
5. What is the element form for the matrix representation of $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$?
6. Write out the 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ in full, i.e. in terms of A_{11}, A_{12}, etc. and verify that $[\mathbf{AB}]_{ij} = A_{ik}B_{kj}$ for $i = 2,\ j = 1$.
7. What is the element form for
   (i) $[\mathbf{A}][\mathbf{B}^{\mathrm{T}}]$
   (ii) $[\mathbf{v}^{\mathrm{T}}][\mathbf{A}][\mathbf{v}]$ (there is no ambiguity here, since $([\mathbf{v}^{\mathrm{T}}][\mathbf{A}])[\mathbf{v}] = [\mathbf{v}^{\mathrm{T}}]([\mathbf{A}][\mathbf{v}])$)
   (iii) $[\mathbf{B}^{\mathrm{T}}][\mathbf{A}][\mathbf{B}]$
8. Show that $\delta_{ij}A_{ij} = \mathrm{tr}[\mathbf{A}]$.
9. Show that $\det[\mathbf{A}] = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$.

1.5 Coordinate Transformation of Vector Components

Very often in practical problems, the components of a vector are known in one coordinate system but it is necessary to find them in some other coordinate system.

For example, one might know that the force **f** acting "in the x_1 direction" has a certain value, Fig. 1.5.1 – this is equivalent to knowing the x_1 component of the force, in an $x_1 - x_2$ coordinate system. One might then want to know what force is "acting" in some other direction – for example in the x_1' direction shown – this is equivalent to asking what the x_1' component of the force is in a new $x_1' - x_2'$ coordinate system.

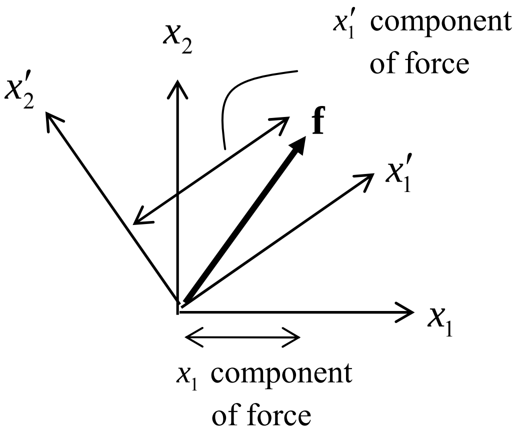


Figure 1.5.1: a vector represented using two different coordinate systems

The relationship between the components in one coordinate system and the components in a second coordinate system are called the **transformation equations**. These transformation equations are derived and discussed in what follows.

1.5.1 Rotations and Translations

Any change of Cartesian coordinate system will be due to a **translation** of the base vectors and a **rotation** of the base vectors. A translation of the base vectors does not change the components of a vector. Mathematically, this can be expressed by saying that the components of a vector **a** are $\mathbf{e}_i \cdot \mathbf{a}$, and these three quantities do not change under a translation of base vectors. Rotation of the base vectors is thus what one is concerned with in what follows.

1.5.2 Components of a Vector in Different Systems

Vectors are mathematical objects which exist *independently of any coordinate system*. Introducing a coordinate system for the purpose of analysis, one could choose, for example, a certain Cartesian coordinate system with base vectors $\mathbf{e}_i$ and origin *o*, Fig.

1.5.2. In that case the vector can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, and u_1, u_2, u_3 are its components.

Now a second coordinate system can be introduced (with the same origin), this time with base vectors $\mathbf{e}'_i$. In that case, the vector can be written as $\mathbf{u} = u'_1\mathbf{e}'_1 + u'_2\mathbf{e}'_2 + u'_3\mathbf{e}'_3$, where u'_1, u'_2, u'_3 are its components in this second coordinate system, as shown in the figure. Thus the *same vector* can be written in more than one way:

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = u'_1\mathbf{e}'_1 + u'_2\mathbf{e}'_2 + u'_3\mathbf{e}'_3$$

The first coordinate system is often referred to as "the $ox_1x_2x_3$ system" and the second as "the $ox'_1x'_2x'_3$ system".

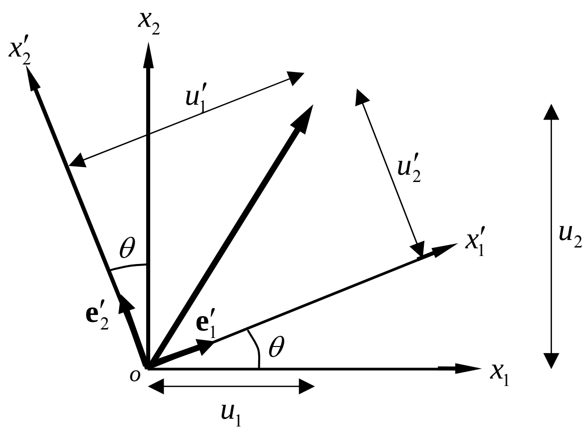


Figure 1.5.2: a vector represented using two different coordinate systems

Note that the new coordinate system is obtained from the first one by a *rotation* of the base vectors. The figure shows a rotation θ about the x_3 axis (the sign convention for rotations is positive counterclockwise).

Two Dimensions

Concentrating for the moment on the two dimensions $x_1 - x_2$, from trigonometry (refer to Fig. 1.5.3),

$$\begin{aligned}\mathbf{u} &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \\ &= \left[|OB| - |AB|\right]\mathbf{e}_1 + \left[|BD| + |CP|\right]\mathbf{e}_2 \\ &= \left[\cos\theta\, u'_1 - \sin\theta\, u'_2\right]\mathbf{e}_1 + \left[\sin\theta\, u'_1 + \cos\theta\, u'_2\right]\mathbf{e}_2\end{aligned}$$

and so

$$
\begin{aligned}
u_1 &= \cos\theta\, u_1' - \sin\theta\, u_2' \\
u_2 &= \sin\theta\, u_1' + \cos\theta\, u_2'
\end{aligned}
$$

vector components in first coordinate system

vector components in second coordinate system

In matrix form, these transformation equations can be written as

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}
$$

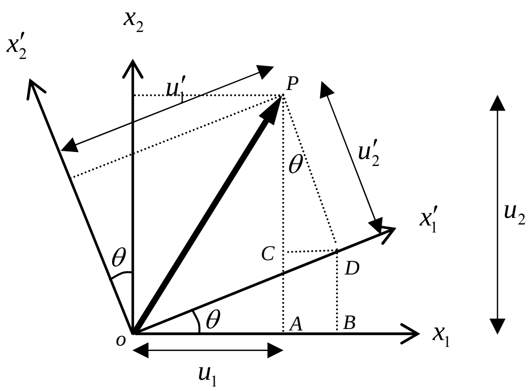


Figure 1.5.3: geometry of the 2D coordinate transformation

The 2×2 matrix is called the **transformation** or **rotation matrix** $[\mathbf{Q}]$. By pre-multiplying both sides of these equations by the inverse of $[\mathbf{Q}]$, $[\mathbf{Q}^{-1}]$, one obtains the transformation equations transforming from $[u_1 \quad u_2]^{\mathrm{T}}$ to $[u_1' \quad u_2']^{\mathrm{T}}$:

$$
\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

An important property of the transformation matrix is that it is **orthogonal**, by which is meant that

$$
\boxed{[\mathbf{Q}^{-1}] = [\mathbf{Q}^{\mathrm{T}}]} \qquad \textbf{Orthogonality of Transformation/Rotation Matrix} \qquad (1.5.1)
$$

Three Dimensions

It is straight forward to show that, in the full three dimensions, Fig. 1.5.4, the components in the two coordinate systems are related through

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix}$$

where $\cos(x_i, x_j')$ is the cosine of the angle between the x_i and x_j' axes. These nine quantities are called the **direction cosines** of the coordinate transformation. Again denoting these by the letter Q, $Q_{11} = \cos(x_1, x_1')$, $Q_{12} = \cos(x_1, x_2')$, etc., so that

$$Q_{ij} = \cos(x_i, x_j'), \tag{1.5.2}$$

one has the matrix equations

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix}$$

or, in element form and short-hand matrix notation,

$$u_i = Q_{ij} u_j' \qquad \ldots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \tag{1.5.3}$$

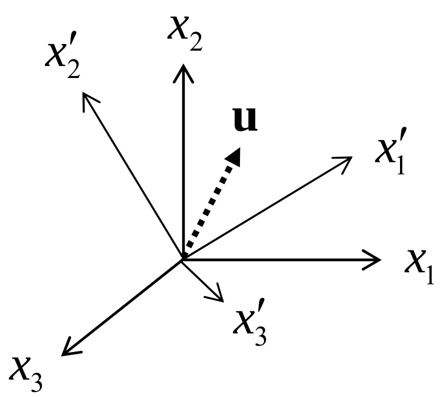


Figure 1.5.4: two different coordinate systems in a 3D space

Note: some authors define the matrix of direction cosines to consist of the components $Q_{ij} = \cos(x_i', x_j)$, so that the subscript i refers to the new coordinate system and the j to the old coordinate system, rather than the other way around as used here.

Transformation of Cartesian Base Vectors

The direction cosines introduced above also relate the base vectors in any two Cartesian coordinate systems. It can be seen that

$$\mathbf{e}_i \cdot \mathbf{e}_j' = Q_{ij} \tag{1.5.4}$$

This relationship is illustrated in Fig. 1.5.5 for $i = 1$.

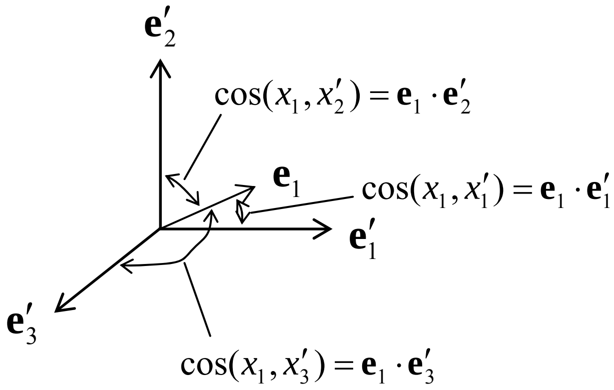


Figure 1.5.5: direction cosines

Formal Derivation of the Transformation Equations

In the above, the transformation equations $u_i = Q_{ij} u_j'$ were derived geometrically. They can also be derived algebraically using the index notation as follows: start with the relations $\mathbf{u} = u_k \mathbf{e}_k = u_j' \mathbf{e}_j'$ and post-multiply both sides by $\mathbf{e}_i$ to get (the corresponding matrix representation is to the right (also, see Problem 3 in §1.4.3)):

$$
\begin{aligned}
& u_k \mathbf{e}_k \cdot \mathbf{e}_i = u_j' \mathbf{e}_j' \cdot \mathbf{e}_i \\
& \rightarrow u_k \delta_{ki} = u_j' Q_{ij} \\
& \rightarrow \quad u_i = u_j' Q_{ij} \qquad \ldots \quad [\mathbf{u}^{\mathrm{T}}] = [\mathbf{u}'^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}] \\
& \rightarrow \quad u_i = Q_{ij} u_j' \qquad \ldots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}']
\end{aligned}
$$

The inverse equations are {▲Problem 3}

$$
u_i' = Q_{ji} u_j \qquad \ldots \quad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \tag{1.5.5}
$$

Orthogonality of the Transformation Matrix $[\mathbf{Q}]$

As in the two dimensional case, the transformation matrix is orthogonal, $[\mathbf{Q}^{\mathrm{T}}] = [\mathbf{Q}^{-1}]$. This follows from 1.5.3, 1.5.5.

Example

Consider a Cartesian coordinate system with base vectors $\mathbf{e}_i$. A coordinate transformation is carried out with the new basis given by

$$\mathbf{e}_1' = n_1^{(1)}\mathbf{e}_1 + n_2^{(1)}\mathbf{e}_2 + n_3^{(1)}\mathbf{e}_3$$
$$\mathbf{e}_2' = n_1^{(2)}\mathbf{e}_1 + n_2^{(2)}\mathbf{e}_2 + n_3^{(2)}\mathbf{e}_3$$
$$\mathbf{e}_3' = n_1^{(3)}\mathbf{e}_1 + n_2^{(3)}\mathbf{e}_2 + n_3^{(3)}\mathbf{e}_3$$

What is the transformation matrix?

Solution

The transformation matrix consists of the direction cosines $Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j'$, so

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix}$$

■

1.5.3 Problems

1. The angles between the axes in two coordinate systems are given in the table below.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

   Construct the corresponding transformation matrix $[\mathbf{Q}]$ and verify that it is orthogonal.
2. The $ox_1'x_2'x_3'$ coordinate system is obtained from the $ox_1x_2x_3$ coordinate system by a positive (counterclockwise) rotation of θ about the x_3 axis. Find the (full three dimensional) transformation matrix $[\mathbf{Q}]$. A further positive rotation β about the x_2 axis is then made to give the $ox_1''x_2''x_3''$ coordinate system. Find the corresponding transformation matrix $[\mathbf{P}]$. Then construct the transformation matrix $[\mathbf{R}]$ for the complete transformation from the $ox_1x_2x_3$ to the $ox_1''x_2''x_3''$ coordinate system.
3. Beginning with the expression $u_j\mathbf{e}_j \cdot \mathbf{e}_i' = u_k'\mathbf{e}_k' \cdot \mathbf{e}_i'$, formally derive the relation $u_i' = Q_{ji}u_j$ ($[\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}]$).

1.6 Vector Calculus 1 - Differentiation

Calculus involving vectors is discussed in this section, rather intuitively at first and more formally toward the end of this section.

1.6.1 The Ordinary Calculus

Consider a **scalar-valued function of a scalar**, for example the time-dependent density of a material $\rho = \rho(t)$. The calculus of scalar valued functions of scalars is just the ordinary calculus. Some of the important concepts of the ordinary calculus are reviewed in Appendix B to this Chapter, §1.B.2.

1.6.2 Vector-valued Functions of a scalar

Consider a **vector-valued function of a scalar**, for example the time-dependent displacement of a particle $\mathbf{u} = \mathbf{u}(t)$. In this case, the derivative is defined in the usual way,

$$\frac{d\mathbf{u}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t+\Delta t) - \mathbf{u}(t)}{\Delta t},$$

which turns out to be simply the derivative of the coefficients[1],

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{e}_1 + \frac{du_2}{dt}\mathbf{e}_2 + \frac{du_3}{dt}\mathbf{e}_3 \equiv \frac{du_i}{dt}\mathbf{e}_i$$

Partial derivatives can also be defined in the usual way. For example, if $\mathbf{u}$ is a function of the coordinates, $\mathbf{u}(x_1, x_2, x_3)$, then

$$\frac{\partial \mathbf{u}}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{\mathbf{u}(x_1 + \Delta x_1, x_2, x_3) - \mathbf{u}(x_1, x_2, x_3)}{\Delta x_1}$$

Differentials of vectors are also defined in the usual way, so that when u_1, u_2, u_3 undergo increments $du_1 = \Delta u_1$, $du_2 = \Delta u_2$, $du_3 = \Delta u_3$, the differential of $\mathbf{u}$ is

$$d\mathbf{u} = du_1\mathbf{e}_1 + du_2\mathbf{e}_2 + du_3\mathbf{e}_3$$

and the differential and actual increment $\Delta\mathbf{u}$ approach one another as $\Delta u_1, \Delta u_2, \Delta u_3 \to 0$.

[1] assuming that the base vectors do not depend on t

Space Curves

The derivative of a vector can be interpreted geometrically as shown in Fig. 1.6.1: $\Delta\mathbf{u}$ is the increment in **u** consequent upon an increment Δt in *t*. As *t* changes, the end-point of the vector $\mathbf{u}(t)$ traces out the dotted curve Γ shown – it is clear that as $\Delta t \to 0$, $\Delta\mathbf{u}$ approaches the tangent to Γ, so that $d\mathbf{u}/dt$ is tangential to Γ. The unit vector tangent to the curve is denoted by $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \frac{d\mathbf{u}/dt}{|d\mathbf{u}/dt|} \tag{1.6.1}$$

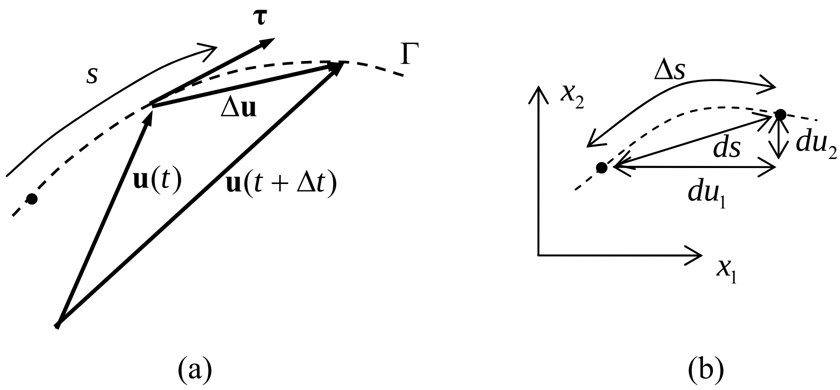


Figure 1.6.1: a space curve; (a) the tangent vector, (b) increment in arc length

Let *s* be a measure of the length of the curve Γ, measured from some fixed point on Γ. Let Δs be the increment in arc-length corresponding to increments in the coordinates, $\Delta\mathbf{u} = [\Delta u_1, \Delta u_2, \Delta u_3]^{\mathrm{T}}$, Fig. 1.6.1b. Then, from the ordinary calculus (see Appendix 1.B),

$$(ds)^2 = (du_1)^2 + (du_2)^2 + (du_3)^2$$

so that

$$\frac{ds}{dt} = \sqrt{\left(\frac{du_1}{dt}\right)^2 + \left(\frac{du_2}{dt}\right)^2 + \left(\frac{du_3}{dt}\right)^2}$$

But

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{e}_1 + \frac{du_2}{dt}\mathbf{e}_2 + \frac{du_3}{dt}\mathbf{e}_3$$

so that

$$\left|\frac{d\mathbf{u}}{dt}\right| = \frac{ds}{dt} \tag{1.6.2}$$

Thus the unit vector tangent to the curve can be written as

$$\boldsymbol{\tau} = \frac{d\mathbf{u}/dt}{ds/dt} = \frac{d\mathbf{u}}{ds} \tag{1.6.3}$$

If **u** is interpreted as the position vector of a particle and t is interpreted as time, then $\mathbf{v} = d\mathbf{u}/dt$ is the velocity vector of the particle as it moves with speed ds/dt along Γ.

Example (of particle motion)

A particle moves along a curve whose parametric equations are $x_1 = 2t^2$, $x_2 = t^2 - 4t$, $x_3 = 3t - 5$ where t is time. Find the component of the velocity at time $t = 1$ in the direction $\mathbf{a} = \mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$.

<u>Solution</u>

The velocity is

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left\{2t^2\mathbf{e}_1 + \left(t^2 - 4t\right)\mathbf{e}_2 + \left(3t - 5\right)\mathbf{e}_3\right\} \\ &= 4\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 \quad \text{at } t = 1\end{aligned}$$

The component in the given direction is $\mathbf{v} \cdot \hat{\mathbf{a}}$, where $\hat{\mathbf{a}}$ is a unit vector in the direction of **a**, giving $8\sqrt{14}/7$.

■

Curvature

The scalar **curvature** $\kappa(s)$ of a space curve is defined to be the magnitude of the rate of change of the unit tangent vector:

$$\kappa(s) = \left|\frac{d\boldsymbol{\tau}}{ds}\right| = \left|\frac{d^2\mathbf{u}}{ds^2}\right| \tag{1.6.4}$$

Note that $\Delta\boldsymbol{\tau}$ is in a direction perpendicular to $\boldsymbol{\tau}$, Fig. 1.6.2. In fact, this can be proved as follows: since $\boldsymbol{\tau}$ is a unit vector, $\boldsymbol{\tau} \cdot \boldsymbol{\tau}$ is a constant ($= 1$), and so $d(\boldsymbol{\tau} \cdot \boldsymbol{\tau})/ds = 0$, but also,

$$\frac{d}{ds}(\boldsymbol{\tau} \cdot \boldsymbol{\tau}) = 2\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds}$$

and so $\boldsymbol{\tau}$ and $d\boldsymbol{\tau}/ds$ are perpendicular. The unit vector defined in this way is called the **principal normal vector**:

$$\mathbf{v} = \frac{1}{\kappa}\frac{d\boldsymbol{\tau}}{ds} \qquad (1.6.5)$$

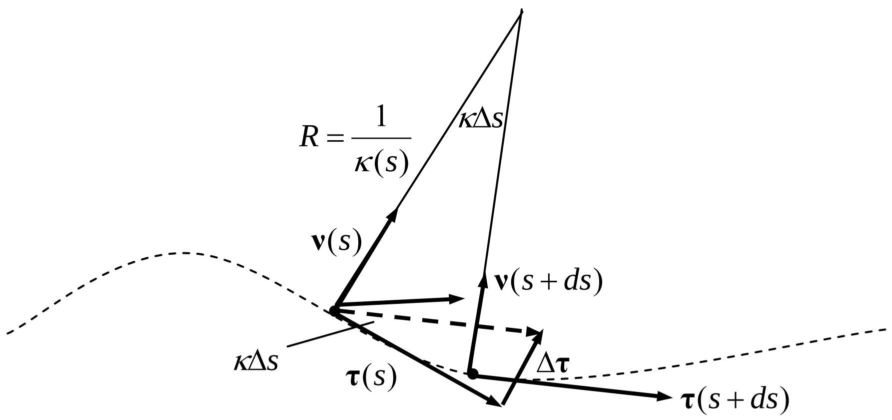


Figure 1.6.2: the curvature

This can be seen geometrically in Fig. 1.6.2: from Eqn. 1.6.5, $\Delta\boldsymbol{\tau}$ is a vector of magnitude $\kappa\Delta s$ in the direction of the vector normal to $\boldsymbol{\tau}$. The **radius of curvature** *R* is defined as the reciprocal of the curvature; it is the radius of the circle which just touches the curve at *s*, Fig. 1.6.2.

Finally, the unit vector perpendicular to both the tangent vector and the principal normal vector is called the **unit binormal vector**:

$$\mathbf{b} = \boldsymbol{\tau} \times \mathbf{v} \qquad (1.6.6)$$

The planes defined by these vectors are shown in Fig. 1.6.3; they are called the **rectifying plane**, the **normal plane** and the **osculating plane**.

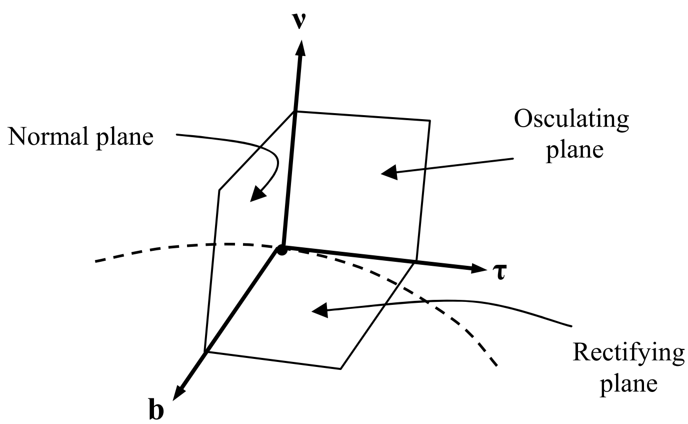


Figure 1.6.3: the unit tangent, principal normal and binormal vectors and associated planes

Rules of Differentiation

The derivative of a vector is also a vector and the usual rules of differentiation apply,

$$\begin{aligned} \frac{d}{dt}(\mathbf{u}+\mathbf{v}) &= \frac{d\mathbf{u}}{dt}+\frac{d\mathbf{v}}{dt} \\ \frac{d}{dt}(\alpha(t)\mathbf{v}) &= \alpha\frac{d\mathbf{v}}{dt}+\mathbf{v}\frac{d\alpha}{dt} \end{aligned} \tag{1.6.7}$$

Also, it is straight forward to show that {▲Problem 2}

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a}) = \mathbf{v}\cdot\frac{d\mathbf{a}}{dt}+\frac{d\mathbf{v}}{dt}\cdot\mathbf{a} \qquad \frac{d}{dt}(\mathbf{v}\times\mathbf{a}) = \mathbf{v}\times\frac{d\mathbf{a}}{dt}+\frac{d\mathbf{v}}{dt}\times\mathbf{a} \tag{1.6.8}$$

(The order of the terms in the cross-product expression is important here.)

1.6.3 Fields

In many applications of vector calculus, a scalar or vector can be associated with each point in space $\mathbf{x}$. In this case they are called **scalar** or **vector fields**. For example

$\theta(\mathbf{x})$	temperature	a scalar field (a scalar-valued function of position)
$\mathbf{v}(\mathbf{x})$	velocity	a vector field (a vector valued function of position)

These quantities will in general depend also on time, so that one writes $\theta(\mathbf{x},t)$ or $\mathbf{v}(\mathbf{x},t)$. Partial differentiation of scalar and vector fields with respect to the variable t is symbolised by $\partial/\partial t$. On the other hand, partial differentiation with respect to the coordinates is symbolised by $\partial/\partial x_i$. The notation can be made more compact by introducing the **subscript comma** to denote partial differentiation with respect to the coordinate variables, in which case $\phi_{,i} = \partial\phi/\partial x_i$, $u_{i,jk} = \partial^2 u_i/\partial x_j\partial x_k$, and so on.

1.6.4 The Gradient of a Scalar Field

Let $\phi(\mathbf{x})$ be a scalar field. The **gradient** of ϕ is a vector field defined by (see Fig. 1.6.4)

$$\boxed{\begin{aligned} \nabla\phi &= \frac{\partial\phi}{\partial x_1}\mathbf{e}_1+\frac{\partial\phi}{\partial x_2}\mathbf{e}_2+\frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \\ &= \frac{\partial\phi}{\partial x_i}\mathbf{e}_i \\ &\equiv \frac{\partial\phi}{\partial\mathbf{x}} \end{aligned}} \qquad \textbf{Gradient of a Scalar Field} \tag{1.6.9}$$

The gradient $\nabla\phi$ is of considerable importance because if one takes the dot product of $\nabla\phi$ with $d\mathbf{x}$, it gives the increment in ϕ:

$$\begin{aligned}\nabla\phi\cdot d\mathbf{x} &= \frac{\partial\phi}{\partial x_i}\mathbf{e}_i\cdot dx_j\mathbf{e}_j \\ &= \frac{\partial\phi}{\partial x_i}dx_i \\ &= d\phi \\ &= \phi(\mathbf{x}+d\mathbf{x})-\phi(d\mathbf{x})\end{aligned} \tag{1.6.10}$$

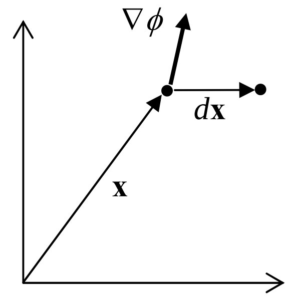


Figure 1.6.4: the gradient of a vector

If one writes $d\mathbf{x}$ as $|d\mathbf{x}|\mathbf{e} = dx\mathbf{e}$, where **e** is a unit vector in the direction of d**x**, then

$$\nabla\phi\cdot\mathbf{e} = \left(\frac{d\phi}{dx}\right)_{\text{in } \mathbf{e} \text{ direction}} \equiv \frac{d\phi}{d\mathbf{n}} \tag{1.6.11}$$

This quantity is called the **directional derivative** of ϕ, in the direction of **e**, and will be discussed further in §1.6.11.

The gradient of a scalar field is also called the **scalar gradient**, to distinguish it from the **vector gradient** (see later)[2], and is also denoted by

$$\text{grad}\,\phi \equiv \nabla\phi \tag{1.6.12}$$

Example (of the Gradient of a Scalar Field)

Consider a two-dimensional temperature field $\theta = x_1^2 + x_2^2$. Then

$$\nabla\theta = 2x_1\mathbf{e}_1 + 2x_2\mathbf{e}_2$$

For example, at $(1,0)$, $\theta = 1$, $\nabla\theta = 2\mathbf{e}_1$ and at $(1,1)$, $\theta = 2$, $\nabla\theta = 2\mathbf{e}_1 + 2\mathbf{e}_2$, Fig. 1.6.5. Note the following:

(i) $\nabla\theta$ points in the direction *normal* to the curve $\theta = \text{const.}$
(ii) the direction of *maximum* rate of change of θ is in the direction of $\nabla\theta$

---

[2] in this context, a *gradient* is a derivative with respect to a position vector, but the term gradient is used more generally than this, e.g. see §1.14

(iii) the direction of zero $d\theta$ is in the direction *perpendicular* to $\nabla\theta$

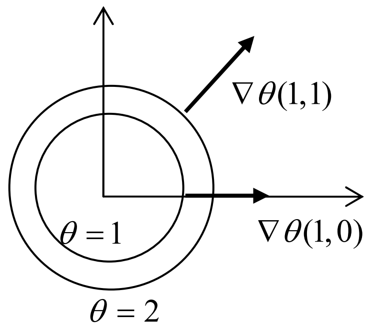


Figure 1.6.5: gradient of a temperature field

The curves $\theta(x_1, x_2) =$ const. are called **isotherms** (curves of constant temperature). In general, they are called **iso-curves** (or **iso-surfaces** in three dimensions).

■

Many physical laws are given in terms of the gradient of a scalar field. For example, **Fourier's law** of heat conduction relates the heat flux **q** (the rate at which heat flows through a surface of unit area[3]) to the temperature gradient through

$$\mathbf{q} = -k\nabla\theta \tag{1.6.13}$$

where k is the **thermal conductivity** of the material, so that heat flows along the direction normal to the isotherms.

The Normal to a Surface

In the above example, it was seen that $\nabla\theta$ points in the direction normal to the curve $\theta =$ const. Here it will be seen generally how and why the gradient can be used to obtain a normal vector to a surface.

Consider a surface represented by the scalar function $f(x_1, x_2, x_3) = c$, c a constant[4], and also a space curve C lying on the surface, defined by the position vector $\mathbf{r} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$. The components of **r** must satisfy the equation of the surface, so $f(x_1(t), x_2(t), x_3(t)) = c$. Differentiation gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} = 0$$

---

[3] the **flux** is the rate of flow of fluid, particles or energy through a given surface; the **flux density** is the flux per unit area but, as here, this is more commonly referred to simply as the flux

[4] a surface can be represented by the equation $f(x_1, x_2, x_3) = c$; for example, the expression $x_1^2 + x_2^2 + x_3^2 = 4$ is the equation for a sphere of radius 2 (with centre at the origin). Alternatively, the surface can be written in the form $x_3 = g(x_1, x_2)$, for example $x_3 = \sqrt{4 - x_1^2 - x_2^2}$

which is equivalent to the equation $\operatorname{grad} f \cdot (d\mathbf{r}/dt) = 0$ and, as seen in §1.6.2, $d\mathbf{r}/dt$ is a vector tangential to the surface. Thus $\operatorname{grad} f$ is normal to the tangent vector; $\operatorname{grad} f$ must be normal to all the tangents to all the curves through *p*, so it must be normal to the plane tangent to the surface.

Taylor's Series

Writing ϕ as a function of three variables (omitting time *t*), so that $\phi = \phi(x_1, x_2, x_3)$, then ϕ can be expanded in a three-dimensional Taylor's series:

$$\phi(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) = \phi(x_1, x_2, x_3) + \left\{ \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 \right\}$$
$$+ \frac{1}{2} \left\{ \frac{\partial^2 \phi}{\partial x_1^2} (dx_1)^2 + \cdots \right\}$$

Neglecting the higher order terms, this can be written as

$$\phi(\mathbf{x} + d\mathbf{x}) = \phi(\mathbf{x}) + \frac{\partial \phi}{\partial \mathbf{x}} \cdot d\mathbf{x}$$

which is equivalent to 1.6.9, 1.6.10.

1.6.5 The Nabla Operator

The symbolic vector operator ∇ is called the **Nabla operator**[5]. One can write this in component form as

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \mathbf{e}_i \frac{\partial}{\partial x_i} \qquad (1.6.14)$$

One can generalise the idea of the gradient of a scalar field by defining the dot product and the cross product of the vector operator ∇ with a vector field $(\bullet)$, according to the rules

$$\nabla \cdot (\bullet) = \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot (\bullet), \quad \nabla \times (\bullet) = \mathbf{e}_i \frac{\partial}{\partial x_i} \times (\bullet) \qquad (1.6.15)$$

The following terminology is used:

$$\begin{aligned} \operatorname{grad} \phi &= \nabla \phi \\ \operatorname{div} \mathbf{u} &= \nabla \cdot \mathbf{u} \\ \operatorname{curl} \mathbf{u} &= \nabla \times \mathbf{u} \end{aligned} \qquad (1.6.16)$$

[5] or **del** or the **Gradient operator**

These latter two are discussed in the following sections.

1.6.6 The Divergence of a Vector Field

From the definition (1.6.15), the **divergence** of a vector field $\mathbf{a}(\mathbf{x})$ is the scalar field

$$\boxed{\begin{aligned}\operatorname{div}\mathbf{a} = \nabla\cdot\mathbf{a} &= \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right)\cdot\left(a_j\mathbf{e}_j\right) = \frac{\partial a_i}{\partial x_i} \\ &= \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\end{aligned}}$$

Divergence of a Vector Field (1.6.17)

Differential Elements & Physical interpretations of the Divergence

Consider a flowing compressible[6] material with velocity field $\mathbf{v}(x_1, x_2, x_3)$. Consider now a **differential element** of this material, with dimensions $\Delta x_1, \Delta x_2, \Delta x_3$, with bottom left-hand corner at (x_1, x_2, x_3), fixed in space and through which the material flows[7], Fig. 1.6.6.

The component of the velocity in the x_1 direction, v_1, will vary over a face of the element but, *if the element is small*, the velocities will vary linearly as shown; only the components at the four corners of the face are shown for clarity.

Since [distance = time $\times$ velocity], the volume of material flowing through the right-hand face in time Δt is Δt times the "volume" bounded by the four corner velocities (between the right-hand face and the plane surface denoted by the dotted lines); it is straightforward to show that this volume is equal to the volume shown to the right, Fig. 1.6.6b, with constant velocity equal to the average velocity v_{ave}, which occurs at the centre of the face. Thus the volume of material flowing out is[8] $\Delta x_2 \Delta x_3 v_{ave} \Delta t$ and the **volume flux**, i.e. the *rate* of volume flow, is $\Delta x_2 \Delta x_3 v_{ave}$. Now

$$v_{ave} = v_1(x_1 + \Delta x_1, x_2 + \tfrac{1}{2}\Delta x_2, x_3 + \tfrac{1}{2}\Delta x_3)$$

Using a Taylor's series expansion, and neglecting higher order terms,

$$v_{ave} \approx v_1(x_1, x_2, x_3) + \Delta x_1 \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial v_1}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial v_1}{\partial x_3}$$

---

[6] that is, it can be compressed or expanded
[7] this type of fixed volume in space, used in analysis, is called a **control volume**
[8] the velocity will change by a small amount during the time interval Δt. One could use the average velocity in the calculation, i.e. $\frac{1}{2}\left(v_1(\mathbf{x},t) + v_1(\mathbf{x}, t+\Delta t)\right)$, but in the limit as $\Delta t \to 0$, this will reduce to $v_1(\mathbf{x},t)$

with the partial derivatives evaluated at (x_1, x_2, x_3), so the volume flux out is

$$\Delta x_2 \Delta x_3 \left\{ v_1(x_1, x_2, x_3) + \Delta x_1 \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2} \Delta x_2 \frac{\partial v_1}{\partial x_2} + \tfrac{1}{2} \Delta x_3 \frac{\partial v_1}{\partial x_3} \right\}$$

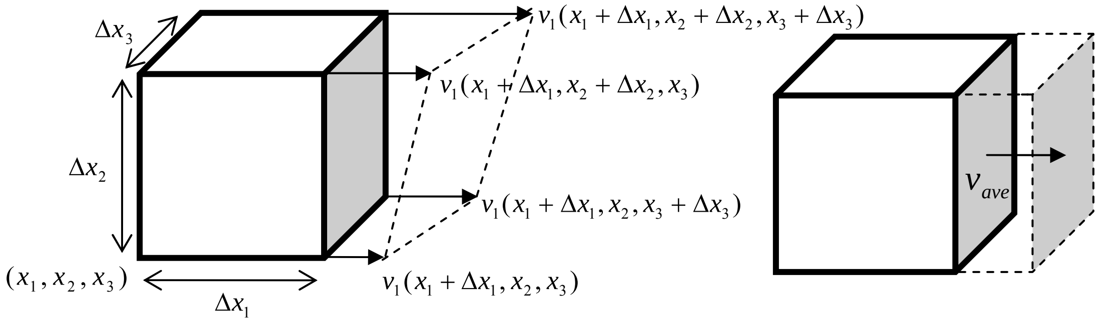


Figure 1.6.6: a differential element; (a) flow through a face, (b) volume of material flowing through the face

The net volume flux out (rate of volume flow out through the right-hand face minus the rate of volume flow in through the left-hand face) is then $\Delta x_1 \Delta x_2 \Delta x_3 (\partial v_1 / \partial x_1)$ and the net volume flux per unit volume is $\partial v_1 / \partial x_1$. Carrying out a similar calculation for the other two coordinate directions leads to

$$\textit{net unit volume flux out of an elemental volume}: \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \equiv \operatorname{div} \mathbf{v} \qquad (1.6.18)$$

which is the physical meaning of the divergence of the velocity field.

If $\operatorname{div}\mathbf{v} > 0$, there is a net flow out and the density of material is decreasing. On the other hand, if $\operatorname{div}\mathbf{v} = 0$, the inflow equals the outflow and the density remains constant – such a material is called **incompressible**[9]. A flow which is divergence free is said to be **isochoric**. A vector $\mathbf{v}$ for which $\operatorname{div}\mathbf{v} = 0$ is said to be **solenoidal**.

Notes:

- The above result holds only in the limit when the element shrinks to zero size – so that the extra terms in the Taylor series tend to zero and the velocity field varies in a linear fashion over a face
- consider the velocity at a fixed point in space, $\mathbf{v}(\mathbf{x}, t)$. The velocity at a later time, $\mathbf{v}(\mathbf{x}, t + \Delta t)$, actually gives the velocity of a different material particle. This is shown in Fig. 1.6.7 below: the material particles $1,2,3$ are moving through space and whereas $\mathbf{v}(\mathbf{x}, t)$ represents the velocity of particle 2, $\mathbf{v}(\mathbf{x}, t + \Delta t)$ now represents the velocity of particle 1, which has moved into position $\mathbf{x}$. This point is important in the consideration of the kinematics of materials, to be discussed in Chapter 2

[9] a **liquid**, such as water, is a material which is incompressible

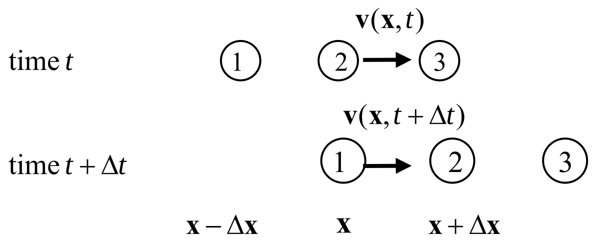


Figure 1.6.7: moving material particles

Another example would be the divergence of the heat flux vector **q**. This time suppose also that there is some generator of heat inside the element (a **source**), generating at a rate of r per unit volume, r being a scalar field. Again, assuming the element to be small, one takes r to be acting at the mid-point of the element, and one considers $r(x_1 + \frac{1}{2}\Delta x_1, \cdots)$. Assume a **steady-state** heat flow, so that the (heat) energy within the elemental volume remains constant with time – the law of balance of (heat) energy then requires that the net flow of heat out must equal the heat generated within, so

$$\begin{aligned}\Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial q_1}{\partial x_1} + \Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial q_2}{\partial x_2} + \Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial q_3}{\partial x_3} \\ = \Delta x_1 \Delta x_2 \Delta x_3 \left\{ r(x_1, x_2, x_3) + \tfrac{1}{2}\Delta x_1 \frac{\partial r}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial r}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial r}{\partial x_3} \right\}\end{aligned}$$

Dividing through by $\Delta x_1 \Delta x_2 \Delta x_3$ and taking the limit as $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$, one obtains

$$\mathrm{div}\mathbf{q} = r \tag{1.6.19}$$

Here, the divergence of the heat flux vector field can be interpreted as the heat generated (or absorbed) per unit volume per unit time in a temperature field. If the divergence is zero, there is no heat being generated (or absorbed) and the heat leaving the element is equal to the heat entering it.

1.6.7 The Laplacian

Combining Fourier's law of heat conduction (1.6.13), $\mathbf{q} = -k\nabla\theta$, with the energy balance equation (1.6.19), $\mathrm{div}\mathbf{q} = r$, and assuming the conductivity is constant, leads to $-k\,\nabla\cdot\nabla\theta = r$. Now

$$\begin{aligned}\nabla\cdot\nabla\theta &= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \left(\frac{\partial \theta}{\partial x_j} \mathbf{e}_j \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial \theta}{\partial x_j} \right) \delta_{ij} = \frac{\partial^2 \theta}{\partial x_i^2} \\ &= \frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2} + \frac{\partial^2 \theta}{\partial x_3^2}\end{aligned} \tag{1.6.20}$$

This expression is called the **Laplacian** of θ. By introducing the Laplacian operator $\nabla^2 \equiv \nabla \cdot \nabla$, one has

$$\nabla^2 \theta = -\frac{r}{k} \tag{1.6.21}$$

This equation governs the steady state heat flow for constant conductivity. In general, the equation $\nabla^2 \phi = a$ is called **Poisson's equation**. When there are no heat sources (or sinks), one has **Laplace's equation**, $\nabla^2 \theta = 0$. Laplace's and Poisson's equation arise in many other mathematical models in mechanics, electromagnetism, etc.

1.6.8 The Curl of a Vector Field

From the definition 1.6.15 and 1.6.14, the **curl** of a vector field $\mathbf{a}(\mathbf{x})$ is the vector field

$$\boxed{\begin{aligned}\operatorname{curl}\mathbf{a} = \nabla \times \mathbf{a} &= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(a_j \mathbf{e}_j\right) \\ &= \frac{\partial a_j}{\partial x_i} \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \mathbf{e}_k\end{aligned}} \qquad \textbf{Curl of a Vector Field} \tag{1.6.22}$$

It can also be expressed in the form

$$\begin{aligned}\operatorname{curl}\mathbf{a} = \nabla \times \mathbf{a} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \mathbf{e}_k = \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j} \mathbf{e}_i = \varepsilon_{ijk} \frac{\partial a_i}{\partial x_k} \mathbf{e}_j\end{aligned} \tag{1.6.23}$$

Note: the divergence and curl of a vector field are independent of any coordinate system (for example, the divergence of a vector and the length and direction of $\operatorname{curl}\mathbf{a}$ are independent of the coordinate system in use) – these will be re-defined without reference to any particular coordinate system when discussing tensors (see §1.14).

Physical interpretation of the Curl

Consider a particle with position vector **r** and moving with velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, that is, with an angular velocity ω about an axis in the direction of $\boldsymbol{\omega}$. Then {▲Problem 7}

$$\operatorname{curl}\mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega} \tag{1.6.24}$$

Thus the curl of a vector field is associated with rotational properties. In fact, if **v** is the velocity of a moving fluid, then a small paddle wheel placed in the fluid would tend to rotate in regions where $\operatorname{curl}\mathbf{v} \neq 0$, in which case the velocity field **v** is called a **vortex**

field. The paddle wheel would remain stationary in regions where $\text{curl}\,\mathbf{v} = 0$, in which case the velocity field $\mathbf{v}$ is called **irrotational**.

1.6.9 Identities

Here are some important identities of vector calculus {▲Problem 8}:

$$\begin{aligned} \text{grad}(\phi + \psi) &= \text{grad}\,\phi + \text{grad}\,\psi \\ \text{div}(\mathbf{u} + \mathbf{v}) &= \text{div}\,\mathbf{u} + \text{div}\,\mathbf{v} \\ \text{curl}(\mathbf{u} + \mathbf{v}) &= \text{curl}\,\mathbf{u} + \text{curl}\,\mathbf{v} \end{aligned} \tag{1.6.25}$$

$$\begin{aligned} \text{grad}(\phi\psi) &= \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi \\ \text{div}(\phi\mathbf{u}) &= \phi\,\text{div}\,\mathbf{u} + \text{grad}\,\phi \cdot \mathbf{u} \\ \text{curl}(\phi\mathbf{u}) &= \phi\,\text{curl}\,\mathbf{u} + \text{grad}\,\phi \times \mathbf{u} \\ \text{div}(\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v} \\ \text{curl}(\text{grad}\,\phi) &= \mathbf{o} \\ \text{div}(\text{curl}\,\mathbf{u}) &= 0 \\ \text{div}(\lambda\,\text{grad}\,\phi) &= \lambda\nabla^2\phi + \text{grad}\,\lambda \cdot \text{grad}\,\phi \end{aligned} \tag{1.6.26}$$

1.6.10 Cylindrical and Spherical Coordinates

Cartesian coordinates have been used exclusively up to this point. In many practical problems, it is easier to carry out an analysis in terms of cylindrical or spherical coordinates. Differentiation in these coordinate systems is discussed in what follows[10].

Cylindrical Coordinates

Cartesian and cylindrical coordinates are related through (see Fig. 1.6.8)

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta\,, & \theta &= \tan^{-1}(y/x) \\ z &= z & z &= z \end{aligned} \tag{1.6.27}$$

Then the Cartesian partial derivatives become

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned} \tag{1.6.28}$$

[10] this section also serves as an introduction to the more general topic of **Curvilinear Coordinates** covered in §1.16-§1.19

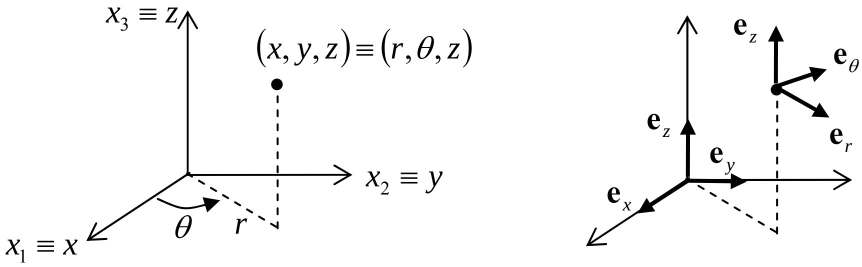


Figure 1.6.8: cylindrical coordinates

The base vectors are related through

$$\begin{aligned} \mathbf{e}_x &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \qquad & \mathbf{e}_r &= \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta + \mathbf{e}_\theta \cos\theta, \qquad & \mathbf{e}_\theta &= -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta \\ \mathbf{e}_z &= \mathbf{e}_z & \mathbf{e}_z &= \mathbf{e}_z \end{aligned} \tag{1.6.29}$$

so that from Eqn. 1.6.14, after some algebra, the Nabla operator in cylindrical coordinates reads as {▲Problem 9}

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{1.6.30}$$

which allows one to take the gradient of a scalar field in cylindrical coordinates:

$$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\mathbf{e}_z \tag{1.6.31}$$

Cartesian base vectors are independent of position. However, the cylindrical base vectors, although they are always of unit magnitude, change direction with position. In particular, the directions of the base vectors $\mathbf{e}_r, \mathbf{e}_\theta$ depend on θ, and so these base vectors have derivatives with respect to θ: from Eqn. 1.6.29,

$$\begin{aligned} \frac{\partial}{\partial\theta}\mathbf{e}_r &= \mathbf{e}_\theta \\ \frac{\partial}{\partial\theta}\mathbf{e}_\theta &= -\mathbf{e}_r \end{aligned} \tag{1.6.32}$$

with all other derivatives of the base vectors with respect to r, θ, z equal to zero.

The divergence can now be evaluated:

$$\nabla \cdot \mathbf{v} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right)$$
$$= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \tag{1.6.33}$$

Similarly the curl of a vector and the Laplacian of a scalar are {▲Problem 10}

$$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right)\mathbf{e}_\theta + \left[\frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta} \right) \right]\mathbf{e}_z$$
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{1.6.34}$$

Spherical Coordinates

Cartesian and spherical coordinates are related through (see Fig. 1.6.9)

$$\begin{aligned} x &= r \sin\theta \cos\phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin\theta \sin\phi\,, & \theta &= \tan^{-1}\left(\sqrt{x^2 + y^2} / z \right) \\ z &= r \cos\theta & \phi &= \tan^{-1}(y/x) \end{aligned} \tag{1.6.35}$$

and the base vectors are related through

$$\begin{aligned} \mathbf{e}_x &= \mathbf{e}_r \sin\theta\cos\phi + \mathbf{e}_\theta \cos\theta\cos\phi - \mathbf{e}_\phi \sin\phi \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta\sin\phi + \mathbf{e}_\theta \cos\theta\sin\phi + \mathbf{e}_\phi \cos\phi \\ \mathbf{e}_z &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \\ \\ \mathbf{e}_r &= \mathbf{e}_x \sin\theta\cos\phi + \mathbf{e}_y \sin\theta\sin\phi + \mathbf{e}_z \cos\theta \\ \mathbf{e}_\theta &= \mathbf{e}_x \cos\theta\cos\phi + \mathbf{e}_y \cos\theta\sin\phi - \mathbf{e}_z \sin\theta \\ \mathbf{e}_\phi &= -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi \end{aligned} \tag{1.6.36}$$

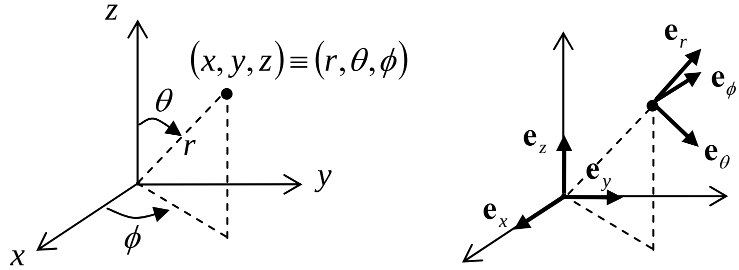


Figure 1.6.9: spherical coordinates

In this case the non-zero derivatives of the base vectors are

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathbf{e}_r &= \mathbf{e}_\theta \\ \frac{\partial}{\partial\theta}\mathbf{e}_\theta &= -\mathbf{e}_r\end{aligned}, \qquad \begin{aligned}\frac{\partial}{\partial\phi}\mathbf{e}_r &= \sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial\phi}\mathbf{e}_\theta &= \cos\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial\phi}\mathbf{e}_\phi &= -\sin\theta\mathbf{e}_r - \cos\theta\mathbf{e}_\theta\end{aligned} \tag{1.6.37}$$

and it can then be shown that {▲Problem 11}

$$\begin{aligned}\nabla\varphi &= \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{e}_\phi \\ \nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \\ \nabla^2\varphi &= \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\end{aligned} \tag{1.6.38}$$

1.6.11 The Directional Derivative

Consider a function $\phi(\mathbf{x})$. The directional derivative of φ in the direction of some vector $\mathbf{w}$ is the change in φ in that direction. Now the difference between its values at position $\mathbf{x}$ and $\mathbf{x}+\mathbf{w}$ is, Fig. 1.6.10,

$$d\phi = \phi(\mathbf{x}+\mathbf{w}) - \phi(\mathbf{x}) \tag{1.6.39}$$

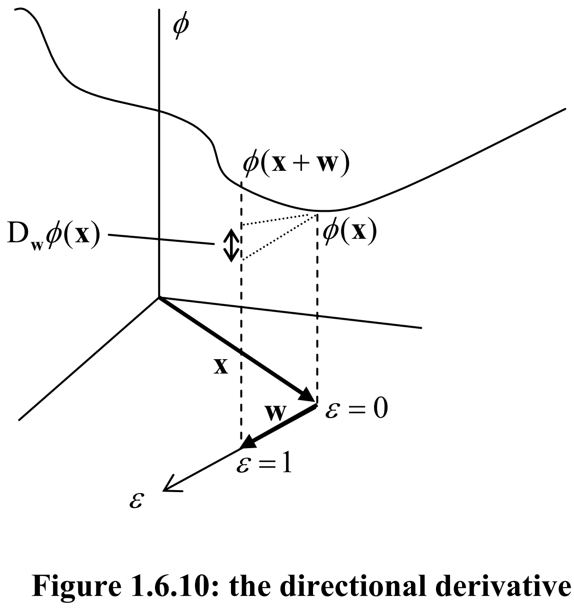


Figure 1.6.10: the directional derivative

An approximation to $d\phi$ can be obtained by introducing a parameter ε and by considering the function $\phi(\mathbf{x}+\varepsilon\mathbf{w})$; one has $\phi(\mathbf{x}+\varepsilon\mathbf{w})_{\varepsilon=0} = \phi(\mathbf{x})$ and $\phi(\mathbf{x}+\varepsilon\mathbf{w})_{\varepsilon=1} = \phi(\mathbf{x}+\mathbf{w})$.

If one treats ϕ as a function of ε, a Taylor's series about $\varepsilon = 0$ gives

$$\phi(\varepsilon) = \phi(0) + \varepsilon \left.\frac{d\phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left.\frac{d^2\phi(\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} + \cdots$$

or, writing it as a function of $\mathbf{x}+\varepsilon\mathbf{w}$,

$$\phi(\mathbf{x}+\varepsilon\mathbf{w}) = \phi(\mathbf{x}) + \varepsilon \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{x}+\varepsilon\mathbf{w}) + \cdots$$

By setting $\varepsilon = 1$, the derivative here can be seen to be a linear approximation to the increment $d\phi$, Eqn. 1.6.39. This is defined as the **directional derivative** of the function $\phi(\mathbf{x})$ at the point **x** in the direction of **w**, and is denoted by

$$\boxed{\partial_{\mathbf{x}}\phi[\mathbf{w}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{x}+\varepsilon\mathbf{w})} \qquad \textbf{The Directional Derivative} \qquad (1.6.40)$$

The directional derivative is also written as $\mathrm{D}_{\mathbf{w}}\phi(\mathbf{x})$.

The power of the directional derivative as defined by Eqn. 1.6.40 is its generality, as seen in the following example.

Example (the Directional Derivative of the Determinant)

Consider the directional derivative of the determinant of the 2×2 matrix **A**, in the direction of a second matrix **T** (the word "direction" is obviously used loosely in this context). One has

$$\begin{aligned}
\partial_{\mathbf{A}}(\det\mathbf{A})[\mathbf{T}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \det(\mathbf{A}+\varepsilon\mathbf{T}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[(A_{11}+\varepsilon T_{11})(A_{22}+\varepsilon T_{22}) - (A_{12}+\varepsilon T_{12})(A_{21}+\varepsilon T_{21})\right] \\
&= A_{11}T_{22} + A_{22}T_{11} - A_{12}T_{21} - A_{21}T_{12}
\end{aligned}$$

■

The Directional Derivative and The Gradient

Consider a scalar-valued function ϕ of a vector **z**. Let **z** be a function of a parameter ε, $\phi \equiv \phi(z_1(\varepsilon), z_2(\varepsilon), z_3(\varepsilon))$. Then

$$\frac{d\phi}{d\varepsilon} = \frac{\partial \phi}{\partial z_i}\frac{dz_i}{d\varepsilon} = \frac{\partial \phi}{\partial \mathbf{z}} \cdot \frac{d\mathbf{z}}{d\varepsilon}$$

Thus, with $\mathbf{z} = \mathbf{x} + \varepsilon \mathbf{w}$,

$$\partial_{\mathbf{x}}\phi[\mathbf{w}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{z}(\varepsilon)) = \left(\frac{\partial \phi}{\partial \mathbf{z}} \cdot \frac{d\mathbf{z}}{d\varepsilon}\right)_{\varepsilon=0} = \frac{\partial \phi}{\partial \mathbf{x}} \cdot \mathbf{w} \tag{1.6.41}$$

which can be compared with Eqn. 1.6.11. Note that for Eqns. 1.6.11 and 1.6.41 to be consistent definitions of the directional derivative, **w** here should be a *unit* vector.

1.6.12 Formal Treatment of Vector Calculus

The calculus of vectors is now treated more formally in what follows, following on from the introductory section in §1.2. Consider a vector **h**, an element of the Euclidean vector space *E*, $\mathbf{h} \in E$. In order to be able to speak of limits as elements become "small" or "close" to each other in this space, one requires a norm. Here, take the standard Euclidean norm on *E*, Eqn. 1.2.8,

$$\|\mathbf{h}\| \equiv \sqrt{\langle \mathbf{h}, \mathbf{h} \rangle} = \sqrt{\mathbf{h} \cdot \mathbf{h}} \tag{1.6.42}$$

Consider next a scalar function $f : E \to R$. If there is a constant $M > 0$ such that $|f(\mathbf{h})| \le M\|\mathbf{h}\|$ as $\mathbf{h} \to \mathbf{o}$, then one writes

$$f(\mathbf{h}) = O(\|\mathbf{h}\|) \quad \text{as} \quad \mathbf{h} \to \mathbf{o} \tag{1.6.43}$$

This is called the **Big Oh** (or **Landau**) notation. Eqn. 1.6.43 states that $|f(\mathbf{h})|$ goes to zero at least as fast as $\|\mathbf{h}\|$. An expression such as

$$f(\mathbf{h}) = g(\mathbf{h}) + O(\|\mathbf{h}\|) \tag{1.6.44}$$

then means that $|f(\mathbf{h}) - g(\mathbf{h})|$ is smaller than $\|\mathbf{h}\|$ for $\mathbf{h}$ sufficiently close to **o**.

Similarly, if

$$\frac{f(\mathbf{h})}{\|\mathbf{h}\|} \to 0 \quad \text{as} \quad \mathbf{h} \to \mathbf{o} \tag{1.6.45}$$

then one writes $f(\mathbf{h}) = o(\|\mathbf{h}\|)$ as $\mathbf{h} \to \mathbf{o}$. This implies that $|f(\mathbf{h})|$ goes to zero faster than $\|\mathbf{h}\|$.

A **field** is a function which is defined in a Euclidean (point) space E^3. A **scalar field** is then a function $f : E^3 \to R$. A scalar field is **differentiable** at a point $\mathbf{x} \in E^3$ if there exists a vector $Df(\mathbf{x}) \in E$ such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + Df(\mathbf{x}) \cdot \mathbf{h} + o(\|\mathbf{h}\|) \quad \text{for all} \quad \mathbf{h} \in E \tag{1.6.46}$$

In that case, the vector $Df(\mathbf{x})$ is called the **derivative** (or **gradient**) of *f* at **x** (and is given the symbol $\nabla f(\mathbf{x})$).

Now setting $\mathbf{h} = \varepsilon \mathbf{w}$ in 1.6.46, where $\mathbf{w} \in E$ is a unit vector, dividing through by ε and taking the limit as $\varepsilon \to 0$, one has the equivalent statement

$$\nabla f(\mathbf{x}) \cdot \mathbf{w} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\mathbf{x}+\varepsilon\mathbf{w}) \quad \text{for all} \quad \mathbf{w} \in E \tag{1.6.47}$$

which is 1.6.41. In other words, for the derivative to exist, the scalar field must have a directional derivative in all directions at **x**.

Using the chain rule as in §1.6.11, Eqn. 1.6.47 can be expressed in terms of the Cartesian basis $\{\mathbf{e}_i\}$,

$$\nabla f(\mathbf{x}) \cdot \mathbf{w} = \frac{\partial f}{\partial x_i} w_i = \frac{\partial f}{\partial x_i} \mathbf{e}_i \cdot w_j \mathbf{e}_j \tag{1.6.48}$$

This must be true for all **w** and so, in a Cartesian basis,

$$\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x_i} \mathbf{e}_i \tag{1.6.49}$$

which is Eqn. 1.6.9.

1.6.13 Problems

1. A particle moves along a curve in space defined by
$$\mathbf{r} = (t^3 - 4t)\mathbf{e}_1 + (t^2 + 4t)\mathbf{e}_2 + (8t^2 - 3t^3)\mathbf{e}_3$$
Here, *t* is time. Find
(i) a unit tangent vector at $t = 2$
(ii) the magnitudes of the tangential and normal components of acceleration at $t = 2$
2. Use the index notation (1.3.12) to show that $\frac{d}{dt}(\mathbf{v} \times \mathbf{a}) = \mathbf{v} \times \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt} \times \mathbf{a}$. Verify this result for $\mathbf{v} = 3t\mathbf{e}_1 - t^2\mathbf{e}_3$, $\mathbf{a} = t^2\mathbf{e}_1 + t\mathbf{e}_2$. [Note: the permutation symbol and the unit vectors are independent of *t*; the components of the vectors are scalar functions of *t* which can be differentiated in the usual way, for example by using the product rule of differentiation.]

3. The density distribution throughout a material is given by $\rho = 1 + \mathbf{x} \cdot \mathbf{x}$.
   (i) what sort of function is this?
   (ii) the density is given in symbolic notation - write it in index notation
   (iii) evaluate the gradient of ρ
   (iv) give a unit vector in the direction in which the density is increasing the most
   (v) give a unit vector in *any* direction in which the density is not increasing
   (vi) take any unit vector other than the base vectors and the other vectors you used above and calculate $d\rho / dx$ in the direction of this unit vector
   (vii) evaluate and sketch all these quantities for the point (2,1).
   In parts (iii-iv), give your answer in (a) symbolic, (b) index, and (c) full notation.
4. Consider the scalar field defined by $\phi = x^2 + 3yx + 2z$.
   (i) find the unit normal to the surface of constant ϕ at the origin (0,0,0)
   (ii) what is the maximum value of the directional derivative of ϕ at the origin?
   (iii) evaluate $d\phi / dx$ at the origin if $d\mathbf{x} = ds(\mathbf{e}_1 + \mathbf{e}_3)$.
5. If $\mathbf{u} = x_1 x_2 x_3 \mathbf{e}_1 + x_1 x_2 \mathbf{e}_2 + x_1 \mathbf{e}_3$, determine $\operatorname{div}\mathbf{u}$ and $\operatorname{curl}\mathbf{u}$.
6. Determine the constant *a* so that the vector
$$\mathbf{v} = (x_1 + 3x_2)\mathbf{e}_1 + (x_2 - 2x_3)\mathbf{e}_2 + (x_1 + a x_3)\mathbf{e}_3$$
   is solenoidal.
7. Show that $\operatorname{curl}\mathbf{v} = 2\boldsymbol{\omega}$ (see also Problem 9 in §1.1).
8. Verify the identities (1.6.25-26).
9. Use (1.6.14) to derive the Nabla operator in cylindrical coordinates (1.6.30).
10. Derive Eqn. (1.6.34), the curl of a vector and the Laplacian of a scalar in the cylindrical coordinates.
11. Derive (1.6.38), the gradient, divergence and Laplacian in spherical coordinates.
12. Show that the directional derivative $\mathrm{D}_{\mathbf{v}}\phi(\mathbf{u})$ of the scalar-valued function of a vector $\phi(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u}$, in the direction $\mathbf{v}$, is $2\mathbf{u} \cdot \mathbf{v}$.
13. Show that the directional derivative of the functional
$$U(v(x)) = \frac{1}{2}\int_0^l EI\left(\frac{d^2 v}{dx^2}\right)^2 dx - \int_0^l p(x)v(x)dx$$
   in the direction of $\omega(x)$ is given by
$$\int_0^l EI \frac{d^2 v(x)}{dx^2}\frac{d^2 \omega(x)}{dx^2} dx - \int_0^l p(x)\omega(x)dx .$$

1.7 Vector Calculus 2 - Integration

1.7.1 Ordinary Integrals of a Vector

A vector can be integrated in the ordinary way to produce another vector, for example

$$\int_1^2 \left\{ \left(t - t^2\right)\mathbf{e}_1 + 2t^2\mathbf{e}_2 - 3\mathbf{e}_3 \right\} dt = -\frac{5}{6}\mathbf{e}_1 + \frac{15}{2}\mathbf{e}_2 - 3\mathbf{e}_3$$

1.7.2 Line Integrals

Discussed here is the notion of a definite integral involving a vector function that generates a scalar.

Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ be a position vector tracing out the curve C between the points p_1 and p_2. Let **f** be a vector field. Then

$$\int_{p_1}^{p_2} \mathbf{f} \cdot d\mathbf{x} = \int_C \mathbf{f} \cdot d\mathbf{x} = \int_C \left\{ f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \right\}$$

is an example of a line integral.

Example (of a Line Integral)

A particle moves along a path C from the point $(0,0,0)$ to $(1,1,1)$, where C is the straight line joining the points, Fig. 1.7.1. The particle moves in a force field given by

$$\mathbf{f} = \left(3x_1^2 + 6x_2\right)\mathbf{e}_1 - 14x_2x_3\mathbf{e}_2 + 20x_1x_3^2\mathbf{e}_3$$

What is the work done on the particle?

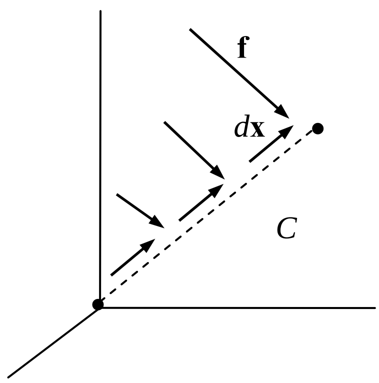


Figure 1.7.1: a particle moving in a force field

Solution
The work done is

$$W = \int_C \mathbf{f} \cdot d\mathbf{x} = \int_C \left\{ \left(3x_1^2 + 6x_2\right)dx_1 - 14x_2x_3dx_2 + 20x_1x_3^2dx_3 \right\}$$

The straight line can be written in the parametric form $x_1 = t, x_2 = t, x_3 = t$, so that

$$W = \int_0^1 \left(20t^3 - 11t^2 + 6t\right)dt = \frac{13}{3} \quad \text{or} \quad W = \int_C \mathbf{f} \cdot \frac{d\mathbf{x}}{dt} dt = \int_C \mathbf{f} \cdot \left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right)dt = \frac{13}{3}$$

■

If C is a closed curve, i.e. a loop, the line integral is often denoted $\oint_C \mathbf{v} \cdot d\mathbf{x}$.

Note: in fluid mechanics and aerodynamics, when **v** is the velocity field, this integral $\oint_C \mathbf{v} \cdot d\mathbf{x}$ is called the **circulation** of **v** about C.

1.7.3 Conservative Fields

If for a vector **f** one can find a scalar ϕ such that

$$\mathbf{f} = \nabla\phi \tag{1.7.1}$$

then

(1) $\int_{p_1}^{p_2} \mathbf{f} \cdot d\mathbf{x}$ is independent of the path C joining p_1 and p_2

(2) $\oint_C \mathbf{f} \cdot d\mathbf{x} = 0$ around any closed curve C

In such a case, **f** is called a **conservative vector field** and ϕ is its **scalar potential**[1]. For example, the work done by a conservative force field **f** is

$$\int_{p_1}^{p_2} \mathbf{f} \cdot d\mathbf{x} = \int_{p_1}^{p_2} \nabla\phi \cdot d\mathbf{x} = \int_{p_1}^{p_2} \frac{\partial\phi}{\partial x_i} dx_i = \int_{p_1}^{p_2} d\phi = \phi(p_2) - \phi(p_1)$$

which clearly depends only on the values at the end-points p_1 and p_2, and not on the path taken between them.

It can be shown that a vector **f** is conservative if and only if $\text{curl}\mathbf{f} = \mathbf{o}$ {▲Problem 3}.

[1] in general, of course, there does not exist a scalar field ϕ such that $\mathbf{f} = \nabla\phi$; this is not surprising since a vector field has three scalar components whereas $\nabla\phi$ is determined from just one

Example (of a Conservative Force Field)

The gravitational force field $\mathbf{f} = -mg\mathbf{e}_3$ is an example of a conservative vector field. Clearly, $\text{curl}\mathbf{f} = \mathbf{o}$, and the gravitational scalar potential is $\phi = -mgx_3$:

$$W = -\int_{p_1}^{p_2} mg\mathbf{e}_3 \cdot d\mathbf{x} = -mg\int_{p_1}^{p_2} dx_3 = -mg[x_3(p_2) - x_3(p_1)] = \phi(p_2) - \phi(p_1)$$

■

Example (of a Conservative Force Field)

Consider the force field

$$\mathbf{f} = (2x_1x_2 + x_3^3)\mathbf{e}_1 + x_1^2\mathbf{e}_2 + 3x_1x_3^2\mathbf{e}_3$$

Show that it is a conservative force field, find its scalar potential and find the work done in moving a particle in this field from $(1,-2,1)$ to $(3,1,4)$.

<u>Solution</u>

One has

$$\text{curl}\mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ 2x_1x_2 + x_3^3 & x_1^2 & 3x_1x_3^2 \end{vmatrix} = \mathbf{o}$$

so the field is conservative.

To determine the scalar potential, let

$$f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3 = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3.$$

Equating coefficients and integrating leads to

$$\begin{aligned} \phi &= x_1^2x_2 + x_1x_3^3 + p(x_2, x_3) \\ \phi &= x_1^2x_2 \qquad\qquad + q(x_1, x_3) \\ \phi &= \qquad\quad\; x_1x_3^3 + r(x_1, x_1) \end{aligned}$$

which agree if one chooses $p = 0,\ q = x_1x_3^3,\ r = x_1^2x_2$, so that $\phi = x_1^2x_2 + x_1x_3^3$, to which may be added a constant.

The work done is

$$W = \phi(3,1,4) - \phi(1,-2,1) = 202$$

■

Helmholtz Theory

As mentioned, a conservative vector field which is irrotational, i.e. $\mathbf{f} = \nabla\phi$, implies $\nabla \times \mathbf{f} = \mathbf{o}$, and *vice versa*. Similarly, it can be shown that if one can find a vector **a** such that $\mathbf{f} = \nabla \times \mathbf{a}$, where **a** is called the **vector potential**, then **f** is solenoidal, i.e. $\nabla \cdot \mathbf{f} = 0$ {▲Problem 4}.

Helmholtz showed that a vector can always be represented in terms of a scalar potential ϕ and a vector potential **a**:[2]

Type of Vector	Condition	Representation
General		$\mathbf{f} = \nabla\phi + \nabla \times \mathbf{a}$
Irrotational (conservative)	$\nabla \times \mathbf{f} = \mathbf{o}$	$\mathbf{f} = \nabla\phi$
Solenoidal	$\nabla \cdot \mathbf{f} = 0$	$\mathbf{f} = \nabla \times \mathbf{a}$

1.7.4 Double Integrals

The most elementary type of two-dimensional integral is that over a plane region. For example, consider the integral over a region *R* in the $x_1 - x_2$ plane, Fig. 1.7.2. The integral

$$\iint_R dx_1 dx_2$$

then gives the area of *R* and, just as the one dimensional integral of a function gives the area under the curve, the integral

$$\iint_R f(x_1, x_2) dx_1 dx_2$$

gives the volume under the (in general, curved) surface $x_3 = f(x_1, x_2)$. These integrals are called **double integrals**.

[2] this decomposition can be made unique by requiring that $\mathbf{f} \to \mathbf{0}$ as $\mathbf{x} \to \infty$; in general, if one is given **f**, then ϕ and **a** can be obtained by solving a number of differential equations

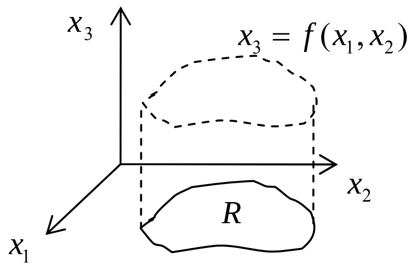


Figure 1.7.2: integration over a region

Change of variables in Double Integrals

To evaluate integrals of the type $\iint_R f(x_1, x_2)dx_1 dx_2$, it is often convenient to make a change of variable. To do this, one must find an elemental surface area in terms of the new variables, t_1, t_2 say, equivalent to that in the x_1, x_2 coordinate system, $dS = dx_1 dx_2$.

The region *R* over which the integration takes place is the plane surface $g(x_1, x_2) = 0$. Just as a curve can be represented by a position vector of one single parameter *t* (*cf.* §1.6.2), this surface can be represented by a position vector with two parameters[3], t_1 and t_2:

$$\mathbf{x} = x_1(t_1, t_2)\mathbf{e}_1 + x_2(t_1, t_2)\mathbf{e}_2$$

Parameterising the plane surface in this way, one can calculate the element of surface dS in terms of t_1, t_2 by considering curves of constant t_1, t_2, as shown in Fig. 1.7.3. The vectors bounding the element are

$$d\mathbf{x}^{(1)} = d\mathbf{x}\big|_{t_2 \text{ const}} = \frac{\partial \mathbf{x}}{\partial t_1} dt_1, \qquad d\mathbf{x}^{(2)} = d\mathbf{x}\big|_{t_1 \text{ const}} = \frac{\partial \mathbf{x}}{\partial t_2} dt_2 \tag{1.7.2}$$

so the area of the element is given by

$$dS = \left| d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \right| = \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| dt_1 dt_2 = J\, dt_1 dt \tag{1.7.3}$$

where *J* is the **Jacobian** of the transformation,

---

[3] for example, the unit circle $x_1^2 + x_2^2 - 1 = 0$ can be represented by $\mathbf{x} = t_1 \cos t_2 \mathbf{e}_1 + t_1 \sin t_2 \mathbf{e}_2$, $0<t_1\leq 1$, $0 < t_2 \leq 2\pi$ (t_1, t_2 being in this case the polar coordinates r, θ, respectively)

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} \\ \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} \quad \text{or} \quad J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} \\ \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} \tag{1.7.4}$$

The Jacobian is also often written using the notation

$$dx_1 dx_2 = J dt_1 dt_2, \qquad J = \left| \frac{\partial(x_1, x_2)}{\partial(t_1, t_2)} \right|$$

The integral can now be written as

$$\iint_R f(t_1, t_2) J dt_1 dt_2$$

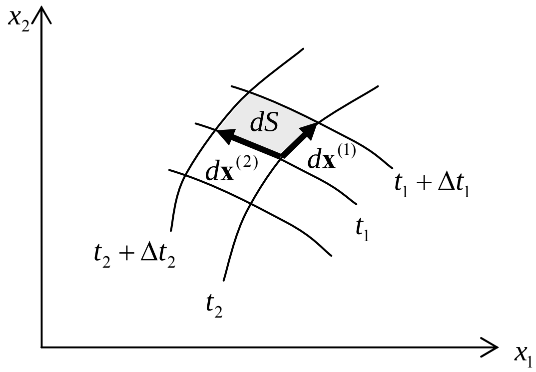


Figure 1.7.3: a surface element

Example

Consider a region R, the quarter unit-circle in the first quadrant, $0 \le x_2 \le \sqrt{1 - x_1^2}$, $0 \le x_1 \le 1$. The moment of inertia about the x_1 – axis is defined by

$$I_{x_1} \equiv \iint_R x_2^2 dx_1 dx_2$$

Transform the integral into the new coordinate system t_1, t_2 by making the substitutions[4] $x_1 = t_1 \cos t_2$, $x_2 = t_1 \sin t_2$. Then

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} \\ \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} = \begin{vmatrix} \cos t_2 & -t_1 \sin t_2 \\ \sin t_2 & t_1 \cos t_2 \end{vmatrix} = t_1$$

[4] these are the polar coordinates, t_1, t_2 equal to r, θ, respectively

so

$$I_{x_1} = \int_0^{\pi/2}\int_0^1 t_1^3 \sin^2 t_2 dt_1 dt_2 = \frac{\pi}{16}$$

■

1.7.5 Surface Integrals

Up to now, double integrals over a plane region have been considered. In what follows, consideration is given to integrals over more complex, curved, surfaces in space, such as the surface of a sphere.

Surfaces

Again, a curved surface can be parameterized by t_1, t_2, now by the position vector

$$\mathbf{x} = x_1(t_1, t_2)\mathbf{e}_1 + x_2(t_1, t_2)\mathbf{e}_2 + x_3(t_1, t_2)\mathbf{e}_3$$

One can generate a curve *C* on the surface *S* by taking $t_1 = t_1(s)$, $t_2 = t_2(s)$ so that *C* has position vector, Fig. 1.7.4,

$$\mathbf{x}(s) = \mathbf{x}(t_1(s), t_2(s))$$

A vector tangent to *C* at a point *p* on *S* is, from Eqn. 1.6.3,

$$\frac{d\mathbf{x}}{ds} = \frac{\partial \mathbf{x}}{\partial t_1}\frac{dt_1}{ds} + \frac{\partial \mathbf{x}}{\partial t_2}\frac{dt_2}{ds}$$

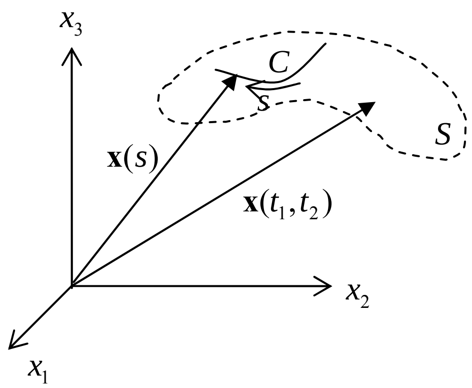


Figure 1.7.4: a curved surface

Many different curves *C* pass through *p*, and hence there are many different tangents, with different corresponding values of dt_1/ds, dt_2/ds. Thus the partial derivatives $\partial\mathbf{x}/\partial t_1$, $\partial\mathbf{x}/\partial t_2$ must also both be tangential to *C* and so a normal to the surface at *p* is given by their cross-product, and a unit normal is

$$\mathbf{n} = \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) \Big/ \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| \tag{1.7.5}$$

In some cases, it is possible to use a non-parametric form for the surface, for example $g(x_1, x_2, x_3) = c$, in which case the normal can be obtained simply from $\mathbf{n} = \operatorname{grad} g / |\operatorname{grad} g|$.

Example (Parametric Representation and the Normal to a Sphere)

The surface of a sphere of radius a can be parameterised as[5]

$$\mathbf{x} = a\{\sin t_1 \cos t_2 \mathbf{e}_1 + \sin t_1 \sin t_2 \mathbf{e}_2 + \cos t_1 \mathbf{e}_3\}, \qquad 0 \le t_1 \le \pi,\ 0 \le t_2 \le 2\pi$$

Here, lines of $t_1 = \text{const}$ are parallel to the $x_1 - x_2$ plane ("parallels"), whereas lines of $t_2 = \text{const}$ are "meridian" lines, Fig. 1.7.5. If one takes the simple expressions $t_1 = s, t_2 = \pi/2 - s$, over $0 \le s \le \pi/2$, one obtains a curve C_1 joining $(0,0,1)$ and $(1,0,0)$, and passing through $(1/2, 1/2, 1/\sqrt{2})$, as shown.

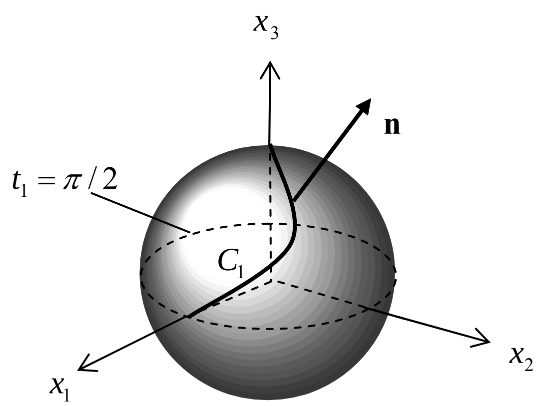


Figure 1.7.5: a sphere

The partial derivatives with respect to the parameters are

$$\frac{\partial \mathbf{x}}{\partial t_1} = a\{\cos t_1 \cos t_2 \mathbf{e}_1 + \cos t_1 \sin t_2 \mathbf{e}_2 - \sin t_1 \mathbf{e}_3\}$$
$$\frac{\partial \mathbf{x}}{\partial t_2} = a\{-\sin t_1 \sin t_2 \mathbf{e}_1 + \sin t_1 \cos t_2 \mathbf{e}_2\}$$

so that

$$\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} = a^2\{\sin^2 t_1 \cos t_2 \mathbf{e}_1 + \sin^2 t_1 \sin t_2 \mathbf{e}_2 + \sin t_1 \cos t_1 \mathbf{e}_3\}$$

---

[5] these are the **spherical coordinates** (see §1.6.10); $t_1 = \theta, t_2 = \phi$

and a unit normal to the spherical surface is

$$\mathbf{n} = \sin t_1 \cos t_2 \mathbf{e}_1 + \sin t_1 \sin t_2 \mathbf{e}_2 + \cos t_1 \mathbf{e}_3$$

For example, at $t_1 = t_2 = \pi / 4$ (this is on the curve C_1), one has

$$\mathbf{n}(\pi / 4, \pi / 4) = \tfrac{1}{2}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + \tfrac{1}{\sqrt{2}}\mathbf{e}_3$$

and, as expected, it is in the same direction as **r**.

■

Surface Integrals

Consider now the integral $\iint_S \mathbf{f} dS$ where **f** is a vector function and S is some curved surface. As for the integral over the plane region,

$$dS = \left| d\mathbf{x}\big|_{t_2 \text{ const}} \times d\mathbf{x}\big|_{t_1 \text{ const}} \right| = \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| dt_1 dt_2 \, ,$$

only now dS is not "flat" and **x** is three dimensional. The integral can be evaluated if one parameterises the surface with t_1, t_2 and then writes

$$\iint_S \mathbf{f} \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| dt_1 dt_2$$

One way to evaluate this cross product is to use the relation (**Lagrange's identity**, Problem 15, §1.3)

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \qquad (1.7.6)$$

so that

$$\left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right|^2 = \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) \cdot \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) = \left(\frac{\partial \mathbf{x}}{\partial t_1} \cdot \frac{\partial \mathbf{x}}{\partial t_1} \right)\left(\frac{\partial \mathbf{x}}{\partial t_2} \cdot \frac{\partial \mathbf{x}}{\partial t_2} \right) - \left(\frac{\partial \mathbf{x}}{\partial t_1} \cdot \frac{\partial \mathbf{x}}{\partial t_2} \right)^2 \qquad (1.7.7)$$

Example (Surface Area of a Sphere)

Using the parametric form for a sphere given above, one obtains

$$\left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right|^2 = a^4 \sin^2 t_1$$

so that

$$\text{area} = \iint_S dS = a^2 \int_0^{2\pi} \int_0^{\pi} \sin t_1 dt_1 dt_2 = 4\pi a^2$$

■

Flux Integrals

Surface integrals often involve the normal to the surface, as in the following example.

Example

If $\mathbf{f} = 4x_1 x_3 \mathbf{e}_1 - x_2^2 \mathbf{e}_2 + x_2 x_3 \mathbf{e}_3$, evaluate $\iint_S \mathbf{f} \cdot \mathbf{n} dS$, where S is the surface of the cube bounded by $x_1 = 0,1;\ x_2 = 0,1;\ x_3 = 0,1$, and $\mathbf{n}$ is the unit outward normal, Fig. 1.7.6.

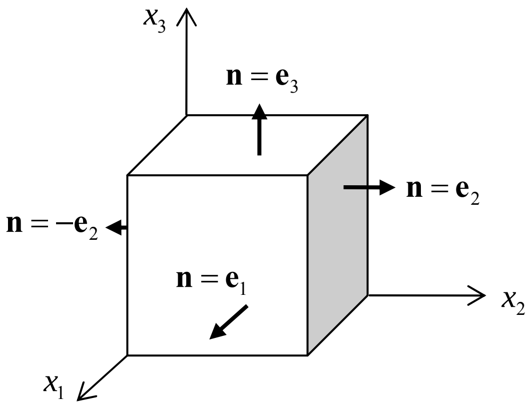


Figure 1.7.6: the unit cube

<u>Solution</u>

The integral needs to be evaluated over the six faces. For the face with $\mathbf{n} = +\mathbf{e}_1$, $x_1 = 1$ and

$$\iint_S \mathbf{f} \cdot \mathbf{n} dS = \int_0^1 \int_0^1 \left(4x_3 \mathbf{e}_1 - x_2^2 \mathbf{e}_2 + x_2 x_3 \mathbf{e}_3\right) \cdot \mathbf{e}_1 dx_2 dx_3 = 4 \int_0^1 \int_0^1 x_3 dx_2 dx_3 = 2$$

Similarly for the other five sides, whence $\iint_S \mathbf{f} \cdot \mathbf{n} dS = \frac{3}{2}$.

■

Integrals of the form $\iint_S \mathbf{f} \cdot \mathbf{n} dS$ are known as **flux integrals** and arise quite often in applications. For example, consider a material flowing with velocity **v**, in particular the flow through a small surface element dS with outward unit normal **n**, Fig. 1.7.7. The volume of material flowing through the surface in time dt is equal to the volume of the slanted cylinder shown, which is the base dS times the height. The slanted height is (=

velocity × time) is $|\mathbf{v}|dt$, and the vertical height is then $\mathbf{v}\cdot\mathbf{n}dt$. Thus the *rate* of flow is the **volume flux** (volume per unit time) through the surface element: $\mathbf{v}\cdot\mathbf{n}dS$.

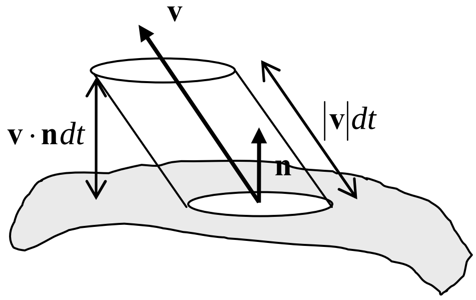


Figure 1.7.7: flow through a surface element

The total (volume) flux *out* of a surface S is then[6]

$$\textit{volume flux}: \quad \iint_S \mathbf{v}\cdot\mathbf{n}dS \tag{1.7.8}$$

Similarly, the **mass flux** is given by

$$\textit{mass flux}: \quad \iint_S \rho\mathbf{v}\cdot\mathbf{n}dS \tag{1.7.9}$$

For more complex surfaces, one can write using Eqn. 1.7.3, 1.7.5,

$$\iint_S \mathbf{f}\cdot\mathbf{n}dS = \iint_S \mathbf{f}\cdot\left(\frac{\partial\mathbf{x}}{\partial t_1}\times\frac{\partial\mathbf{x}}{\partial t_2}\right)dt_1dt_2$$

Example (of a Flux Integral)

Compute the flux integral $\iint_S \mathbf{f}\cdot\mathbf{n}dS$, where S is the parabolic cylinder represented by

$$x_2 = x_1^2, \quad 0 \le x_1 \le 2, \quad 0 \le x_3 \le 3$$

and $\mathbf{f} = x_2\mathbf{e}_1 + 2\mathbf{e}_2 + x_1x_3\mathbf{e}_3$, Fig. 1.7.8.

<u>Solution</u>

Making the substitutions $x_1 = t_1$, $x_3 = t_2$, so that $x_2 = t_1^2$, the surface can be represented by the position vector

[6] if **v** acts in the same direction as **n**, i.e. pointing outward, the dot product is positive and this integral is positive; if, on the other hand, material is flowing *in* through the surface, **v** and **n** are in opposite directions and the dot product is negative, so the integral is negative

$$\mathbf{x} = t_1\mathbf{e}_1 + t_1^2\mathbf{e}_2 + t_2\mathbf{e}_3, \qquad 0 \le t_1 \le 2,\ 0 \le t_2 \le 3$$

Then $\partial\mathbf{x}/\partial t_1 = \mathbf{e}_1 + 2t_1\mathbf{e}_2$, $\partial\mathbf{x}/\partial t_2 = \mathbf{e}_3$ and

$$\frac{\partial\mathbf{x}}{\partial t_1}\times\frac{\partial\mathbf{x}}{\partial t_2} = 2t_1\mathbf{e}_1 - \mathbf{e}_2$$

so the integral becomes

$$\int_0^3\int_0^2 \left(t_1^2\mathbf{e}_1 + 2\mathbf{e}_2 + t_1t_2\mathbf{e}_3\right)\cdot\left(2t_1\mathbf{e}_1 - \mathbf{e}_2\right)dt_1dt_2 = 12$$

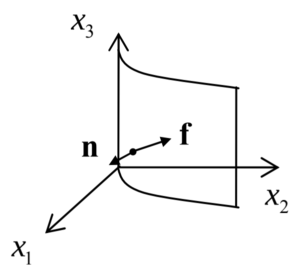


Figure 1.7.8: flux through a parabolic cylinder

Note: in this example, the value of the integral depends on the choice of **n**. If one chooses $-\mathbf{n}$ instead of **n**, one would obtain -12. The normal in the opposite direction (on the "other side" of the surface) can be obtained by simply switching t_1 and t_2, since $\partial\mathbf{x}/\partial t_1 \times \partial\mathbf{x}/\partial t_2 = -\partial\mathbf{x}/\partial t_2 \times \partial\mathbf{x}/\partial t_1$.

■

Surface flux integrals can also be evaluated by first converting them into double integrals over a plane region. For example, if a surface *S* has a projection *R* on the $x_1 - x_2$ plane, then an element of surface dS is related to the projected element dx_1dx_2 through (see Fig. 1.7.9)

$$\cos\theta dS = \left(\mathbf{n}\cdot\mathbf{e}_3\right)dS = dx_1dx_2$$

and so

$$\iint_S \mathbf{f}\cdot\mathbf{n}dS = \iint_R \mathbf{f}\cdot\mathbf{n}\frac{1}{\left|\mathbf{n}\cdot\mathbf{e}_3\right|}dx_1dx_2$$

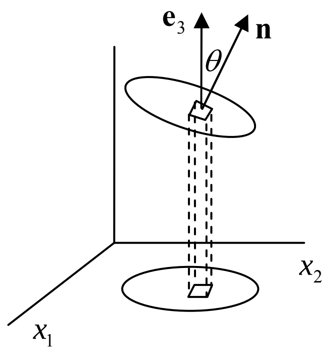


Figure 1.7.9: projection of a surface element onto a plane region

The Normal and Surface Area Elements

It is sometimes convenient to associate a special vector $d\mathbf{S}$ with a differential element of surface area dS, where

$$d\mathbf{S} = \mathbf{n}\, dS$$

so that $d\mathbf{S}$ is the vector with magnitude dS and direction of the unit normal to the surface. Flux integrals can then be written as

$$\iint_S \mathbf{f} \cdot \mathbf{n} dS = \iint_S \mathbf{f} \cdot d\mathbf{S}$$

1.7.6 Volume Integrals

The volume integral, or triple integral, is a generalisation of the double integral.

Change of Variable in Volume Integrals

For a volume integral, it is often convenient to make the change of variables $(x_1, x_2, x_3) \to (t_1, t_2, t_3)$. The volume of an element dV is given by the triple scalar product (Eqns. 1.1.5, 1.3.17)

$$dV = \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) \cdot \frac{\partial \mathbf{x}}{\partial t_3} dt_1 dt_2 dt_3 = J dt_1 dt_2 dt_3 \tag{1.7.10}$$

where the Jacobian is now

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_3}{\partial t_1} \\ \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_2}{\partial t_2} & \dfrac{\partial x_3}{\partial t_2} \\ \dfrac{\partial x_1}{\partial t_3} & \dfrac{\partial x_2}{\partial t_3} & \dfrac{\partial x_3}{\partial t_3} \end{vmatrix} \quad \text{or} \quad J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_1}{\partial t_3} \\ \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} & \dfrac{\partial x_2}{\partial t_3} \\ \dfrac{\partial x_3}{\partial t_1} & \dfrac{\partial x_3}{\partial t_2} & \dfrac{\partial x_3}{\partial t_3} \end{vmatrix} \tag{1.7.11}$$

so that

$$\iiint_V \mathbf{f}(x_1, x_2, x_3)dxdydz = \iiint_V \mathbf{f}\left(x_1(t_1, t_2, t_3), x_2(t_1, t_2, t_3), x_3(t_1, t_2, t_3)\right)J\,dt_1 dt_2 dt_3$$

1.7.7 Integral Theorems

A number of integral theorems and relations are presented here (without proof), the most important of which is the divergence theorem. These theorems can be used to simplify the evaluation of line, double, surface and triple integrals. They can also be used in various proofs of other important results.

The Divergence Theorem

Consider an arbitrary differentiable vector field $\mathbf{v}(\mathbf{x}, t)$ defined in some finite region of physical space. Let V be a volume in this space with a closed surface S bounding the volume, and let the outward normal to this bounding surface be $\mathbf{n}$. The **divergence theorem of Gauss** states that (in symbolic and index notation)

$$\boxed{\int_S \mathbf{v}\cdot\mathbf{n}\,dS = \int_V \operatorname{div}\mathbf{v}\,dV \qquad \int_S v_i n_i\,dS = \int_V \frac{\partial v_i}{\partial x_i}\,dV} \quad \textbf{Divergence Theorem} \tag{1.7.12}$$

and one has the following useful identities {▲Problem 10}

$$\begin{aligned} \int_S \phi\mathbf{u}\cdot\mathbf{n}\,dS &= \int_V \operatorname{div}(\phi\mathbf{u})\,dV \\ \int_S \phi\mathbf{n}\,dS &= \int_V \operatorname{grad}\phi\,dV \\ \int_S \mathbf{n}\times\mathbf{u}\,dS &= \int_V \operatorname{curl}\mathbf{u}\,dV \end{aligned} \tag{1.7.13}$$

By applying the divergence theorem to a very small volume, one finds that

$$\operatorname{div}\mathbf{v} = \lim_{V\to 0}\frac{\int_S \mathbf{v}\cdot\mathbf{n}\,dS}{V}$$

that is, the divergence is equal to the outward flux per unit volume, the result 1.6.18.

Stoke's Theorem

Stoke's theorem transforms line integrals into surface integrals and *vice versa*. It states that

$$\iint_S (\mathrm{curl}\,\mathbf{f}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{f} \cdot \boldsymbol{\tau}\, ds \tag{1.7.14}$$

Here C is the boundary of the surface S, $\mathbf{n}$ is the unit outward normal and $\boldsymbol{\tau} = d\mathbf{r}/ds$ is the unit tangent vector.

As has been seen, Eqn. 1.6.24, the curl of the velocity field is a measure of how much a fluid is rotating. The direction of this vector is along the direction of the local axis of rotation and its magnitude measures the local angular velocity of the fluid. Stoke's theorem then states that the amount of rotation of a fluid can be measured by integrating the tangential velocity around a curve (the line integral), or by integrating the amount of vorticity "moving through" a surface bounded by the same curve.

Green's Theorem and Related Identities

Green's theorem relates a line integral to a double integral, and states that

$$\oint_C \{\psi_1 dx_1 + \psi_2 dx_2\} = \iint_R \left(\frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \right) dx_1 dx_2 \,, \tag{1.7.15}$$

where R is a region in the $x_1 - x_2$ plane bounded by the curve C. In vector form, Green's theorem reads as

$$\oint_C \mathbf{f} \cdot d\mathbf{x} = \iint_R \mathrm{curl}\,\mathbf{f} \cdot \mathbf{e}_3 dx_1 dx_2 \quad \text{where} \quad \mathbf{f} = \psi_1 \mathbf{e}_1 + \psi_2 \mathbf{e}_2 \tag{1.7.16}$$

from which it can be seen that Green's theorem is a special case of Stoke's theorem, for the case of a plane surface (region) in the $x_1 - x_2$ plane.

It can also be shown that (this is **Green's first identity**)

$$\iint_S \psi (\mathbf{n} \cdot \mathrm{grad}\phi) dS = \iiint_V \{\psi \nabla^2 \phi + \mathrm{grad}\,\psi \cdot \mathrm{grad}\phi\} dV \tag{1.7.17}$$

Note that the term $\mathbf{n} \cdot \mathrm{grad}\phi$ is the directional derivative of ϕ in the direction of the outward unit normal. This is often denoted as $\partial\phi / \partial n$. Green's first identity can be regarded as a multi-dimensional "integration by parts" – compare the rule $\int u dv = uv - \int v du$ with the identity re-written as

$$\iiint_V \psi (\nabla \cdot \nabla \phi) dV = \iint_S \psi (\nabla \phi \cdot \mathbf{n}) dS - \iiint_V (\nabla \psi) \cdot (\nabla \phi) dV \tag{1.7.18}$$

or

$$\iiint_V \psi(\nabla \cdot \mathbf{u})dV = \iint_S \psi(\mathbf{u} \cdot \mathbf{n})dS - \iiint_V (\nabla \psi) \cdot \mathbf{u}dV \tag{1.7.18}$$

One also has the relation (this is **Green's second identity**)

$$\iint_S \{\psi(\mathbf{n} \cdot \operatorname{grad}\phi) - \phi(\mathbf{n} \cdot \operatorname{grad}\psi)\}dS = \iiint_V \{\psi\nabla^2\phi - \phi\nabla^2\psi\}dV \tag{1.7.19}$$

1.7.8 Problems

1. Find the work done in moving a particle in a force field given by $\mathbf{f} = 3x_1x_2\mathbf{e}_1 - 5x_3\mathbf{e}_2 + 10x_1\mathbf{e}_1$ along the curve $x_1 = t^2 + 1$, $x_2 = 2t^2$, $x_3 = t^3$, from $t = 1$ to $t = 2$. (Plot the curve.)
2. Show that the following vectors are conservative and find their scalar potentials:
   (i) $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$
   (ii) $\mathbf{v} = e^{x_1x_2}(x_2\mathbf{e}_1 + x_1\mathbf{e}_2)$
   (iii) $\mathbf{u} = (1/x_2)\mathbf{e}_1 - (x_1/x_2^2)\mathbf{e}_2 + x_3\mathbf{e}_3$
3. Show that if $\mathbf{f} = \nabla\phi$ then $\operatorname{curl}\mathbf{f} = \mathbf{o}$.
4. Show that if $\mathbf{f} = \nabla \times \mathbf{a}$ then $\nabla \cdot \mathbf{f} = 0$.
5. Find the volume beneath the surface $x_1^2 + x_2^2 - x_3 = 0$ and above the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ in the $x_1 - x_2$ plane.
6. Find the Jacobian (and sketch lines of constant t_1, t_2) for the rotation
$$x_1 = t_1\cos\theta - t_2\sin\theta$$
$$x_2 = t_1\sin\theta + t_2\cos\theta$$
7. Find a unit normal to the circular cylinder with parametric representation
$$\mathbf{x}(t_1,t_2) = a\cos t_1\mathbf{e}_1 + a\sin t_1\mathbf{e}_2 + t_2\mathbf{e}_3, \quad 0 \le t_1 \le 2\pi, \quad 0 \le t_1 \le 1$$
8. Evaluate $\int_S \psi dS$ where $\psi = x_1 + x_2 + x_3$ and S is the plane surface $x_3 = x_1 + x_2$, $0 \le x_2 \le x_1$, $0 \le x_1 \le 1$.
9. Evaluate the flux integral $\int_S \mathbf{f} \cdot \mathbf{n}dS$ where $\mathbf{f} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3$ and S is the cone $x_3 = a(x_1^2 + x_2^2)$, $x_3 \le a$ [Hint: first parameterise the surface with t_1, t_2.]
10. Prove the relations in (1.7.13). [Hint: first write the expressions in index notation.]
11. Use the divergence theorem to show that
$$\int_S \mathbf{x} \cdot \mathbf{n}dS = 3V,$$
where V is the volume enclosed by S (and $\mathbf{x}$ is the position vector).
12. Verify the divergence theorem for $\mathbf{v} = x_1^3\mathbf{e}_1 + x_2^3\mathbf{e}_2 + x_3^3\mathbf{e}_3$ where S is the surface of the sphere $x_1^2 + x_2^2 + x_3^2 = a^2$.
13. Interpret the divergence theorem (1.7.12) for the case when $\mathbf{v}$ is the velocity field. See (1.6.18, 1.7.8). Interpret also the case of $\operatorname{div}\mathbf{v} = 0$.

14. Verify Stoke's theorem for $\mathbf{f} = x_2\mathbf{e}_1 + x_3\mathbf{e}_2 + x_1\mathbf{e}_3$ where *S* is $x_3 = 1 - x_1^2 - x_2^2 \geq 0$ (so that *C* is the circle of radius 1 in the $x_1 - x_2$ plane).
15. Verify Green's theorem for the case of $\psi_1 = x_1^2 - 2x_2, \psi_2 = x_1 + x_2$, with *C* the unit circle $x_1^2 + x_2^2 = 1$. The following relations might be useful:
$$\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \pi, \quad \int_0^{2\pi} \sin\theta\cos\theta d\theta = \int_0^{2\pi} \sin\theta\cos^2\theta d\theta = 0$$
16. Evaluate $\oint_C \mathbf{f} \cdot d\mathbf{x}$ using Green's theorem, where $\mathbf{f} = -x_2^3\mathbf{e}_1 + x_1^3\mathbf{e}_2$ and *C* is the circle $x_1^2 + x_2^2 = 4$.
17. Use Green's theorem to show that the double integral of the Laplacian of *p* over a region *R* is equivalent to the integral of $\partial p / \partial n = \operatorname{grad} p \cdot \mathbf{n}$ around the curve *C* bounding the region:
$$\iint_R \nabla^2 p dx_1 dx_2 = \oint_C \frac{\partial p}{\partial n} ds$$
[Hint: Let $\psi_1 = -\partial p / \partial x_2, \psi_2 = +\partial p / \partial x_1$. Also, show that
$$\mathbf{n} = \frac{dx_2}{ds}\mathbf{e}_1 - \frac{dx_1}{ds}\mathbf{e}_2$$
is a unit normal to *C*, Fig. 1.7.10]

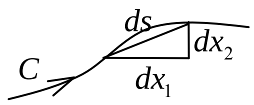


Figure 1.7.10: projection of a surface element onto a plane region

1.8 Tensors

Here the concept of the **tensor** is introduced. Tensors can be of different **orders** – zeroth-order tensors, first-order tensors, second-order tensors, and so on. Apart from the zeroth and first order tensors (see below), the second-order tensors are the most important tensors from a practical point of view, being important quantities in, amongst other topics, continuum mechanics, relativity, electromagnetism and quantum theory.

1.8.1 Zeroth and First Order Tensors

A **tensor of order zero** is simply another name for a scalar α .

A **first-order tensor** is simply another name for a vector **u**.

1.8.2 Second Order Tensors

Notation

Vectors:	lowercase bold-face Latin letters, e.g. **a**, **r**, **q**
2^{nd} order Tensors:	uppercase bold-face Latin letters, e.g. **F**, **T**, **S**

Tensors as Linear Operators

A *second*-order tensor **T** may be *defined* as an operator that acts on a vector **u** generating another vector **v**, so that $\mathbf{T}(\mathbf{u}) = \mathbf{v}$, or[1]

$$\boxed{\mathbf{T} \cdot \mathbf{u} = \mathbf{v} \quad \text{or} \quad \mathbf{Tu} = \mathbf{v}} \qquad \textbf{Second-order Tensor} \qquad (1.8.1)$$

The second-order tensor **T** is a **linear operator** (or **linear transformation**)[2], which means that

$$\begin{aligned} \mathbf{T}(\mathbf{a} + \mathbf{b}) &= \mathbf{Ta} + \mathbf{Tb} && \ldots \text{ distributive} \\ \mathbf{T}(\alpha \mathbf{a}) &= \alpha(\mathbf{Ta}) && \ldots \text{ associative} \end{aligned}$$

This linearity can be viewed geometrically as in Fig. 1.8.1.

Note: the vector may also be defined in this way, as a mapping **u** that acts on a vector **v**, this time generating a scalar α, $\mathbf{u} \cdot \mathbf{v} = \alpha$. This transformation (the dot product) is linear (see properties (2,3) in §1.1.4). Thus a first-order tensor (vector) maps a first-order tensor into a zeroth-order tensor (scalar), whereas a second-order tensor maps a first-order tensor into a first-order tensor. It will be seen that a third-order tensor maps a first-order tensor into a second-order tensor, and so on.

[1] both these notations for the tensor operation are used; here, the convention of omitting the “dot” will be used

[2] An operator or transformation is a special function which maps elements of one type into elements of a similar type; here, vectors into vectors

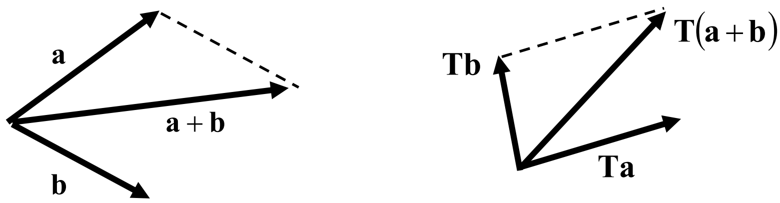


Figure 1.8.1: Linearity of the second order tensor

Further, two tensors **T** and **S** are said to be equal if and only if

$$\mathbf{Sv} = \mathbf{Tv}$$

for all vectors **v**.

Example (of a Tensor)

Suppose that **F** is an operator which transforms every vector into its mirror-image with respect to a given plane, Fig. 1.8.2. **F** transforms a vector into another vector and the transformation is linear, as can be seen geometrically from the figure. Thus **F** is a second-order tensor.

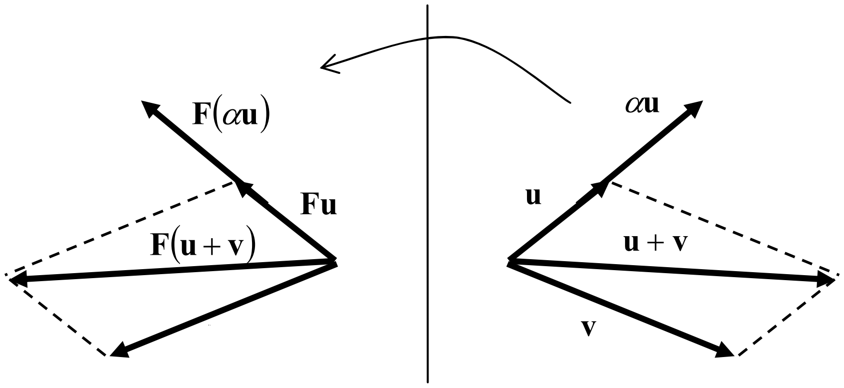


Figure 1.8.2: Mirror-imaging of vectors as a second order tensor mapping

■

Example (of a Tensor)

The combination $\mathbf{u}\times$ linearly transforms a vector into another vector and is thus a second-order tensor[3]. For example, consider a force **f** applied to a spanner at a distance **r** from the centre of the nut, Fig. 1.8.3. Then it can be said that the tensor $(\mathbf{r}\times)$ maps the force **f** into the (moment/torque) vector $\mathbf{r}\times\mathbf{f}$.

[3] Some authors use the notation $\tilde{\mathbf{u}}$ to denote $\mathbf{u}\times$

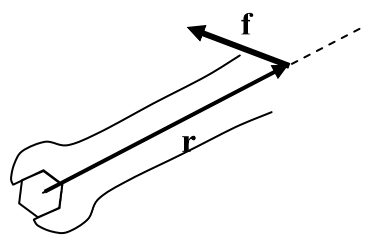


Figure 1.8.3: the force on a spanner

■

1.8.3 The Dyad (the tensor product)

The vector *dot product* and vector *cross product* have been considered in previous sections. A third vector product, the **tensor product** (or **dyadic product**), is important in the analysis of tensors of order 2 or more. The tensor product of two vectors **u** and **v** is written as[4]

$$\boxed{\mathbf{u}\otimes\mathbf{v}} \qquad \textbf{Tensor Product} \qquad (1.8.2)$$

This tensor product is itself a tensor of order two, and is called **dyad**:

$\mathbf{u}\cdot\mathbf{v}$	is a scalar	(a zeroth order tensor)
$\mathbf{u}\times\mathbf{v}$	is a vector	(a first order tensor)
$\mathbf{u}\otimes\mathbf{v}$	is a dyad	(a second order tensor)

It is best to *define* this dyad by what it *does*: it transforms a vector **w** into another vector with the direction of **u** according to the rule[5]

$$\boxed{(\mathbf{u}\otimes\mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v}\cdot\mathbf{w})} \qquad \textbf{The Dyad Transformation} \qquad (1.8.3)$$

This relation defines the symbol "$\otimes$".

The length of the new vector is $|\mathbf{u}|$ times $\mathbf{v}\cdot\mathbf{w}$, and the new vector has the same direction as $\mathbf{u}$, Fig. 1.8.4. It can be seen that the dyad is a second order tensor, because it operates linearly on a vector to give another vector {▲Problem 2}.

Note that the dyad is not commutative, $\mathbf{u}\otimes\mathbf{v} \neq \mathbf{v}\otimes\mathbf{u}$. Indeed it can be seen clearly from the figure that $(\mathbf{u}\otimes\mathbf{v})\mathbf{w} \neq (\mathbf{v}\otimes\mathbf{u})\mathbf{w}$.

[4] many authors omit the $\otimes$ and write simply **uv**
[5] note that it is the two vectors that are beside each other (separated by a bracket) that get "dotted" together

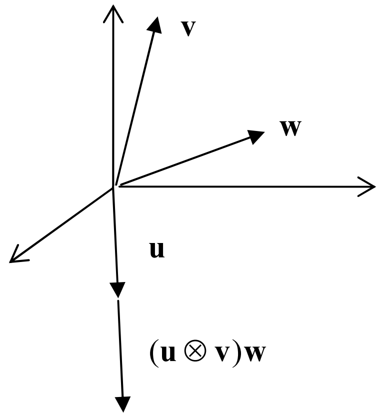


Figure 1.8.4: the dyad transformation

The following important relations follow from the above definition {▲Problem 4},

$$\begin{aligned} (\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) &= (\mathbf{v} \cdot \mathbf{w})(\mathbf{u} \otimes \mathbf{x}) \\ \mathbf{u}(\mathbf{v} \otimes \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \end{aligned} \tag{1.8.4}$$

It can be seen from these that the operation of the dyad on a vector is not commutative:

$$\mathbf{u}(\mathbf{v} \otimes \mathbf{w}) \neq (\mathbf{v} \otimes \mathbf{w})\mathbf{u} \tag{1.8.5}$$

Example (The Projection Tensor)

Consider the dyad $\mathbf{e} \otimes \mathbf{e}$. From the definition 1.8.3, $(\mathbf{e} \otimes \mathbf{e})\mathbf{u} = (\mathbf{e} \cdot \mathbf{u})\mathbf{e}$. But $\mathbf{e} \cdot \mathbf{u}$ is the projection of **u** onto a line through the unit vector **e**. Thus $(\mathbf{e} \cdot \mathbf{u})\mathbf{e}$ is the vector projection of **u** on **e**. For this reason $\mathbf{e} \otimes \mathbf{e}$ is called the **projection tensor**. It is usually denoted by **P**.

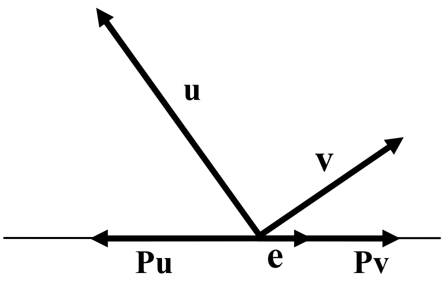


Figure 1.8.5: the projection tensor

■

1.8.4 Dyadics

A **dyadic** is a linear combination of these dyads (with scalar coefficients). An example might be

$$5(\mathbf{a} \otimes \mathbf{b}) + 3(\mathbf{c} \otimes \mathbf{d}) - 2(\mathbf{e} \otimes \mathbf{f})$$

This is clearly a second-order tensor. It will be seen in §1.9 that *every second-order tensor can be represented by a* dyadic, that is

$$\mathbf{T} = \alpha(\mathbf{a} \otimes \mathbf{b}) + \beta(\mathbf{c} \otimes \mathbf{d}) + \gamma(\mathbf{e} \otimes \mathbf{f}) + \cdots \tag{1.8.6}$$

Note: second-order tensors cannot, in general, be written as a dyad, $\mathbf{T} = \mathbf{a} \otimes \mathbf{b}$ – when they can, they are called **simple tensors**.

Example (Angular Momentum and the Moment of Inertia Tensor)

Suppose a rigid body is rotating so that every particle in the body is instantaneously moving in a circle about some axis fixed in space, Fig. 1.8.6.

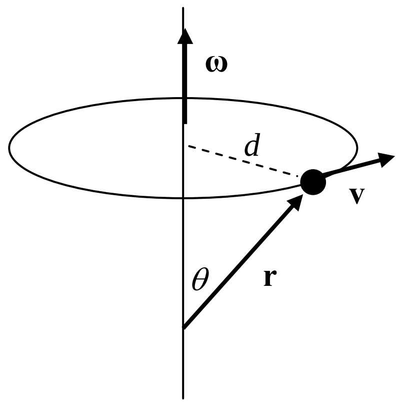


Figure 1.8.6: a particle in motion about an axis

The body's angular velocity $\boldsymbol{\omega}$ is defined as the vector whose magnitude is the angular speed ω and whose direction is along the axis of rotation. Then a particle's linear velocity is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

where $v = wd$ is the linear speed, d is the distance between the axis and the particle, and $\mathbf{r}$ is the position vector of the particle from a fixed point O on the axis. The particle's **angular momentum** (or moment of momentum) $\mathbf{h}$ about the point O is defined to be

$$\mathbf{h} = m\mathbf{r} \times \mathbf{v}$$

where m is the mass of the particle. The angular momentum can be written as

$$\mathbf{h} = \hat{\mathbf{I}}\boldsymbol{\omega} \tag{1.8.8}$$

where $\hat{\mathbf{I}}$, a second-order tensor, is the **moment of inertia** of the particle about the point O, given by

$$\hat{\mathbf{I}} = m\left(|\mathbf{r}|^2 \mathbf{I} - \mathbf{r} \otimes \mathbf{r}\right) \tag{1.8.9}$$

where $\mathbf{I}$ is the identity tensor, i.e. $\mathbf{Ia} = \mathbf{a}$ for all vectors $\mathbf{a}$.

To show this, it must be shown that $\mathbf{r} \times \mathbf{v} = \left(|\mathbf{r}|^2 \mathbf{I} - \mathbf{r} \otimes \mathbf{r}\right)\boldsymbol{\omega}$. First examine $\mathbf{r} \times \mathbf{v}$. It is evidently a vector perpendicular to both $\mathbf{r}$ and $\mathbf{v}$ and in the plane of $\mathbf{r}$ and $\boldsymbol{\omega}$; its magnitude is

$$|\mathbf{r} \times \mathbf{v}| = |\mathbf{r}||\mathbf{v}| = |\mathbf{r}|^2 |\boldsymbol{\omega}| \sin\theta$$

Now (see Fig. 1.8.7)

$$\begin{aligned} \left(|\mathbf{r}|^2 \mathbf{I} - \mathbf{r} \otimes \mathbf{r}\right)\boldsymbol{\omega} &= |\mathbf{r}|^2 \boldsymbol{\omega} - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}) \\ &= |\mathbf{r}|^2 |\boldsymbol{\omega}| \left(\mathbf{e}_\omega - \cos\theta \mathbf{e}_r\right) \end{aligned}$$

where $\mathbf{e}_\omega$ and $\mathbf{e}_r$ are unit vectors in the directions of $\boldsymbol{\omega}$ and $\mathbf{r}$ respectively. From the diagram, this is equal to $|\mathbf{r}|^2 |\boldsymbol{\omega}| \sin\theta \mathbf{e}_h$. Thus both expressions are equivalent, and one can indeed write $\mathbf{h} = \hat{\mathbf{I}}\boldsymbol{\omega}$ with $\hat{\mathbf{I}}$ defined by Eqn. 1.8.9: the second-order tensor $\hat{\mathbf{I}}$ maps the angular velocity vector $\boldsymbol{\omega}$ into the angular momentum vector $\mathbf{h}$ of the particle.

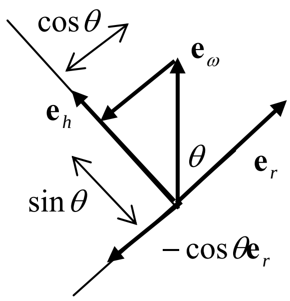


Figure 1.8.7: geometry of unit vectors for angular momentum calculation

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1.8.5 The Vector Space of Second Order Tensors

The vector space of vectors and associated spaces were discussed in §1.2. Here, spaces of second order tensors are discussed.

As mentioned above, the second order tensor is a mapping on the vector space *V*,

$$\mathbf{T}: V \to V \tag{1.8.10}$$

and follows the rules

$$\begin{aligned}\mathbf{T}(\mathbf{a}+\mathbf{b}) &= \mathbf{Ta}+\mathbf{Tb}\\ \mathbf{T}(\alpha\mathbf{a}) &= \alpha(\mathbf{Ta})\end{aligned} \tag{1.8.11}$$

for all $\mathbf{a}, \mathbf{b} \in V$ and $\alpha \in R$.

Denote the set of all second order tensors by V^2. Define then the sum of two tensors $\mathbf{S}, \mathbf{T} \in V^2$ through the relation

$$(\mathbf{S}+\mathbf{T})\mathbf{v} = \mathbf{Sv}+\mathbf{Tv} \tag{1.8.12}$$

and the product of a scalar $\alpha \in R$ and a tensor $\mathbf{T} \in V^2$ through

$$(\alpha\mathbf{T})\mathbf{v} = \alpha\mathbf{Tv} \tag{1.8.13}$$

Define an identity tensor $\mathbf{I} \in V^2$ through

$$\mathbf{Iv} = \mathbf{v}, \quad \text{for all } \mathbf{v} \in V \tag{1.8.14}$$

and a zero tensor $\mathbf{O} \in V^2$ through

$$\mathbf{Ov} = \mathbf{o}, \quad \text{for all } \mathbf{v} \in V \tag{1.8.15}$$

It follows from the definition 1.8.11 that V^2 has the structure of a real vector space, that is, the sum $\mathbf{S}+\mathbf{T} \in V^2$, the product $\alpha\mathbf{T} \in V^2$, and the following 8 axioms hold:

1. for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V^2$, one has $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$
2. there exists an element $\mathbf{O} \in V^2$ such that $\mathbf{T}+\mathbf{O} = \mathbf{O}+\mathbf{T} = \mathbf{T}$ for every $\mathbf{T} \in V^2$
3. for each $\mathbf{T} \in V^2$ there exists an element $-\mathbf{T} \in V^2$, called the negative of **T**, such that $\mathbf{T}+(-\mathbf{T}) = (-\mathbf{T})+\mathbf{T} = 0$
4. for any $\mathbf{S}, \mathbf{T} \in V^2$, one has $\mathbf{S}+\mathbf{T} = \mathbf{T}+\mathbf{S}$
5. for any $\mathbf{S}, \mathbf{T} \in V^2$ and scalar $\alpha \in R$, $\alpha(\mathbf{S}+\mathbf{T}) = \alpha\mathbf{S}+\alpha\mathbf{T}$
6. for any $\mathbf{T} \in V^2$ and scalars $\alpha, \beta \in R$, $(\alpha+\beta)\mathbf{T} = \alpha\mathbf{T}+\beta\mathbf{T}$
7. for any $\mathbf{T} \in V^2$ and scalars $\alpha, \beta \in R$, $\alpha(\beta\mathbf{T}) = (\alpha\beta)\mathbf{T}$
8. for the unit scalar $1 \in R$, $1\mathbf{T} = \mathbf{T}$ for any $\mathbf{T} \in V^2$.

1.8.6 Problems

1. Consider the function **f** which transforms a vector **v** into $\mathbf{a}\cdot\mathbf{v}+\beta$. Is **f** a tensor (of order one)? [Hint: test to see whether the transformation is linear, by examining $\mathbf{f}(\alpha\mathbf{u}+\mathbf{v})$.]
2. Show that the dyad is a linear operator, in other words, show that $(\mathbf{u}\otimes\mathbf{v})(\alpha\mathbf{w}+\beta\mathbf{x})=\alpha(\mathbf{u}\otimes\mathbf{v})\mathbf{w}+\beta(\mathbf{u}\otimes\mathbf{v})\mathbf{x}$
3. When is $\mathbf{a}\otimes\mathbf{b}=\mathbf{b}\otimes\mathbf{a}$?
4. Prove that
   (i) $(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})=(\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\otimes\mathbf{x})$ [Hint: post-"multiply" both sides of the definition (1.8.3) by $\otimes\mathbf{x}$; then show that $((\mathbf{u}\otimes\mathbf{v})\mathbf{w})\otimes\mathbf{x}=(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})$.]
   (ii) $\mathbf{u}(\mathbf{v}\otimes\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ [hint: pre "multiply" both sides by $\mathbf{x}\otimes$ and use the result of (i)]
5. Consider the dyadic (tensor) $\mathbf{a}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{b}$. Show that this tensor orthogonally projects every vector **v** onto the plane formed by **a** and **b** (sketch a diagram).
6. Draw a sketch to show the meaning of $\mathbf{u}\cdot(\mathbf{P}\mathbf{v})$, where **P** is the projection tensor. What is the order of the resulting tensor?
7. Prove that $\mathbf{a}\otimes\mathbf{b}-\mathbf{b}\otimes\mathbf{a}=(\mathbf{b}\times\mathbf{a})\times$.

1.9 Cartesian Tensors

As with the vector, a (higher order) tensor is a mathematical object which represents many physical phenomena and which exists independently of any coordinate system. In what follows, a Cartesian coordinate system is used to describe tensors.

1.9.1 Cartesian Tensors

A second order tensor and the vector it operates on can be described in terms of Cartesian components. For example, $(\mathbf{a} \otimes \mathbf{b})\mathbf{c}$, with $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{c} = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, is

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = 4\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$$

Example (The Unit Dyadic or Identity Tensor)

The **identity tensor**, or **unit tensor**, **I**, which maps every vector onto itself, has been introduced in the previous section. The Cartesian representation of **I** is

$$\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 \equiv \mathbf{e}_i \otimes \mathbf{e}_i \tag{1.9.1}$$

This follows from

$$\begin{aligned}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{u} &= (\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u} + (\mathbf{e}_2 \otimes \mathbf{e}_2)\mathbf{u} + (\mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{u} \\ &= \mathbf{e}_1(\mathbf{e}_1 \cdot \mathbf{u}) + \mathbf{e}_2(\mathbf{e}_2 \cdot \mathbf{u}) + \mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{u}) \\ &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \\ &= \mathbf{u}\end{aligned}$$

Note also that the identity tensor can be written as $\mathbf{I} = \delta_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)$, in other words the Kronecker delta gives the components of the identity tensor in a Cartesian coordinate system.

■

Second Order Tensor as a Dyadic

In what follows, it will be shown that a second order tensor can always be written as a dyadic involving the Cartesian base vectors $\mathbf{e}_i$ [1].

Consider an arbitrary second-order tensor **T** which operates on **a** to produce **b**, $\mathbf{T}(\mathbf{a}) = \mathbf{b}$, or $\mathbf{T}(a_i\mathbf{e}_i) = \mathbf{b}$. From the linearity of **T**,

[1] this can be generalised to the case of non-Cartesian base vectors, which might not be orthogonal nor of unit magnitude (see §1.16)

$$a_1\mathbf{T}(\mathbf{e}_1) + a_2\mathbf{T}(\mathbf{e}_2) + a_3\mathbf{T}(\mathbf{e}_3) = \mathbf{b}$$

Just as **T** transforms **a** into **b**, it transforms the base vectors $\mathbf{e}_i$ into some other vectors; suppose that $\mathbf{T}(\mathbf{e}_1) = \mathbf{u}$, $\mathbf{T}(\mathbf{e}_2) = \mathbf{v}$, $\mathbf{T}(\mathbf{e}_3) = \mathbf{w}$, then

$$\begin{aligned}\mathbf{b} &= a_1\mathbf{u} + a_2\mathbf{v} + a_3\mathbf{w} \\ &= (\mathbf{a}\cdot\mathbf{e}_1)\mathbf{u} + (\mathbf{a}\cdot\mathbf{e}_2)\mathbf{v} + (\mathbf{a}\cdot\mathbf{e}_3)\mathbf{w} \\ &= (\mathbf{u}\otimes\mathbf{e}_1)\mathbf{a} + (\mathbf{v}\otimes\mathbf{e}_2)\mathbf{a} + (\mathbf{w}\otimes\mathbf{e}_3)\mathbf{a} \\ &= [\mathbf{u}\otimes\mathbf{e}_1 + \mathbf{v}\otimes\mathbf{e}_2 + \mathbf{w}\otimes\mathbf{e}_3]\mathbf{a}\end{aligned}$$

and so

$$\mathbf{T} = \mathbf{u}\otimes\mathbf{e}_1 + \mathbf{v}\otimes\mathbf{e}_2 + \mathbf{w}\otimes\mathbf{e}_3 \qquad (1.9.2)$$

which is indeed a dyadic.

Cartesian components of a Second Order Tensor

The second order tensor **T** can be written in terms of components and base vectors as follows: write the vectors **u**, **v** and **w** in (1.9.2) in component form, so that

$$\begin{aligned}\mathbf{T} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)\otimes\mathbf{e}_1 + (\cdots)\otimes\mathbf{e}_2 + (\cdots)\otimes\mathbf{e}_3 \\ &= u_1\mathbf{e}_1\otimes\mathbf{e}_1 + u_2\mathbf{e}_2\otimes\mathbf{e}_1 + u_3\mathbf{e}_3\otimes\mathbf{e}_1 + \cdots\end{aligned}$$

Introduce nine scalars T_{ij} by letting $u_i = T_{i1}$, $v_i = T_{i2}$, $w_i = T_{i3}$, so that

$$\boxed{\begin{aligned}\mathbf{T} = T_{11}\mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}\mathbf{e}_1\otimes\mathbf{e}_2 + T_{13}\mathbf{e}_1\otimes\mathbf{e}_3 \\ + T_{21}\mathbf{e}_2\otimes\mathbf{e}_1 + T_{22}\mathbf{e}_2\otimes\mathbf{e}_2 + T_{23}\mathbf{e}_2\otimes\mathbf{e}_3 \\ + T_{31}\mathbf{e}_3\otimes\mathbf{e}_1 + T_{32}\mathbf{e}_3\otimes\mathbf{e}_2 + T_{33}\mathbf{e}_3\otimes\mathbf{e}_3\end{aligned}}$$ **Second-order Cartesian Tensor** (1.9.3)

These nine scalars T_{ij} are the components of the second order tensor **T** in the Cartesian coordinate system. In index notation,

$$\mathbf{T} = T_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j)$$

Thus whereas a vector has three components, a second order tensor has *nine* components. Similarly, whereas the three vectors $\{\mathbf{e}_i\}$ form a basis for the space of vectors, the nine dyads $\{\mathbf{e}_i\otimes\mathbf{e}_j\}$ from a basis for the space of tensors, i.e. all second order tensors can be expressed as a linear combination of these basis tensors.

It can be shown that the components of a second-order tensor can be obtained directly from {▲Problem 1}

$$\boxed{T_{ij} = \mathbf{e}_i\mathbf{T}\mathbf{e}_j}$$ **Components of a Tensor** (1.9.4)

which is the tensor expression analogous to the vector expression $u_i = \mathbf{e}_i \cdot \mathbf{u}$. Note that, in Eqn. 1.9.4, the components can be written simply as $\mathbf{e}_i \mathbf{T} \mathbf{e}_j$ (without a "dot"), since $\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \mathbf{e}_i \mathbf{T} \cdot \mathbf{e}_j$.

Example (The Stress Tensor)

Define the traction vector **t** acting on a surface element within a material to be the force acting on that element[2] divided by the area of the element, Fig. 1.9.1. Let **n** be a vector normal to the surface. The **stress** $\boldsymbol{\sigma}$ is defined to be that second order tensor which maps **n** onto **t**, according to

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}} \qquad \textbf{The Stress Tensor} \qquad (1.9.5)$$

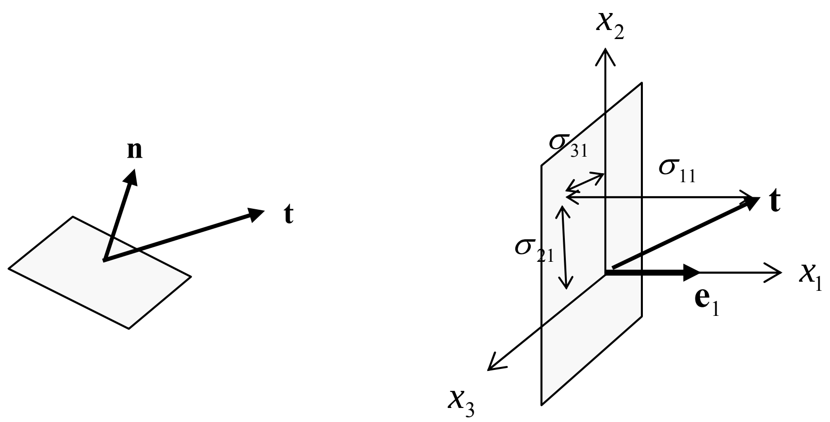

Figure 1.9.1: stress acting on a plane

If one now considers a coordinate system with base vectors $\mathbf{e}_i$, then $\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and, for example,

$$\boldsymbol{\sigma}\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3$$

Thus the components σ_{11}, σ_{21} and σ_{31} of the stress tensor are the three components of the traction vector which acts on the plane with normal $\mathbf{e}_1$.

Augustin-Louis Cauchy was the first to regard stress as a linear map of the normal vector onto the traction vector; hence the name "tensor", from the French for stress, *tension*.

■

[2] this force would be due, for example, to intermolecular forces within the material: the particles on one side of the surface element exert a force on the particles on the other side

Higher Order Tensors

The above can be generalised to tensors of order three and higher. The following notation will be used:

α, β, γ	…	0th-order tensors	("scalars")
$\mathbf{a}, \mathbf{b}, \mathbf{c}$	…	1st-order tensors	("vectors")
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	…	2nd-order tensors	("dyadics")
$\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}$	…	3rd-order tensors	("triadics")
$\mathsf{A}, \mathsf{B}, \mathsf{C}$	…	4th-order tensors	("tetradics")

An important third-order tensor is the **permutation tensor**, defined by

$$\boldsymbol{\mathcal{E}} = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{1.9.6}$$

whose components are those of the permutation symbol, Eqns. 1.3.10-1.3.13.

A fourth-order tensor can be written as

$$\mathsf{A} = A_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \tag{1.9.7}$$

It can be seen that a zeroth-order tensor (scalar) has $3^0 = 1$ component, a first-order tensor has $3^1 = 3$ components, a second-order tensor has $3^2 = 9$ components, so $\boldsymbol{\mathcal{A}}$ has $3^3 = 27$ components and A has 81 components.

1.9.2 Simple Contraction

Tensor/vector operations can be written in component form, for example,

$$\begin{aligned} \mathbf{Ta} &= T_{ij} \left(\mathbf{e}_i \otimes \mathbf{e}_j \right) a_k \mathbf{e}_k \\ &= T_{ij} a_k \left[\left(\mathbf{e}_i \otimes \mathbf{e}_j \right) \mathbf{e}_k \right] \\ &= T_{ij} a_k \delta_{jk} \mathbf{e}_i \\ &= T_{ij} a_j \mathbf{e}_i \end{aligned} \tag{1.9.8}$$

This operation is called **simple contraction**, because the order of the tensors is contracted – to begin there was a tensor of order 2 and a tensor of order 1, and to end there is a tensor of order 1 (it is called "simple" to distinguish it from "double" contraction – see below). This is always the case – when a tensor operates on another in this way, the order of the result will be *two* less than the sum of the original orders.

An example of simple contraction of two second order tensors has already been seen in Eqn. 1.8.4a; the tensors there were simple tensors (dyads). Here is another example:

$$\begin{aligned}
\mathbf{TS} &= T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)S_{kl}(\mathbf{e}_k \otimes \mathbf{e}_l) \\
&= T_{ij}S_{kl}[(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_k \otimes \mathbf{e}_l)] \\
&= T_{ij}S_{kl}\delta_{jk}(\mathbf{e}_i \otimes \mathbf{e}_l) \\
&= T_{ij}S_{jl}(\mathbf{e}_i \otimes \mathbf{e}_l)
\end{aligned} \tag{1.9.9}$$

From the above, the simple contraction of two second order tensors results in another second order tensor. If one writes $\mathbf{A} = \mathbf{TS}$, then the components of the new tensor are related to those of the original tensors through $A_{ij} = T_{ik}S_{kj}$.

Note that, in general,

$$\begin{aligned}
\mathbf{AB} &\neq \mathbf{BA} \\
(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}) && \ldots \text{ associative} \\
\mathbf{A}(\mathbf{B}+\mathbf{C}) &= \mathbf{AB} + \mathbf{AC} && \ldots \text{ distributive}
\end{aligned} \tag{1.9.10}$$

The associative and distributive properties follow from the fact that a tensor is by definition a linear operator, §1.8.2; they apply to tensors of any order, for example,

$$(\mathbf{AB})\mathbf{v} = \mathbf{A}(\mathbf{Bv}) \tag{1.9.11}$$

To deal with tensors of any order, all one has to remember is how simple tensors operate on each other – the two vectors which are beside each other are the ones which are “dotted” together:

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b})\mathbf{c} &= (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \\
(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) &= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d}) \\
(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) &= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d} \otimes \mathbf{e}) \\
(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})(\mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}) &= (\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{e} \otimes \mathbf{f})
\end{aligned} \tag{1.9.12}$$

An example involving a higher order tensor is

$$\begin{aligned}
\boldsymbol{\mathsf{A}} \cdot \mathbf{E} &= A_{ijkl}E_{mn}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)(\mathbf{e}_m \otimes \mathbf{e}_n) \\
&= A_{ijkl}E_{ln}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_n)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \alpha \\
\mathbf{AB} &= \mathbf{C} \\
\mathbf{Au} &= \mathbf{v} \\
\boldsymbol{\mathsf{A}}\mathbf{b} &= \mathbf{C} \\
\boldsymbol{\mathsf{AB}} &= \boldsymbol{\mathsf{C}}
\end{aligned}$$

Note the relation {▲Problem 10}

$$\mathbf{A}(\mathbf{B} \otimes \mathbf{C})\mathbf{D} = (\mathbf{AB}) \otimes (\mathbf{CD}) \tag{1.9.13}$$

Powers of Tensors

Integral powers of tensors are defined inductively by $\mathbf{T}^0 = \mathbf{I}$, $\mathbf{T}^n = \mathbf{T}^{n-1}\mathbf{T}$, so, for example,

$$\boxed{\mathbf{T}^2 = \mathbf{TT}} \qquad \textbf{The Square of a Tensor} \tag{1.9.14}$$

$\mathbf{T}^3 = \mathbf{TTT}$, etc.

1.9.3 Double Contraction

Double contraction, as the name implies, contracts the tensors twice as much a simple contraction. Thus, where the sum of the orders of two tensors is reduced by two in the simple contraction, the sum of the orders is reduced by four in double contraction. The double contraction is denoted by a colon (:), e.g. $\mathbf{T} : \mathbf{S}$.

First, define the double contraction of simple tensors (dyads) through

$$(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \tag{1.9.15}$$

So in double contraction, one takes the scalar product of four vectors which are adjacent to each other, according to the following rule:

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) : (\mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}) = (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})(\mathbf{a} \otimes \mathbf{f})$$

For example,

$$\begin{aligned} \mathbf{T} : \mathbf{S} &= T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) : S_{kl}(\mathbf{e}_k \otimes \mathbf{e}_l) \\ &= T_{ij}S_{kl}\left[(\mathbf{e}_i \cdot \mathbf{e}_k)(\mathbf{e}_j \cdot \mathbf{e}_l)\right] \\ &= T_{ij}S_{ij} \end{aligned} \tag{1.9.16}$$

which is, as expected, a scalar.

Here is another example, the contraction of the two second order tensors **I** (see Eqn. 1.9.1) and $\mathbf{u} \otimes \mathbf{v}$,

$$\begin{aligned} \mathbf{I} : \mathbf{u} \otimes \mathbf{v} &= (\mathbf{e}_i \otimes \mathbf{e}_i) : (\mathbf{u} \otimes \mathbf{v}) \\ &= (\mathbf{e}_i \cdot \mathbf{u})(\mathbf{e}_i \cdot \mathbf{v}) \\ &= u_i v_i \\ &= \mathbf{u} \cdot \mathbf{v} \end{aligned} \tag{1.9.17}$$

so that the scalar product of two vectors can be written in the form of a double contraction involving the Identity Tensor.

An example of double contraction involving the permutation tensor 1.9.6 is {▲Problem 11}

$$\mathbf{E} : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{v} \times \mathbf{u} \qquad (1.9.18)$$

It can be shown that the components of a fourth order tensor are given by (compare with Eqn. 1.9.4)

$$A_{ijkl} = (\mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\mathsf{A}} : (\mathbf{e}_k \otimes \mathbf{e}_l) \qquad (1.9.19)$$

In summary then,

$$\begin{aligned} \mathbf{A} : \mathbf{B} &= \beta \\ \boldsymbol{\mathcal{A}} : \mathbf{b} &= \gamma \\ \boldsymbol{\mathcal{A}} : \mathbf{B} &= \mathbf{c} \\ \boldsymbol{\mathsf{A}} : \mathbf{B} &= \mathbf{C} \end{aligned}$$

Note the following identities:

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} &= \mathbf{A}(\mathbf{B} : \mathbf{C}) = (\mathbf{B} : \mathbf{C})\mathbf{A} \\ \mathbf{A} : (\mathbf{B} \otimes \mathbf{C}) &= \mathbf{C}(\mathbf{A} : \mathbf{B}) = (\mathbf{A} : \mathbf{B})\mathbf{C} \\ (\mathbf{A} \otimes \mathbf{B}) : (\mathbf{C} \otimes \mathbf{D}) &= (\mathbf{B} : \mathbf{C})(\mathbf{A} \otimes \mathbf{D}) = (\mathbf{A} \otimes \mathbf{D})(\mathbf{B} : \mathbf{C}) \end{aligned} \qquad (1.9.20)$$

Note: There are many operations that can be defined and performed with tensors. The two most important operations, the ones which arise most in practice, are the simple and double contractions defined above. Other possibilities are:

(a) double contraction with two "horizontal" dots, $\mathbf{T} \cdot\cdot \mathbf{S}$, $\boldsymbol{\mathsf{A}} \cdot\cdot \mathbf{b}$, etc., which is based on the definition of the following operation as applied to simple tensors:

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) \cdot\cdot (\mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}) \equiv (\mathbf{b} \cdot \mathbf{e})(\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \otimes \mathbf{f})$$

(b) operations involving one cross $(\times)$: $(\mathbf{a} \otimes \mathbf{b}) \times (\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \otimes \mathbf{d}) \otimes (\mathbf{b} \times \mathbf{c})$

(c) "double" operations involving the cross $(\times)$ and dot:

$$(\mathbf{a} \otimes \mathbf{b})\,{}^{\times}_{\times}\,(\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \times \mathbf{c}) \otimes (\mathbf{b} \times \mathbf{d})$$

$$(\mathbf{a} \otimes \mathbf{b})\,{}^{\times}_{\cdot}\,(\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \times \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$$

$$(\mathbf{a} \otimes \mathbf{b})\,{}^{\cdot}_{\times}\,(\mathbf{c} \otimes \mathbf{d}) \equiv (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \times \mathbf{d})$$

1.9.4 Index Notation

The index notation for single and double contraction of tensors of any order can easily be remembered. From the above, a single contraction of two tensors implies that the indices "beside each other" are the same[3], and a double contraction implies that a pair of indices is repeated. Thus, for example, in both symbolic and index notation:

$$
\begin{aligned}
\mathbf{AB} = \mathbf{C} \qquad & A_{ijm}B_{mk} = C_{ijk} \\
\mathbf{A} : \mathbf{B} = \mathbf{c} \qquad & A_{ijk}B_{jk} = c_i
\end{aligned}
\tag{1.9.21}
$$

1.9.5 Matrix Notation

Here the matrix notation of §1.4 is extended to include second-order tensors[4]. The Cartesian components of a second-order tensor can conveniently be written as a 3×3 matrix,

$$
[\mathbf{T}] = \begin{bmatrix} \mathrm{T}_{11} & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{21} & \mathrm{T}_{22} & \mathrm{T}_{13} \\ \mathrm{T}_{31} & \mathrm{T}_{32} & \mathrm{T}_{33} \end{bmatrix}
$$

The operations involving vectors and second-order tensors can now be written in terms of matrices, for example,

$$
\mathbf{Tu} = [\mathbf{T}][\mathbf{u}] = \begin{bmatrix} \mathrm{T}_{11} & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{21} & \mathrm{T}_{22} & \mathrm{T}_{13} \\ \mathrm{T}_{31} & \mathrm{T}_{32} & \mathrm{T}_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11}u_1 + T_{12}u_2 + T_{13}u_3 \\ T_{21}u_1 + T_{22}u_2 + T_{23}u_3 \\ T_{31}u_1 + T_{32}u_2 + T_{33}u_3 \end{bmatrix}
$$

symbolic notation

"short" matrix notation

"full" matrix notation

The tensor product can be written as (see §1.4.1)

$$
\mathbf{u} \otimes \mathbf{v} = [\mathbf{u}][\mathbf{v}^{\mathrm{T}}] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}
\tag{1.9.22}
$$

which is consistent with the definition of the dyadic transformation, Eqn. 1.8.3.

---

[3] compare with the "beside each other rule" for matrix multiplication given in §1.4.1

[4] the matrix notation cannot be used for higher-order tensors

1.9.6 Problems

1. Use Eqn. 1.9.3 to show that the component T_{11} of a tensor $\mathbf{T}$ can be evaluated from $\mathbf{e}_1\mathbf{T}\mathbf{e}_1$, and that $T_{12} = \mathbf{e}_1\mathbf{T}\mathbf{e}_2$ (and so on, so that $T_{ij} = \mathbf{e}_i\mathbf{T}\mathbf{e}_j$).
2. Evaluate $\mathbf{aT}$ using the index notation (for a Cartesian basis). What is this operation called? Is your result equal to $\mathbf{Ta}$, in other words is this operation commutative? Now carry out this operation for two vectors, i.e. $\mathbf{a}\cdot\mathbf{b}$. Is it commutative in this case?
3. Evaluate the simple contractions $\mathbf{Ab}$ and $\mathbf{AB}$, with respect to a Cartesian coordinate system (use index notation).
4. Evaluate the double contraction $\mathbf{A}:\mathbf{B}$ (use index notation).
5. Show that, using a Cartesian coordinate system and the index notation, that the double contraction $\mathbf{A}:\mathbf{b}$ is a scalar. Write this scalar out in full in terms of the components of $\mathbf{A}$ and $\mathbf{b}$.
6. Consider the second-order tensors
$$\mathbf{D} = 3\mathbf{e}_1\otimes\mathbf{e}_1 + 2\mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_2\otimes\mathbf{e}_3 + 5\mathbf{e}_3\otimes\mathbf{e}_3$$
$$\mathbf{F} = 4\mathbf{e}_1\otimes\mathbf{e}_3 + 6\mathbf{e}_2\otimes\mathbf{e}_2 - 3\mathbf{e}_3\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3$$
Compute $\mathbf{DF}$ and $\mathbf{F}:\mathbf{D}$.
7. Consider the second-order tensor
$$\mathbf{D} = 3\mathbf{e}_1\otimes\mathbf{e}_1 - 4\mathbf{e}_1\otimes\mathbf{e}_2 + 2\mathbf{e}_2\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3 .$$
Determine the image of the vector $\mathbf{r} = 4\mathbf{e}_1 + 2\mathbf{e}_2 + 5\mathbf{e}_3$ when $\mathbf{D}$ operates on it.
8. Write the following out in full – are these the components of scalars, vectors or second order tensors?
   (a) B_{ii}
   (b) C_{kkj}
   (c) B_{mn}
   (d) $a_i b_j A_{ij}$
9. Write $(\mathbf{a}\otimes\mathbf{b}):(\mathbf{c}\otimes\mathbf{d})$ in terms of the components of the four vectors. What is the order of the resulting tensor?
10. Verify Eqn. 1.9.13.
11. Show that $\mathbf{E}:(\mathbf{u}\otimes\mathbf{v}) = \mathbf{v}\times\mathbf{u}$ – see (1.9.6, 1.9.18). [Hint: use the definition of the cross product in terms of the permutation symbol, (1.3.14), and the fact that $\varepsilon_{ijk} = -\varepsilon_{kji}$.]

1.10 Special Second Order Tensors & Properties of Second Order Tensors

In this section will be examined a number of special second order tensors, and special properties of second order tensors, which play important roles in tensor analysis. Many of the concepts will be familiar from Linear Algebra and Matrices. The following will be discussed:

- The Identity tensor
- Transpose of a tensor
- Trace of a tensor
- Norm of a tensor
- Determinant of a tensor
- Inverse of a tensor
- Orthogonal tensors
- Rotation Tensors
- Change of Basis Tensors
- Symmetric and Skew-symmetric tensors
- Axial vectors
- Spherical and Deviatoric tensors
- Positive Definite tensors

1.10.1 The Identity Tensor

The linear transformation which transforms every tensor into itself is called the **identity tensor**. This special tensor is denoted by **I** so that, for example,

$$\mathbf{Ia} = \mathbf{a} \quad \text{for any vector a}$$

In particular, $\mathbf{Ie}_1 = \mathbf{e}_1, \mathbf{Ie}_2 = \mathbf{e}_2, \mathbf{Ie}_3 = \mathbf{e}_3$, from which it follows that, for a Cartesian coordinate system, $I_{ij} = \delta_{ij}$. In matrix form,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.10.1}$$

1.10.2 The Transpose of a Tensor

The **transpose** of a second order tensor **A** with components A_{ij} is the tensor $\mathbf{A}^{\mathrm{T}}$ with components A_{ji}; so the transpose swaps the indices,

$$\boxed{\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{A}^{\mathrm{T}} = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j} \quad \textbf{Transpose of a Second-Order Tensor} \quad (1.10.2)$$

In matrix notation,

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \qquad [\mathbf{A}^{\mathrm{T}}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Some useful properties and relations involving the transpose are {▲Problem 2}:

$$\begin{aligned} (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} &= \mathbf{A} \\ (\alpha\mathbf{A} + \beta\mathbf{B})^{\mathrm{T}} &= \alpha\mathbf{A}^{\mathrm{T}} + \beta\mathbf{B}^{\mathrm{T}} \\ (\mathbf{u} \otimes \mathbf{v})^{\mathrm{T}} &= \mathbf{v} \otimes \mathbf{u} \\ \mathbf{Tu} = \mathbf{uT}^{\mathrm{T}}, \quad \mathbf{uT} &= \mathbf{T}^{\mathrm{T}}\mathbf{u} \\ (\mathbf{AB})^{\mathrm{T}} &= \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \\ \mathbf{A} : \mathbf{B} &= \mathbf{A}^{\mathrm{T}} : \mathbf{B}^{\mathrm{T}} \\ (\mathbf{u} \otimes \mathbf{v})\mathbf{A} &= \mathbf{u} \otimes (\mathbf{A}^{\mathrm{T}}\mathbf{v}) \\ \mathbf{A} : (\mathbf{BC}) = (\mathbf{B}^{\mathrm{T}}\mathbf{A}) &: \mathbf{C} = (\mathbf{AC}^{\mathrm{T}}) : \mathbf{B} \end{aligned} \tag{1.10.3}$$

A formal definition of the transpose which does not rely on any particular coordinate system is as follows: the transpose of a second-order tensor is that tensor which satisfies the identity[1]

$$\mathbf{u} \cdot \mathbf{Av} = \mathbf{v} \cdot \mathbf{A}^{\mathrm{T}}\mathbf{u} \tag{1.10.4}$$

for all vectors **u** and **v**. To see that Eqn. 1.10.4 implies 1.10.2, first note that, for the present purposes, a convenient way of writing the components A_{ij} of the second-order tensor **A** is $(\mathbf{A})_{ij}$. From Eqn. 1.9.4, $(\mathbf{A})_{ij} = \mathbf{e}_i \cdot \mathbf{Ae}_j$ and the components of the transpose can be written as $(\mathbf{A}^{\mathrm{T}})_{ij} = \mathbf{e}_i \cdot \mathbf{A}^{\mathrm{T}}\mathbf{e}_j$. Then, from 1.10.4, $(\mathbf{A}^{\mathrm{T}})_{ij} = \mathbf{e}_i \cdot \mathbf{A}^{\mathrm{T}}\mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{Ae}_i = (\mathbf{A})_{ji} = A_{ji}$.

1.10.3 The Trace of a Tensor

The **trace** of a second order tensor **A**, denoted by $\mathrm{tr}\mathbf{A}$, is a scalar equal to the sum of the diagonal elements of its matrix representation. Thus (see Eqn. 1.4.3)

$$\boxed{\mathrm{tr}\mathbf{A} = A_{ii}} \qquad \textbf{Trace} \tag{1.10.5}$$

A more formal definition, again not relying on any particular coordinate system, is

$$\boxed{\mathrm{tr}\mathbf{A} = \mathbf{I} : \mathbf{A}} \qquad \textbf{Trace} \tag{1.10.6}$$

[1] as mentioned in §1.9, from the linearity of tensors, $\mathbf{uA} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{Av}$ and, for this reason, this expression is usually written simply as $\mathbf{uAv}$

and Eqn. 1.10.5 follows from 1.10.6 {▲Problem 4}. For the dyad $\mathbf{u} \otimes \mathbf{v}$ {▲Problem 5},

$$\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \tag{1.10.7}$$

Another example is

$$\begin{aligned} \mathrm{tr}(\mathbf{E}^2) &= \mathbf{I} : \mathbf{E}^2 \\ &= \delta_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) : E_{pq}E_{qr}(\mathbf{e}_p \otimes \mathbf{e}_r) \\ &= E_{iq}E_{qi} \end{aligned} \tag{1.10.8}$$

This and other important traces, and functions of the trace are listed here {▲Problem 6}:

$$\begin{aligned} \mathrm{tr}\mathbf{A} &= A_{ii} \\ \mathrm{tr}\mathbf{A}^2 &= A_{ij}A_{ji} \\ \mathrm{tr}\mathbf{A}^3 &= A_{ij}A_{jk}A_{ki} \\ (\mathrm{tr}\mathbf{A})^2 &= A_{ii}A_{jj} \\ (\mathrm{tr}\mathbf{A})^3 &= A_{ii}A_{jj}A_{kk} \end{aligned} \tag{1.10.9}$$

Some useful properties and relations involving the trace are {▲Problem 7}:

$$\begin{aligned} \mathrm{tr}\mathbf{A}^{\mathrm{T}} &= \mathrm{tr}\mathbf{A} \\ \mathrm{tr}(\mathbf{AB}) &= \mathrm{tr}(\mathbf{BA}) \\ \mathrm{tr}(\mathbf{A} + \mathbf{B}) &= \mathrm{tr}\mathbf{A} + \mathrm{tr}\mathbf{B} \\ \mathrm{tr}(\alpha\mathbf{A}) &= \alpha\mathrm{tr}\mathbf{A} \\ \mathbf{A} : \mathbf{B} &= \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \mathrm{tr}(\mathbf{AB}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = \mathrm{tr}(\mathbf{BA}^{\mathrm{T}}) \end{aligned} \tag{1.10.10}$$

The double contraction of two tensors was earlier defined with respect to Cartesian coordinates, Eqn. 1.9.16. This last expression allows one to re-define the double contraction in terms of the trace, independent of any coordinate system.

Consider again the real vector space of second order tensors V^2 introduced in §1.8.5. The double contraction of two tensors as defined by 1.10.10e clearly satisfies the requirements of an inner product listed in §1.2.2. Thus this scalar quantity serves as an inner product for the space V^2:

$$\langle \mathbf{A}, \mathbf{B} \rangle \equiv \mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) \tag{1.10.11}$$

and generates an inner product space.

Just as the base vectors $\{\mathbf{e}_i\}$ form an orthonormal set in the inner product (vector dot product) of the space of vectors V, so the base dyads $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ form an orthonormal set in the inner product 1.10.11 of the space of second order tensors V^2. For example,

$$\langle \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_1 \rangle = (\mathbf{e}_1 \otimes \mathbf{e}_1) : (\mathbf{e}_1 \otimes \mathbf{e}_1) = 1 \tag{1.10.12}$$

Similarly, just as the dot product is zero for orthogonal vectors, when the double contraction of two tensors **A** and **B** is zero, one says that the tensors are **orthogonal**,

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right) = 0, \qquad \mathbf{A}, \mathbf{B} \text{ orthogonal} \tag{1.10.13}$$

1.10.4 The Norm of a Tensor

Using 1.2.8 and 1.10.11, the **norm** of a second order tensor **A**, denoted by $|\mathbf{A}|$ (or $\|\mathbf{A}\|$), is defined by

$$|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}} \tag{1.10.14}$$

This is analogous to the norm $|\mathbf{a}|$ of a vector **a**, $\sqrt{\mathbf{a} \cdot \mathbf{a}}$.

1.10.5 The Determinant of a Tensor

The **determinant** of a second order tensor **A** is defined to be the determinant of the matrix $[\mathbf{A}]$ of components of the tensor:

$$\begin{aligned} \det \mathbf{A} &= \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \\ &= \varepsilon_{ijk} A_{i1} A_{j2} A_{k3} \\ &= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} \end{aligned} \tag{1.10.15}$$

Some useful properties of the determinant are {▲Problem 8}

$$\begin{aligned} \det(\mathbf{AB}) &= \det \mathbf{A} \det \mathbf{B} \\ \det \mathbf{A}^{\mathrm{T}} &= \det \mathbf{A} \\ \det(\alpha \mathbf{A}) &= \alpha^3 \det \mathbf{A} \\ \det(\mathbf{u} \otimes \mathbf{v}) &= 0 \\ \varepsilon_{pqr}(\det \mathbf{A}) &= \varepsilon_{ijk} A_{ip} A_{jq} A_{kr} \\ (\mathbf{Ta} \times \mathbf{Tb}) \cdot \mathbf{Tc} &= (\det \mathbf{T})[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}] \end{aligned} \tag{1.10.16}$$

Note that $\det \mathbf{A}$, like $\mathrm{tr}\mathbf{A}$, is independent of the choice of coordinate system / basis.

1.10.6 The Inverse of a Tensor

The **inverse** of a second order tensor **A**, denoted by $\mathbf{A}^{-1}$, is defined by

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} \tag{1.10.17}$$

The inverse of a tensor exists only if it is **non-singular** (a **singular** tensor is one for which $\det \mathbf{A} = 0$), in which case it is said to be **invertible**.

Some useful properties and relations involving the inverse are:

$$\begin{aligned} (\mathbf{A}^{-1})^{-1} &= \mathbf{A} \\ (\alpha\mathbf{A})^{-1} &= (1/\alpha)\mathbf{A}^{-1} \\ (\mathbf{A}\mathbf{B})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ \det(\mathbf{A}^{-1}) &= (\det \mathbf{A})^{-1} \end{aligned} \tag{1.10.18}$$

Since the inverse of the transpose is equivalent to the transpose of the inverse, the following notation is used:

$$\mathbf{A}^{-\mathrm{T}} \equiv (\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1} \tag{1.10.19}$$

1.10.7 Orthogonal Tensors

An **orthogonal** tensor **Q** is a linear vector transformation satisfying the condition

$$\mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} \tag{1.10.20}$$

for all vectors **u** and **v**. Thus **u** is transformed to $\mathbf{Q}\mathbf{u}$, **v** is transformed to $\mathbf{Q}\mathbf{v}$ and the dot product $\mathbf{u} \cdot \mathbf{v}$ is invariant under the transformation. Thus the magnitude of the vectors and the angle between the vectors is preserved, Fig. 1.10.1.

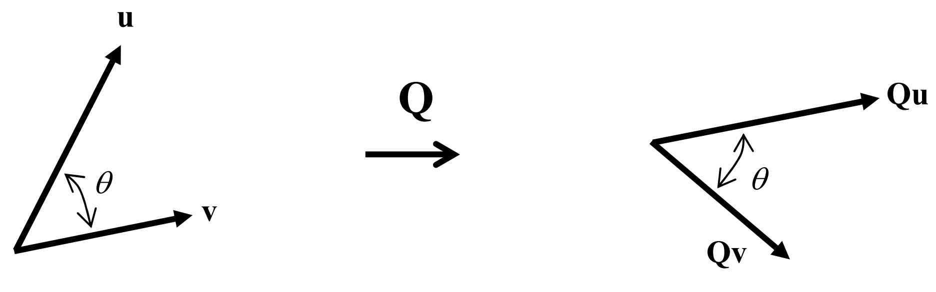

Figure 1.10.1: An orthogonal tensor

Since

$$\mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{v} = \mathbf{u}\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{Q}\mathbf{v} = \mathbf{u} \cdot \left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) \cdot \mathbf{v} \tag{1.10.21}$$

it follows that for $\mathbf{u}\cdot\mathbf{v}$ to be preserved under the transformation, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}=\mathbf{I}$, which is also used as the definition of an orthogonal tensor. Some useful properties of orthogonal tensors are {▲Problem 10}:

$$\begin{aligned} \mathbf{Q}\mathbf{Q}^{\mathrm{T}} &= \mathbf{I} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}, \qquad Q_{ik}Q_{jk} = \delta_{ij} = Q_{ki}Q_{kj} \\ \mathbf{Q}^{-1} &= \mathbf{Q}^{\mathrm{T}} \\ \det\mathbf{Q} &= \pm 1 \end{aligned} \tag{1.10.22}$$

1.10.8 Rotation Tensors

If for an orthogonal tensor, $\det\mathbf{Q}=+1$, $\mathbf{Q}$ is said to be a **proper** orthogonal tensor, corresponding to a **rotation**. If $\det\mathbf{Q}=-1$, $\mathbf{Q}$ is said to be an **improper** orthogonal tensor, corresponding to a **reflection**. Proper orthogonal tensors are also called **rotation tensors**.

1.10.9 Change of Basis Tensors

Consider a rotation tensor $\mathbf{Q}$ which rotates the base vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ into a second set, $\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3$, Fig. 1.10.2.

$$\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i \qquad i=1,2,3 \tag{1.10.23}$$

Such a tensor can be termed a **change of basis tensor** from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$. The transpose $\mathbf{Q}^{\mathrm{T}}$ rotates the base vectors $\mathbf{e}'_i$ back to $\mathbf{e}_i$ and is thus **change of basis tensor** from $\{\mathbf{e}'_i\}$ to $\{\mathbf{e}_i\}$. The components of $\mathbf{Q}$ in the $\mathbf{e}_i$ coordinate system are, from 1.9.4, $Q_{ij}=\mathbf{e}_i\mathbf{Q}\mathbf{e}_j$ and so, from 1.10.23,

$$\mathbf{Q} = Q_{ij}\mathbf{e}_i\otimes\mathbf{e}_j, \qquad Q_{ij} = \mathbf{e}_i\cdot\mathbf{e}'_j, \tag{1.10.24}$$

which are the direction cosines between the axes (see Fig. 1.5.5).

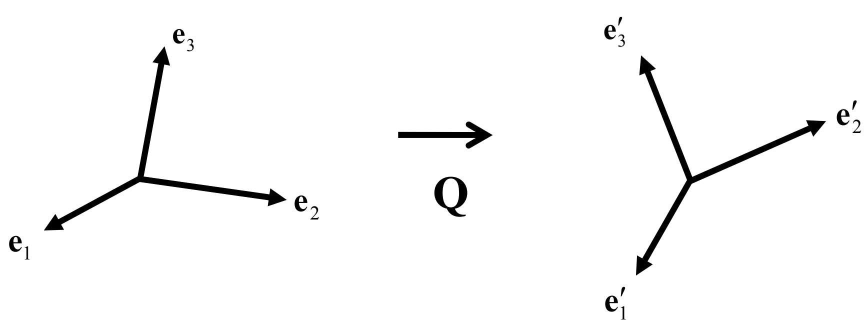

Figure 1.10.2: Rotation of a set of base vectors

The change of basis tensor can also be expressed in terms of the base vectors from *both* bases:

$$\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{e}_i , \tag{1.10.25}$$

from which the above relations can easily be derived, for example $\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i$, $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$, etc.

Consider now the operation of the change of basis tensor on a vector:

$$\mathbf{Q}\mathbf{v} = v_i\left(\mathbf{Q}\mathbf{e}_i\right) = v_i\mathbf{e}'_i \tag{1.10.26}$$

Thus $\mathbf{Q}$ transforms $\mathbf{v}$ into a second vector $\mathbf{v}'$, but this new vector has the *same components* with respect to the basis $\mathbf{e}'_i$, as $\mathbf{v}$ has with respect to the basis $\mathbf{e}_i$, $v'_i = v_i$. Note the difference between this and the coordinate transformations of §1.5: here there are two different vectors, $\mathbf{v}$ and $\mathbf{v}'$.

Example

Consider the two-dimensional rotation tensor

$$\mathbf{Q} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \equiv \mathbf{e}'_i \otimes \mathbf{e}_i$$

which corresponds to a rotation of the base vectors through $\pi/2$. The vector $\mathbf{v} = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$ then transforms into (see Fig. 1.10.3)

$$\mathbf{Q}\mathbf{v} = \begin{bmatrix} -1 \\ +1 \end{bmatrix}\mathbf{e}_i = \begin{bmatrix} +1 \\ +1 \end{bmatrix}\mathbf{e}'_i$$

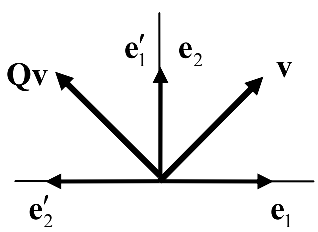

Figure 1.10.3: a rotated vector

■

Similarly, for a second order tensor $\mathbf{A}$, the operation

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\left(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{Q}^{\mathrm{T}} = A_{ij}\left(\mathbf{Q}\mathbf{e}_i \otimes \mathbf{e}_j\mathbf{Q}^{\mathrm{T}}\right) = A_{ij}\left(\mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_j\right) = A_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j \tag{1.10.27}$$

results in a new tensor which has the same components with respect to the $\mathbf{e}'_i$, as $\mathbf{A}$ has with respect to the $\mathbf{e}_i$, $A'_{ij} = A_{ij}$.

1.10.10 Symmetric and Skew Tensors

A tensor $\mathbf{T}$ is said to be **symmetric** if it is identical to the transposed tensor, $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$, and **skew** (**antisymmetric**) if $\mathbf{T} = -\mathbf{T}^{\mathrm{T}}$.

Any tensor $\mathbf{A}$ can be (uniquely) decomposed into a symmetric tensor $\mathbf{S}$ and a skew tensor $\mathbf{W}$, where

$$\begin{aligned} \mathrm{sym}\mathbf{A} &\equiv \mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) \\ \mathrm{skew}\mathbf{A} &\equiv \mathbf{W} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right) \end{aligned} \tag{1.10.28}$$

and

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}}, \qquad \mathbf{W} = -\mathbf{W}^{\mathrm{T}} \tag{1.10.29}$$

In matrix notation one has

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}, \qquad [W] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} \tag{1.10.30}$$

Some useful properties of symmetric and skew tensors are {▲Problem 13}:

$$\begin{aligned} &\mathbf{S} : \mathbf{B} = \mathbf{S} : \mathbf{B}^{\mathrm{T}} = \mathbf{S} : \tfrac{1}{2}\left(\mathbf{B} + \mathbf{B}^{\mathrm{T}}\right) \\ &\mathbf{W} : \mathbf{B} = -\mathbf{W} : \mathbf{B}^{\mathrm{T}} = \mathbf{W} : \tfrac{1}{2}\left(-\mathbf{B}^{\mathrm{T}}\right) \\ &\mathbf{S} : \mathbf{W} = 0 \\ &\mathrm{tr}(\mathbf{SW}) = 0 \\ &\mathbf{v} \cdot \mathbf{W}\mathbf{v} = 0 \\ &\det \mathbf{W} = 0 \quad (\text{has no inverse}) \end{aligned} \tag{1.10.31}$$

where $\mathbf{v}$ and $\mathbf{B}$ denote any arbitrary vector and second-order tensor respectively.

Note that symmetry and skew-symmetry are tensor properties, independent of coordinate system.

1.10.11 Axial Vectors

A skew tensor **W** has only three independent coefficients, so it behaves "like a vector" with three components. Indeed, a skew tensor can always be written in the form

$$\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u} \tag{1.10.32}$$

where **u** is any vector and $\boldsymbol{\omega}$ characterises the **axial** (or **dual**) vector of the skew tensor **W**. The components of **W** can be obtained from the components of $\boldsymbol{\omega}$ through

$$\begin{aligned} W_{ij} &= \mathbf{e}_i \cdot \mathbf{W}\mathbf{e}_j = \mathbf{e}_i \cdot \left(\boldsymbol{\omega} \times \mathbf{e}_j\right) = \mathbf{e}_i \cdot \left(\omega_k \mathbf{e}_k \times \mathbf{e}_j\right) \\ &= \mathbf{e}_i \cdot \left(\omega_k \varepsilon_{kjp} \mathbf{e}_p\right) = \varepsilon_{kji}\omega_k \\ &= -\varepsilon_{ijk}\omega_k \end{aligned} \tag{1.10.33}$$

If one knows the components of **W**, one can find the components of $\boldsymbol{\omega}$ by inverting this equation, whence {▲Problem 14}

$$\boldsymbol{\omega} = -W_{23}\mathbf{e}_1 + W_{13}\mathbf{e}_2 - W_{12}\mathbf{e}_3 \tag{1.10.34}$$

Example (of an Axial Vector)

Decompose the tensor

$$\mathbf{T} = \left[T_{ij}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

into its symmetric and skew parts. Also find the axial vector for the skew part. Verify that $\mathbf{W}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}$ for $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$.

<u>Solution</u>

One has

$$\mathbf{S} = \frac{1}{2}\left[\mathbf{T} + \mathbf{T}^{\mathrm{T}}\right] = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\mathbf{W} = \frac{1}{2}\left[\mathbf{T} - \mathbf{T}^{\mathrm{T}}\right] = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The axial vector is

$$\boldsymbol{\omega} = -W_{23}\mathbf{e}_1 + W_{13}\mathbf{e}_2 - W_{12}\mathbf{e}_3 = \mathbf{e}_2 + \mathbf{e}_3$$

and it can be seen that

$$\begin{aligned}\mathbf{Wa} &= W_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_1 + \mathbf{e}_3) = W_{ij}(\delta_{j1} + \delta_{j3})\mathbf{e}_i = (W_{i1} + W_{i3})\mathbf{e}_i \\ &= (W_{11} + W_{13})\mathbf{e}_1 + (W_{21} + W_{23})\mathbf{e}_2 + (W_{31} + W_{33})\mathbf{e}_3 \\ &= \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\end{aligned}$$

and

$$\boldsymbol{\omega} \times \mathbf{a} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$$

■

The Spin Tensor

The velocity of a particle rotating in a rigid body motion is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$, where $\boldsymbol{\omega}$ is the angular velocity vector and **x** is the position vector relative to the origin on the axis of rotation (see Problem 9, §1.1). If the velocity can be written in terms of a skew-symmetric second order tensor $\mathbf{w}$, such that $\mathbf{wx} = \mathbf{v}$, then it follows from $\mathbf{wx} = \boldsymbol{\omega} \times \mathbf{x}$ that the angular velocity vector $\boldsymbol{\omega}$ is the axial vector of $\mathbf{w}$. In this context, $\mathbf{w}$ is called the **spin tensor**.

1.10.12 Spherical and Deviatoric Tensors

Every tensor **A** can be decomposed into its so-called **spherical** part and its **deviatoric** part, i.e.

$$\mathbf{A} = \text{sph}\mathbf{A} + \text{dev}\mathbf{A} \tag{1.10.35}$$

where

$$\begin{aligned}\text{sph}\mathbf{A} &= \tfrac{1}{3}(\text{tr}\mathbf{A})\mathbf{I} \\ &= \begin{bmatrix} \frac{1}{3}(A_{11} + A_{22} + A_{33}) & 0 & 0 \\ 0 & \frac{1}{3}(A_{11} + A_{22} + A_{33}) & 0 \\ 0 & 0 & \frac{1}{3}(A_{11} + A_{22} + A_{33}) \end{bmatrix} \\ \text{dev}\mathbf{A} &= \mathbf{A} - \text{sph}\mathbf{A} \\ &= \begin{bmatrix} A_{11} - \frac{1}{3}(A_{11} + A_{22} + A_{33}) & A_{12} & A_{13} \\ A_{21} & A_{22} - \frac{1}{3}(A_{11} + A_{22} + A_{33}) & A_{23} \\ A_{31} & A_{32} & A_{33} - \frac{1}{3}(A_{11} + A_{22} + A_{33}) \end{bmatrix}\end{aligned} \tag{1.10.36}$$

Any tensor of the form $\alpha\mathbf{I}$ is known as a **spherical tensor**, while $\text{dev}\mathbf{A}$ is known as a deviator of **A**, or a **deviatoric tensor**.

Some important properties of the spherical and deviatoric tensors are

$$\begin{aligned} \text{tr}(\text{dev}\mathbf{A}) &= 0 \\ \text{sph}(\text{dev}\mathbf{A}) &= 0 \\ \text{dev}\mathbf{A} : \text{sph}\mathbf{B} &= 0 \end{aligned} \tag{1.10.37}$$

1.10.13 Positive Definite Tensors

A **positive definite** tensor **A** is one which satisfies the relation

$$\mathbf{vAv} > 0, \qquad \forall\, \mathbf{v} \neq \mathbf{o} \tag{1.10.38}$$

The tensor is called **positive semi-definite** if $\mathbf{vAv} \geq 0$.

In component form,

$$v_i A_{ij} v_j = A_{11}v_1^2 + A_{12}v_1v_2 + A_{13}v_1v_3 + A_{21}v_2v_1 + A_{22}v_2^2 + \cdots \tag{1.10.39}$$

and so the diagonal elements of the matrix representation of a positive definite tensor must always be positive.

It can be shown that the following conditions are necessary for a tensor **A** to be positive definite (although they are not sufficient):

(i) the diagonal elements of $[\mathbf{A}]$ are positive
(ii) the largest element of $[\mathbf{A}]$ lies along the diagonal
(iii) $\det\mathbf{A} > 0$
(iv) $A_{ii} + A_{jj} > 2A_{ij}$ for $i \neq j$ (no sum over i, j)

These conditions are seen to hold for the following matrix representation of an example positive definite tensor:

$$[\mathbf{A}] = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A necessary and sufficient condition for a tensor to be positive definite is given in the next section, during the discussion of the eigenvalue problem.

One of the key properties of a positive definite tensor is that, since $\det\mathbf{A} > 0$, positive definite tensors are always invertible.

An alternative definition of positive definiteness is the equivalent expression

$$\mathbf{A} : \mathbf{v} \otimes \mathbf{v} > 0 \tag{1.10.40}$$

1.10.14 Problems

1. Show that the components of the (second-order) identity tensor are given by $I_{ij} = \delta_{ij}$.
2. Show that
   (a) $(\mathbf{u} \otimes \mathbf{v})\mathbf{A} = \mathbf{u} \otimes (\mathbf{A}^{\mathrm{T}} \mathbf{v})$
   (b) $\mathbf{A} : (\mathbf{BC}) = (\mathbf{B}^{\mathrm{T}} \mathbf{A}) : \mathbf{C} = (\mathbf{AC}^{\mathrm{T}}) : \mathbf{B}$
3. Use (1.10.4) to show that $\mathbf{I}^{\mathrm{T}} = \mathbf{I}$.
4. Show that (1.10.6) implies (1.10.5) for the trace of a tensor.
5. Show that $\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.
6. Formally derive the index notation for the functions
   $\mathrm{tr}\mathbf{A}^2, \quad \mathrm{tr}\mathbf{A}^3, \quad (\mathrm{tr}\mathbf{A})^2, \quad (\mathrm{tr}\mathbf{A})^3$
7. Show that $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}} \mathbf{B})$.
8. Prove (1.10.16f), $(\mathbf{Ta} \times \mathbf{Tb}) \cdot \mathbf{Tc} = (\det \mathbf{T})[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$.
9. Show that $(\mathbf{A}^{-1})^{\mathrm{T}} : \mathbf{A} = 3$. [Hint: one way of doing this is using the result from Problem 7.]
10. Use 1.10.16b and 1.10.18d to prove 1.10.22c, $\det \mathbf{Q} = \pm 1$.
11. Use the explicit dyadic representation of the rotation tensor, $\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{e}_i$, to show that the components of $\mathbf{Q}$ in the "second", $ox'_1 x'_2 x'_3$, coordinate system are the same as those in the first system [hint: use the rule $Q'_{ij} = \mathbf{e}'_i \cdot \mathbf{Q}\mathbf{e}'_j$]
12. Consider the tensor $\mathbf{D}$ with components (in a certain coordinate system)
$$\begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \end{bmatrix}$$
    Show that $\mathbf{D}$ is a rotation tensor (just show that $\mathbf{D}$ is proper orthogonal).
13. Show that $\mathrm{tr}(\mathbf{SW}) = 0$.
14. Multiply across (1.10.32), $W_{ij} = -\varepsilon_{ijk}\omega_k$, by ε_{ijp} to show that $\boldsymbol{\omega} = -\frac{1}{2}\varepsilon_{ijk} W_{ij} \mathbf{e}_k$. [Hint: use the relation 1.3.19b, $\varepsilon_{ijp}\varepsilon_{ijk} = 2\delta_{pk}$.]
15. Show that $\frac{1}{2}(\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a})$ is a skew tensor $\mathbf{W}$. Show that its axial vector is $\boldsymbol{\omega} = \frac{1}{2}(\mathbf{b} \times \mathbf{a})$. [Hint: first prove that $(\mathbf{b} \cdot \mathbf{u})\mathbf{a} - (\mathbf{a} \cdot \mathbf{u})\mathbf{b} = \mathbf{u} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{a}) \times \mathbf{u}$.]
16. Find the spherical and deviatoric parts of the tensor $\mathbf{A}$ for which $A_{ij} = 1$.

2 Kinematics

Kinematics is concerned with expressing in mathematical form the deformation and motion of materials. In what follows, a number of important quantities, mainly vectors and second-order tensors, are introduced. Each of these quantities, for example the velocity, deformation gradient or rate of deformation tensor, allows one to describe a particular aspect of a deforming material.

No consideration is given to what is *causing* the deformation and movement – the cause is the action of forces on the material, and these will be discussed in the next chapter.

The first section introduces the material and spatial coordinates and descriptions. The second and third sections discuss the strain tensors. The fourth, fifth and sixth sections deal with rates of deformation and rates of change of kinematic quantities. The theory is specialised to small strain deformations in section 7. The notion of objectivity and the related topic of rigid rotations are discussed in sections 8 and 9. The final sections, 10-13, deal with kinematics using the convected coordinate system, and include the important notions of push-forward, pull-back and the Lie time derivative.

2.1 Motion

2.1.1 The Material Body and Motion

Physical materials in the real world are modeled using an abstract mathematical entity called a **body**. This body consists of an infinite number of **material particles**[1]. Shown in Fig. 2.1.1a is a body *B* with material particle *P*. One distinguishes between this body and the space in which it resides and through which it travels. Shown in Fig. 2.1.1b is a certain **point** $\mathbf{x}$ in Euclidean point space *E*.

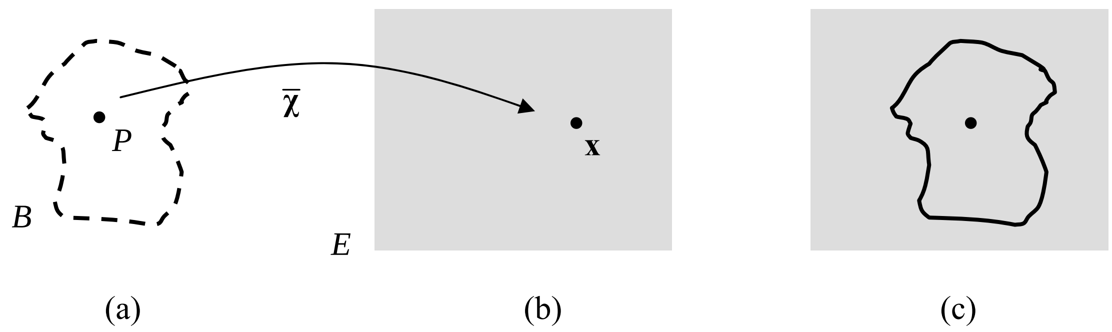

Figure 2.1.1: (a) a material particle in a body, (b) a place in space, (c) a configuration of the body

By fixing the material particles of the body to points in space, one has a **configuration** of the body $\bar{\chi}$, Fig. 2.1.1c. A configuration can be expressed as a mapping of the particles P to the point $\mathbf{x}$,

$$\mathbf{x} = \bar{\chi}(P) \tag{2.1.1}$$

A **motion** of the body is a *family* of configurations parameterised by time *t*,

$$\mathbf{x} = \bar{\chi}(P, t) \tag{2.1.2}$$

At any time *t*, Eqn. 2.1.2 gives the location in space $\mathbf{x}$ of the material particle P, Fig. 2.1.2.

[1] these particles are not the discrete mass particles of Newtonian mechanics, rather they are very small portions of continuous matter; the meaning of particle is made precise in the definitions which follow

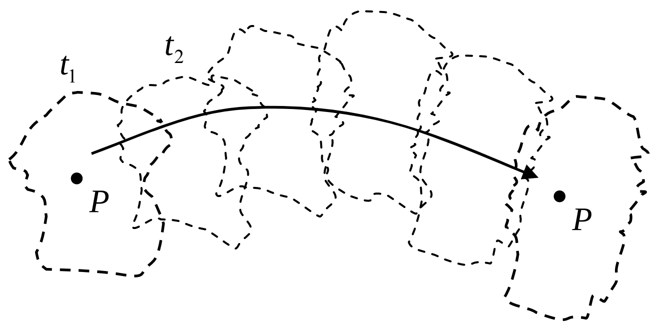

Figure 2.1.2: a motion of material

The Reference and Current Configurations

Choose now some **reference configuration**, Fig. 2.1.3. The motion can then be measured relative to this configuration. The reference configuration might be the configuration occupied by the material at time $t = 0$, in which case it is often called the **initial configuration**. For a solid, it might be natural to choose a configuration for which the material is stress-free, in which case it is often called the **undeformed configuration**. However, the choice of reference configuration is completely arbitrary.

Introduce a Cartesian coordinate system with base vectors $\mathbf{E}_i$ for the reference configuration. A material particle P in the reference configuration can then be assigned a unique position vector $\mathbf{X} = X_i \mathbf{E}_i$ relative to the origin of the axes. The coordinates (X_1, X_2, X_3) of the particle are called **material coordinates** (or **Lagrangian coordinates** or **referential coordinates**).

Some time later, say at time *t*, the material occupies a different configuration, which will be called the **current configuration** (or **deformed configuration**). Introduce a second Cartesian coordinate system with base vectors $\mathbf{e}_i$ for the current configuration, Fig. 2.1.3. In the current configuration, the same particle P now occupies the location $\mathbf{x}$, which can now also be assigned a position vector $\mathbf{x} = x_i \mathbf{e}_i$. The coordinates (x_1, x_2, x_3) are called **spatial coordinates** (or **Eulerian coordinates**).

Each particle thus has two sets of coordinates associated with it. The particle's material coordinates stay with it throughout its motion. The particle's spatial coordinates change as it moves.

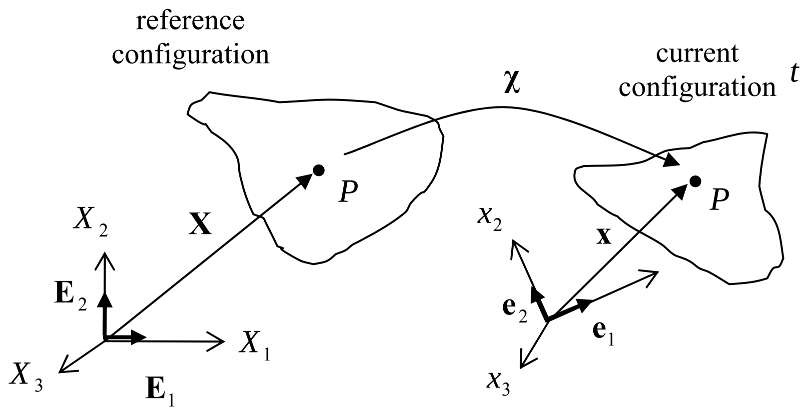

Figure 2.1.3: reference and current configurations

In practice, the material and spatial axes are usually taken to be coincident so that the base vectors $\mathbf{E}_i$ and $\mathbf{e}_i$ are the same, as in Fig. 2.1.4. Nevertheless, the use of different base vectors **E** and **e** for the reference and current configurations is useful even when the material and spatial axes are coincident, since it helps distinguish between quantities associated with the reference configuration and those associated with the spatial configuration (see later).

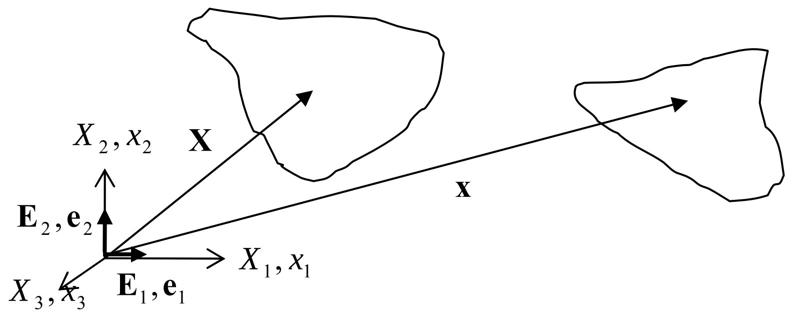

Figure 2.1.4: reference and current configurations with coincident axes

In terms of the position vectors, the motion 2.1.2 can be expressed as a relationship between the material and spatial coordinates,

$$\boxed{\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad x_i = \chi_i\left(X_1, X_2, X_3, t\right)} \qquad \textbf{Material description} \qquad (2.1.3)$$

or the inverse relation

$$\boxed{\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \qquad X_i = \chi_i^{-1}\left(x_1, x_2, x_3, t\right)} \qquad \textbf{Spatial description} \qquad (2.1.4)$$

If one knows the material coordinates of a particle then its position in the current configuration can be determined from 2.1.3. Alternatively, if one focuses on some location in space, in the current configuration, then the material particle occupying that position can be determined from 2.1.4. This is illustrated in the following example.

Example (Extension of a Bar)

Consider the motion

$$x_1 = 3X_1 t + X_1 + t, \quad x_2 = X_2, \quad x_3 = X_3 \tag{2.1.5}$$

These equations are of the form 2.1.3 and say that "the particle that was originally at position **X** is now, at time *t*, at position **x**". They represent a simple translation and uniaxial extension of material as shown in Fig. 2.1.5. Note that $\mathbf{X} = \mathbf{x}$ at $t = 0$.

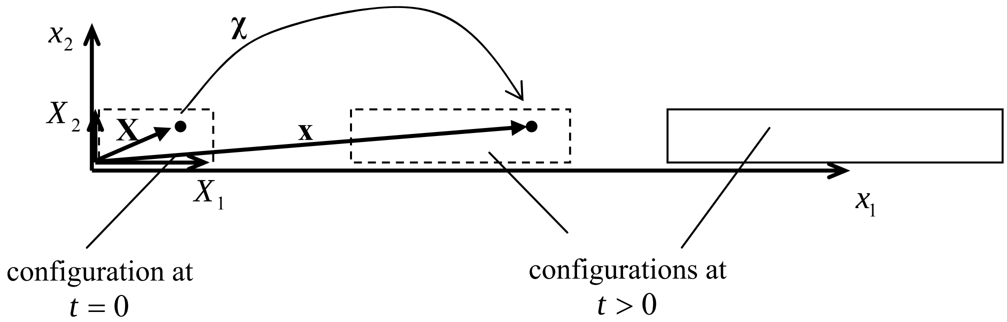

Figure 2.1.5: translation and extension of material

Relations of the form 2.1.4 can be obtained by inverting 2.1.5:

$$X_1 = \frac{x_1 - t}{1 + 3t}, \quad X_2 = x_2, \quad X_3 = x_3$$

These equations say that "the particle that is now, at time *t*, at position **x** was originally at position **X**".

■

Convected Coordinates

The material and spatial coordinate systems used here are fixed Cartesian systems. An alternative method of describing a motion is to *attach* the material coordinate system to the material and let it deform with the material. The motion is then described by defining how this coordinate system changes. This is the **convected coordinate system**. In general, the axes of a convected system will not remain mutually orthogonal and a curvilinear system is required. Convected coordinates will be examined in §2.10.

2.1.2 The Material and Spatial Descriptions

Any physical property (such as density, temperature, etc.) or kinematic property (such as displacement or velocity) of a body can be described in terms of either the material coordinates **X** or the spatial coordinates **x**, since they can be transformed into each other using 2.1.3-4. A **material** (or **Lagrangian**) **description** of events is one where the material coordinates are the independent variables. A **spatial** (or **Eulerian**) description of events is one where the spatial coordinates are used.

Example (Temperature of a Body)

Suppose the temperature θ of a body is, in material coordinates,

$$\theta(\mathbf{X},t) = 3X_1 - X_3 \tag{2.1.6}$$

but, in the spatial description,

$$\theta(\mathbf{x},t) = \frac{x_1}{t} - 1 - x_3 \,. \tag{2.1.7}$$

According to the material description 2.1.6, the temperature is different for different particles, but the temperature of each particle remains constant over time. The spatial description 2.1.7 describes the time-dependent temperature at a specific location in space, **x**, Fig. 2.1.6. Different material particles are flowing through this location over time.

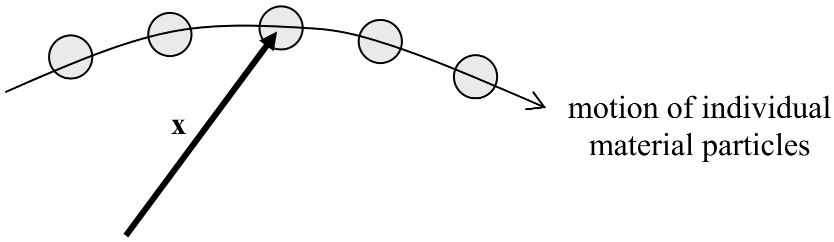

Figure 2.1.6: particles flowing through space

■

In the material description, then, attention is focused on specific *material*. The piece of matter under consideration may change shape, density, velocity, and so on, but it is always the same piece of material. On the other hand, in the spatial description, attention is focused on a fixed location in *space*. Material may pass through this location during the motion, so different material is under consideration at different times.

The spatial description is the one most often used in Fluid Mechanics since there is no natural reference configuration of the material as it is continuously moving. However, both the material and spatial descriptions are used in Solid Mechanics, where the reference configuration is usually the stress-free configuration.

2.1.3 Small Perturbations

A large number of important problems involve materials which deform only by a relatively small amount. An example would be the steel structural columns in a building under modest loading. In this type of problem there is virtually no distinction to be made between the two viewpoints taken above and the analysis is simplified greatly (see later, on Small Strain Theory, §2.7).

2.1.4 Problems

1. The density of a material is given by $\rho = 3X_1 + X_2$ and the motion is given by the equations $X_1 = x_1$, $X_2 = x_2 - t$, $X_3 = x_3 - t$.
   (a) what kind of description is this for the density, and what kind of description is this for the motion?
   (b) re-write the density in terms of **x** – what is the name given to this description of the density?
   (c) is the density of any given material particle changing with time?
   (d) invert the motion equations so that **X** is the independent variable – what is the name given to this description of the motion?
   (e) draw the line element joining the origin to $(1,1,0)$ and sketch the position of this element of material at times $t = 1$ and $t = 2$.

2.2 Deformation and Strain

A number of useful ways of describing and quantifying the deformation of a material are discussed in this section.

Attention is restricted to the reference and current configurations. No consideration is given to the particular sequence by which the current configuration is reached from the reference configuration and so the deformation can be considered to be independent of time. In what follows, particles in the reference configuration will often be termed "undeformed" and those in the current configuration "deformed".

In a change from Chapter 1, lower case letters will now be reserved for both vector- *and* tensor- functions of the spatial coordinates **x**, whereas upper-case letters will be reserved for functions of material coordinates **X**. There will be exceptions to this, but it should be clear from the context what is implied.

2.2.1 The Deformation Gradient

The **deformation gradient F** is the fundamental measure of deformation in continuum mechanics. It is the second order tensor which maps line elements in the reference configuration into line elements (consisting of the *same* material particles) in the current configuration.

Consider a line element $d\mathbf{X}$ emanating from position **X** in the reference configuration which becomes $d\mathbf{x}$ in the current configuration, Fig. 2.2.1. Then, using 2.1.3,

$$\begin{aligned} d\mathbf{x} &= \boldsymbol{\chi}(\mathbf{X}+d\mathbf{X})-\boldsymbol{\chi}(\mathbf{X}) \\ &= (\mathrm{Grad}\,\boldsymbol{\chi})d\mathbf{X} \end{aligned} \tag{2.2.1}$$

A capital G is used on "Grad" to emphasise that this is a gradient with respect to the material coordinates[1], the **material gradient**, $\partial\boldsymbol{\chi}/\partial\mathbf{X}$.

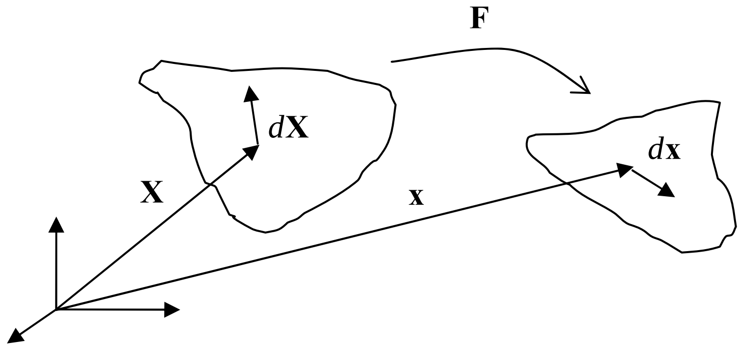

Figure 2.2.1: the Deformation Gradient acting on a line element

[1] one can have material gradients and spatial gradients of material or spatial fields – see later

The motion vector-function χ in 2.1.3, 2.2.1, is a function of the variable **X**, but it is customary to denote this simply by **x**, the value of χ at **X**, i.e. $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, so that

$$\boxed{\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \operatorname{Grad} \mathbf{x}, \qquad F_{iJ} = \frac{\partial x_i}{\partial X_J}} \quad \textbf{Deformation Gradient} \tag{2.2.2}$$

with

$$\boxed{d\mathbf{x} = \mathbf{F}\, d\mathbf{X}, \qquad dx_i = F_{iJ}\, dX_J} \quad \textbf{action of F} \tag{2.2.3}$$

Lower case indices are used in the index notation to denote quantities associated with the spatial basis $\{\mathbf{e}_i\}$ whereas upper case indices are used for quantities associated with the material basis $\{\mathbf{E}_I\}$.

Note that

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X}$$

is a differential quantity and this expression has some error associated with it; the error (due to terms of order $(d\mathbf{X})^2$ and higher, neglected from a Taylor series) tends to zero as the differential $d\mathbf{X} \to 0$. The deformation gradient (whose components are finite) thus characterises the deformation in the *neighbourhood* of a point **X**, mapping infinitesimal line elements $d\mathbf{X}$ emanating from **X** in the reference configuration to the infinitesimal line elements $d\mathbf{x}$ emanating from **x** in the current configuration, Fig. 2.2.2.

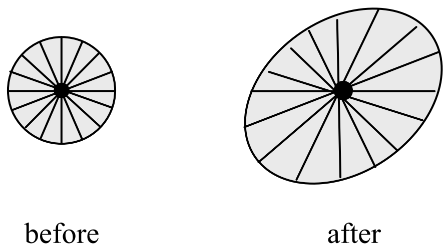

Figure 2.2.2: deformation of a material particle

Example

Consider the cube of material with sides of unit length illustrated by dotted lines in Fig. 2.2.3. It is deformed into the rectangular prism illustrated (this could be achieved, for example, by a continuous rotation and stretching motion). The material and spatial coordinate axes are coincident. The material description of the deformation is

$$\mathbf{x} = \chi(\mathbf{X}) = -6X_2\mathbf{e}_1 + \frac{1}{2}X_1\mathbf{e}_2 + \frac{1}{3}X_3\mathbf{e}_3$$

and the spatial description is

$$\mathbf{X} = \chi^{-1}(\mathbf{x}) = 2x_2\mathbf{E}_1 - \frac{1}{6}x_1\mathbf{E}_2 + 3x_3\mathbf{E}_3$$

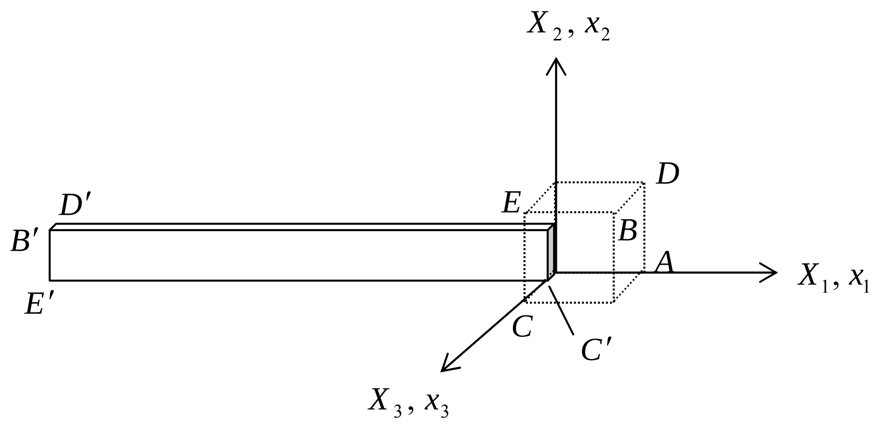

Figure 2.2.3: a deforming cube

Then

$$\mathbf{F} = \frac{\partial x_i}{\partial X_J} = \begin{bmatrix} 0 & -6 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Once **F** is known, the position of any element can be determined. For example, taking a line element $d\mathbf{X} = [da, 0, 0]^{\mathrm{T}}$, $d\mathbf{x} = \mathbf{F}d\mathbf{X} = [0, da/2, 0]^{\mathrm{T}}$.

■

Homogeneous Deformations

A **homogeneous deformation** is one where the deformation gradient is uniform, i.e. independent of the coordinates, and the associated motion is termed **affine**. Every part of the material deforms as the whole does, and straight parallel lines in the reference configuration map to straight parallel lines in the current configuration, as in the above example. Most examples to be considered in what follows will be of homogeneous deformations; this keeps the algebra to a minimum, but homogeneous deformation analysis is very useful in itself since most of the basic experimental testing of materials, e.g. the uniaxial tensile test, involve homogeneous deformations.

Rigid Body Rotations and Translations

One can add a constant vector **c** to the motion, $\mathbf{x} = \mathbf{x} + \mathbf{c}$, without changing the deformation, $\text{Grad}(\mathbf{x} + \mathbf{c}) = \text{Grad}\mathbf{x}$. Thus the deformation gradient does not take into account rigid-body **translations** of bodies in space. If a body only translates as a rigid body in space, then $\mathbf{F} = \mathbf{I}$, and $\mathbf{x} = \mathbf{X} + \mathbf{c}$ (again, note that **F** does not tell us where in space a particle is, only how it has deformed locally). If there is *no* motion, then not only is $\mathbf{F} = \mathbf{I}$, but $\mathbf{x} = \mathbf{X}$.

If the body rotates as a rigid body (with no translation), then $\mathbf{F} = \mathbf{R}$, a rotation tensor (§1.10.8). For example, for a rotation of θ about the X_2 axis,

$$\mathbf{F} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}$$

Note that different particles of the same material body can be translating only, rotating only, deforming only, or any combination of these.

The Inverse of the Deformation Gradient

The inverse deformation gradient $\mathbf{F}^{-1}$ carries the spatial line element d**x** to the material line element d**X**. It is defined as

$$\boxed{\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \operatorname{grad}\mathbf{X}, \qquad F_{Ij}^{-1} = \frac{\partial X_I}{\partial x_j}} \quad \textbf{Inverse Deformation Gradient} \tag{2.2.4}$$

so that

$$\boxed{d\mathbf{X} = \mathbf{F}^{-1}\,d\mathbf{x}, \qquad dX_I = F_{Ij}^{-1}\,dx_j} \quad \textbf{action of } \mathbf{F}^{-1} \tag{2.2.5}$$

with (see Eqn. 1.15.2)

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{F}\mathbf{F}^{-1} = \mathbf{I} \qquad F_{iM}F_{Mj}^{-1} = \delta_{ij} \tag{2.2.6}$$

Cartesian Base Vectors

Explicitly, in terms of the material and spatial base vectors (see 1.14.3),

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial X_J} \otimes \mathbf{E}_J = \frac{\partial x_i}{\partial X_J}\mathbf{e}_i \otimes \mathbf{E}_J \\ \mathbf{F}^{-1} &= \frac{\partial \mathbf{X}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial X_I}{\partial x_j}\mathbf{E}_I \otimes \mathbf{e}_j \end{aligned} \tag{2.2.7}$$

so that, for example, $\mathbf{F}d\mathbf{X} = (\partial x_i / \partial X_J)\mathbf{e}_i \otimes \mathbf{E}_J (dX_M \mathbf{E}_M) = (\partial x_i / \partial X_J) dX_J \mathbf{e}_i = d\mathbf{x}$.

Because **F** and $\mathbf{F}^{-1}$ act on vectors in one configuration to produce vectors in the other configuration, they are termed **two-point tensors**. They are defined in both configurations. This is highlighted by their having both reference and current base vectors **E** and **e** in their Cartesian representation 2.2.7.

Here follow some important relations which relate scalar-, vector- and second-order tensor-valued functions in the material and spatial descriptions, the first two relating the material and spatial gradients {▲Problem 1}.

$$\begin{aligned} \operatorname{grad}\phi &= \operatorname{Grad}\phi\, \mathbf{F}^{-1} \\ \operatorname{grad}\mathbf{v} &= \operatorname{Grad}\mathbf{V}\, \mathbf{F}^{-1} \\ \operatorname{div}\mathbf{a} &= \operatorname{Grad}\mathbf{A} : \mathbf{F}^{-\mathrm{T}} \end{aligned} \qquad (2.2.8)$$

Here, ϕ is a scalar; $\mathbf{V}$ and $\mathbf{v}$ are the *same* vector, the former being a function of the material coordinates, the material description, the latter a function of the spatial coordinates, the spatial description. Similarly, $\mathbf{A}$ is a second order tensor in the material form and $\mathbf{a}$ is the equivalent spatial form.

The first two of 2.2.8 relate the material gradient to the spatial gradient: the gradient of a function is a measure of how the function changes as one moves through space; since the material coordinates and the spatial coordinates differ, the change in a function with respect to a unit change in the material coordinates will differ from the change in the *same* function with respect to a unit change in the spatial coordinates (see also §2.2.7 below).

Example

Consider the deformation

$$\begin{aligned} \mathbf{x} &= (2X_2 - X_3)\mathbf{e}_1 + (-X_2)\mathbf{e}_2 + (X_1 + 3X_2 + X_3)\mathbf{e}_3 \\ \mathbf{X} &= (x_1 + 5x_2 + x_3)\mathbf{E}_1 + (-x_2)\mathbf{E}_2 + (-x_1 - 2x_2)\mathbf{E}_3 \end{aligned}$$

so that

$$\mathbf{F} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \qquad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 5 & 1 \\ 0 & -1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$$

Consider the vector $\mathbf{v}(\mathbf{x}) = (2x_1 - x_2)\mathbf{e}_1 + (-3x_2^2 + x_3)\mathbf{e}_2 + (x_1 + x_3)\mathbf{e}_3$ which, in the material description, is

$$\mathbf{V}(\mathbf{X}) = (5X_2 - 2X_3)\mathbf{E}_1 + (X_1 + 3X_2 + X_3 - 3X_2^2)\mathbf{E}_2 + (X_1 + 5X_2)\mathbf{E}_3$$

The material and spatial gradients are

$$\operatorname{Grad}\mathbf{V} = \begin{bmatrix} 0 & 5 & -2 \\ 1 & 3 - 6X_2 & 1 \\ 1 & 5 & 0 \end{bmatrix}, \qquad \operatorname{grad}\mathbf{v} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -6x_2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and it can be seen that

$$\mathrm{Grad}\mathbf{V}\,\mathbf{F}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 6X_2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -6x_2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \mathrm{grad}\,\mathbf{v}$$

■

2.2.2 The Cauchy-Green Strain Tensors

The deformation gradient describes how a line element in the reference configuration maps into a line element in the current configuration. It has been seen that the deformation gradient gives information about deformation (change of shape) and rigid body rotation, but does not encompass information about possible rigid body translations. The deformation and rigid rotation will be separated shortly (see §2.2.5). To this end, consider the following **strain** tensors; these tensors give direct information about the deformation of the body. Specifically, the **Left Cauchy-Green Strain** and **Right Cauchy-Green Strain** tensors give a measure of how the lengths of line elements and angles between line elements (through the vector dot product) change between configurations.

The Right Cauchy-Green Strain

Consider two line elements in the reference configuration $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$ which are mapped into the line elements $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$ in the current configuration. Then, using 1.10.3d,

$$\boxed{\begin{aligned} d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= \left(\mathbf{F}d\mathbf{X}^{(1)}\right)\cdot\left(\mathbf{F}d\mathbf{X}^{(2)}\right) \\ &= d\mathbf{X}^{(1)}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)d\mathbf{X}^{(2)} \\ &= d\mathbf{X}^{(1)}\mathbf{C}d\mathbf{X}^{(2)} \end{aligned}} \qquad \textbf{action of C} \qquad (2.2.9)$$

where, by definition, $\mathbf{C}$ is the right Cauchy-Green Strain[2]

$$\boxed{\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \qquad C_{IJ} = F_{kI}F_{kJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}} \qquad \textbf{Right Cauchy-Green Strain} \qquad (2.2.10)$$

It is a symmetric, positive definite (which will be clear from Eqn. 2.2.17 below), tensor, which implies that it has real positive eigenvalues (*cf.* §1.11.2), and this has important consequences (see later). Explicitly in terms of the base vectors,

$$\mathbf{C} = \left(\frac{\partial x_k}{\partial X_i}\mathbf{E}_I \otimes \mathbf{e}_k\right)\left(\frac{\partial x_m}{\partial X_J}\mathbf{e}_m \otimes \mathbf{E}_J\right) = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}\mathbf{E}_I \otimes \mathbf{E}_J\,. \qquad (2.2.11)$$

Just as the line element $d\mathbf{X}$ is a vector defined in and associated with the reference configuration, $\mathbf{C}$ is defined in and associated with the reference configuration, acting on vectors in the reference configuration, and so is called a **material tensor**.

[2] "right" because $\mathbf{F}$ is on the right of the formula

The inverse of **C**, $\mathbf{C}^{-1}$, is called the **Piola deformation tensor**.

The Left Cauchy-Green Strain

Consider now the following, using Eqn. 1.10.18c:

$$\boxed{\begin{aligned} d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} &= \left(\mathbf{F}^{-1} d\mathbf{x}^{(1)}\right) \cdot \left(\mathbf{F}^{-1} d\mathbf{x}^{(2)}\right) \\ &= d\mathbf{x}^{(1)} \left(\mathbf{F}^{-\mathrm{T}} \mathbf{F}^{-1}\right) d\mathbf{x}^{(2)} \\ &= d\mathbf{x}^{(1)} \mathbf{b}^{-1} d\mathbf{x}^{(2)} \end{aligned}} \qquad \textbf{action of } \mathbf{b}^{-1} \qquad (2.2.12)$$

where, by definition, **b** is the left Cauchy-Green Strain, also known as the **Finger tensor**:

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \qquad b_{ij} = F_{iK} F_{jK} = \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_K}} \qquad \textbf{Left Cauchy-Green Strain} \qquad (2.2.13)$$

Again, this is a symmetric, positive definite tensor, only here, **b** is defined in the current configuration and so is called a **spatial tensor**.

The inverse of **b**, $\mathbf{b}^{-1}$, is called the **Cauchy deformation tensor**.

It can be seen that the right and left Cauchy-Green tensors are related through

$$\mathbf{C} = \mathbf{F}^{-1}\mathbf{b}\mathbf{F}, \qquad \mathbf{b} = \mathbf{F}\mathbf{C}\mathbf{F}^{-1} \qquad (2.2.14)$$

Note that tensors can be material (e.g. **C**), two-point (e.g. **F**) or spatial (e.g. **b**). Whatever type they are, they can always be described using material or spatial coordinates through the motion mapping 2.1.3, that is, using the material or spatial descriptions. Thus one distinguishes between, for example, a spatial tensor, which is an intrinsic property of a tensor, and the spatial description of a tensor.

The Principal Scalar Invariants of the Cauchy-Green Tensors

Using 1.10.10b,

$$\mathrm{tr}\mathbf{C} = \mathrm{tr}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right) = \mathrm{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = \mathrm{tr}\mathbf{b} \qquad (2.2.15)$$

This holds also for arbitrary powers of these tensors, $\mathrm{tr}\mathbf{C}^n = \mathrm{tr}\mathbf{b}^n$, and therefore, from Eqn. 1.11.17, the invariants of **C** and **b** are equal.

2.2.3 The Stretch

The **stretch** (or the **stretch ratio**) λ is defined as the ratio of the length of a deformed line element to the length of the corresponding undeformed line element:

$$\boxed{\lambda = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}} \qquad \textbf{The Stretch} \qquad (2.2.16)$$

From the relations involving the Cauchy-Green Strains, letting $d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} \equiv d\mathbf{X}$, $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)} \equiv d\mathbf{x}$, and dividing across by the square of the length of $d\mathbf{X}$ or $d\mathbf{x}$,

$$\lambda^2 = \left(\frac{|d\mathbf{x}|}{|d\mathbf{X}|}\right)^2 = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}}, \qquad \lambda^{-2} = \left(\frac{|d\mathbf{X}|}{|d\mathbf{x}|}\right)^2 = d\hat{\mathbf{x}}\mathbf{b}^{-1}d\hat{\mathbf{x}} \qquad (2.2.17)$$

Here, the quantities $d\hat{\mathbf{X}} = d\mathbf{X}/|d\mathbf{X}|$ and $d\hat{\mathbf{x}} = d\mathbf{x}/|d\mathbf{x}|$ are unit vectors in the directions of $d\mathbf{X}$ and $d\mathbf{x}$. Thus, through these relations, **C** and **b** determine how much a line element stretches (and, from 2.2.17, **C** and **b** can be seen to be indeed positive definite).

One says that a line element is **extended**, **unstretched** or **compressed** according to $\lambda > 1$, $\lambda = 1$ or $\lambda < 1$.

Stretching along the Coordinate Axes

Consider three line elements lying along the three coordinate axes[3]. Suppose that the material deforms in a special way, such that these line elements undergo a **pure stretch**, that is, they change length with no change in the right angles between them. If the stretches in these directions are λ_1, λ_2 and λ_3, then

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3 \qquad (2.2.18)$$

and the deformation gradient has only diagonal elements in its matrix form:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad F_{iJ} = \lambda_i \delta_{iJ} \quad \text{(no sum)} \qquad (2.2.19)$$

Whereas material undergoes pure stretch along the coordinate directions, line elements off-axes will in general stretch/contract *and* rotate relative to each other. For example, a line element $d\mathbf{X} = [\alpha, \alpha, 0]^{\mathrm{T}}$ stretches by $\lambda = \sqrt{d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}}} = \sqrt{\left(\lambda_1^2 + \lambda_2^2\right)/2}$ with $d\mathbf{x} = [\lambda_1\alpha, \lambda_2\alpha, 0]^{\mathrm{T}}$, and rotates if $\lambda_1 \neq \lambda_2$.

It will be shown below that, for any deformation, there are always three mutually orthogonal directions along which material undergoes a pure stretch. These directions, the coordinate axes in this example, are called the **principal axes** of the material and the associated stretches are called the **principal stretches**.

[3] with the material and spatial basis vectors coincident

The Case of F Real and Symmetric

Consider now another special deformation, where **F** is a real symmetric tensor, in which case the eigenvalues are real and the eigenvectors form an orthonormal basis (*cf.* §1.11.2)[4]. In any given coordinate system, **F** will in general result in the stretching of line elements and the changing of the angles between line elements. However, if one chooses a coordinate set to be the eigenvectors of **F**, then from Eqn. 1.11.11-12 one can write[5]

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i, \qquad [\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.2.20}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of **F**. The eigenvalues are the principal stretches and the eigenvectors are the principal axes. This indicates that as long as **F** is real and symmetric, one can always find a coordinate system along whose axes the material undergoes a pure stretch, with no rotation. This topic will be discussed more fully in §2.2.5 below.

2.2.4 The Green-Lagrange and Euler-Almansi Strain Tensors

Whereas the left and right Cauchy-Green tensors give information about the change in angle between line elements and the stretch of line elements, the **Green-Lagrange strain** and the **Euler-Almansi strain** tensors directly give information about the change in the squared length of elements.

Specifically, when the Green-Lagrange strain **E** operates on a line element d**X**, it gives (half) the change in the squares of the undeformed and deformed lengths:

$$\boxed{\begin{aligned} \frac{|d\mathbf{x}|^2 - |d\mathbf{X}|^2}{2} &= \frac{1}{2}\{d\mathbf{X}\mathbf{C}d\mathbf{X} - d\mathbf{X}\cdot d\mathbf{X}\} \\ &= \frac{1}{2}\{d\mathbf{X}(\mathbf{C}-\mathbf{I})d\mathbf{X}\} \\ &\equiv d\mathbf{X}\mathbf{E}d\mathbf{X} \end{aligned}} \qquad \textbf{action of E} \tag{2.2.21}$$

where

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I}), \quad E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ})} \qquad \textbf{Green-Lagrange Strain} \tag{2.2.22}$$

It is a symmetric positive definite material tensor. Similarly, the (symmetric spatial) Euler-Almansi strain tensor is defined through

---

[4] in fact, **F** in this case will have to be positive definite, with $\det \mathbf{F} > 0$ (see later in §2.2.8)

[5] $\hat{\mathbf{n}}_i$ are the eigenvectors for the basis $\mathbf{e}_i$, $\hat{\mathbf{N}}_I$ for the basis $\hat{\mathbf{E}}_i$, with $\hat{\mathbf{n}}_i$, $\hat{\mathbf{N}}_I$ coincident; when the bases are not coincident, the notion of rotating line elements becomes ambiguous – this topic will be examined later in the context of *objectivity*

$$\boxed{\frac{|d\mathbf{x}|^2 - |d\mathbf{X}|^2}{2} = d\mathbf{x}\,\mathbf{e}\,d\mathbf{x}} \qquad \textbf{action of e} \tag{2.2.23}$$

and

$$\boxed{\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right)} \qquad \textbf{Euler-Almansi Strain} \tag{2.2.24}$$

Physical Meaning of the Components of E

Take a line element in the 1-direction, $d\mathbf{X}_{(1)} = [dX_1,\ 0,\ 0]^{\mathrm{T}}$, so that $d\hat{\mathbf{X}}_{(1)} = [1,\ 0,\ 0]^{\mathrm{T}}$. The square of the stretch of this element is

$$\lambda_{(1)}^2 = d\hat{\mathbf{X}}_{(1)}\mathbf{C}d\hat{\mathbf{X}}_{(1)} = C_{11} \quad \rightarrow \quad E_{11} = \frac{1}{2}(C_{11} - 1) = \frac{1}{2}\left(\lambda_{(1)}^2 - 1\right)$$

The unit extension is $(|d\mathbf{x}| - |d\mathbf{X}|)/|d\mathbf{X}| = \lambda - 1$. Denoting the unit extension of $d\mathbf{X}_{(1)}$ by $\mathbf{E}_{(1)}$, one has

$$E_{11} = \mathbf{E}_{(1)} + \frac{1}{2}\mathbf{E}_{(1)}^2 \tag{2.2.25}$$

and similarly for the other diagonal elements E_{22}, E_{33}.

When the deformation is small, $\mathbf{E}_{(1)}^2$ is small in comparison to $\mathbf{E}_{(1)}$, so that $E_{11} \approx \mathbf{E}_{(1)}$. For small deformations then, the diagonal terms are equivalent to the unit extensions.

Let θ_{12} denote the angle between the deformed elements which were initially parallel to the X_1 and X_2 axes. Then

$$\begin{aligned}\cos\theta_{12} &= \frac{d\mathbf{x}_{(1)}}{|d\mathbf{x}_{(1)}|}\cdot\frac{d\mathbf{x}_{(2)}}{|d\mathbf{x}_{(2)}|} = \frac{|d\mathbf{X}_{(1)}||d\mathbf{X}_{(2)}|}{|d\mathbf{x}_{(1)}||d\mathbf{x}_{(2)}|}\left\{\frac{d\mathbf{X}_{(1)}}{|d\mathbf{X}_{(1)}|}\cdot\mathbf{C}\frac{d\mathbf{X}_{(2)}}{|d\mathbf{X}_{(2)}|}\right\} = \frac{C_{12}}{\lambda_{(1)}\lambda_{(2)}} \\ &= \frac{2E_{12}}{\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}}\end{aligned} \tag{2.2.26}$$

and similarly for the other off-diagonal elements. Note that if $\theta_{12} = \pi/2$, so that there is no angle change, then $E_{12} = 0$. Again, if the deformation is small, then E_{11}, E_{22} are small, and

$$\frac{\pi}{2} - \theta_{12} \approx \sin\left(\frac{\pi}{2} - \theta_{12}\right) = \cos\theta_{12} \approx 2E_{12} \tag{2.2.27}$$

In words: for small deformations, the component E_{12} gives half the change in the original right angle.

2.2.5 Stretch and Rotation Tensors

The deformation gradient can always be decomposed into the product of two tensors, a stretch tensor and a rotation tensor (in one of two different ways, material or spatial versions). This is known as the **polar decomposition**, and is discussed in §1.11.7. One has

$$\boxed{\mathbf{F} = \mathbf{R}\mathbf{U}} \qquad \textbf{Polar Decomposition (Material)} \qquad (2.2.28)$$

Here, **R** is a proper orthogonal tensor, i.e. $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ with $\det \mathbf{R} = 1$, called *the* **rotation tensor**. It is a measure of the local rotation at **X**.

The decomposition is not unique; it is made unique by choosing **U** to be a *symmetric* tensor, called the **right stretch tensor**. It is a measure of the local stretching (or contraction) of material at **X**. Consider a line element $d\mathbf{X}$. Then

$$\lambda d\hat{\mathbf{x}} = \mathbf{F}d\hat{\mathbf{X}} = \mathbf{R}\mathbf{U}d\hat{\mathbf{X}} \qquad (2.2.29)$$

and so {▲Problem 2}

$$\lambda^2 = d\hat{\mathbf{X}}\mathbf{U} \cdot \mathbf{U}d\hat{\mathbf{X}} \qquad (2.2.30)$$

Thus (this is a definition of **U**)

$$\boxed{\mathbf{U} = \sqrt{\mathbf{C}} \qquad (\mathbf{C} = \mathbf{U}\mathbf{U})} \qquad \textbf{The Right Stretch Tensor} \qquad (2.2.31)$$

From 2.2.30, the right Cauchy-Green strain **C** (and by consequence the Euler-Lagrange strain **E**) only give information about the stretch of line elements; it does not give information about the rotation that is experienced by a particle during motion. The deformation gradient **F**, however, contains information about both the stretch and rotation. It can also be seen from 2.2.30-1 that **U** is a material tensor.

Note that, since

$$d\mathbf{x} = \mathbf{R}(\mathbf{U}d\mathbf{X}),$$

the undeformed line element is *first* stretched by **U** and is *then* rotated by **R** into the deformed element $d\mathbf{x}$ (the element may also undergo a rigid body translation **c**), Fig. 2.2.4. **R** is a two-point tensor.

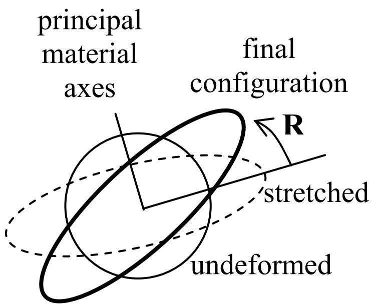

Figure 2.2.4: the polar decomposition

Evaluation of U

In order to evaluate **U**, it is necessary to evaluate $\sqrt{\mathbf{C}}$. To evaluate the square-root, **C** must first be obtained in relation to its principal axes, so that it is diagonal, and then the square root can be taken of the diagonal elements, since its eigenvalues will be positive (see §1.11.6). Then the tensor needs to be transformed back to the original coordinate system.

Example

Consider the motion

$$x_1 = 2X_1 - 2X_2, \quad x_2 = X_1 + X_2, \quad x_3 = X_3$$

The (homogeneous) deformation of a unit square in the $x_1 - x_2$ plane is as shown in Fig. 2.2.5.

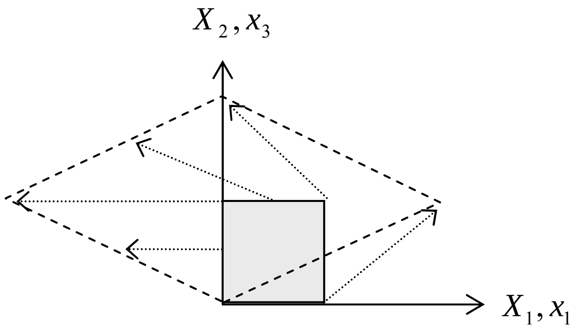

Figure 2.2.5: deformation of a square

One has

$$[\mathbf{F}] = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: (\mathbf{e}_i \otimes \mathbf{E}_j), \qquad [\mathbf{C}] = [\mathbf{F}^{\mathrm{T}}\mathbf{F}] = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: (\mathbf{E}_i \otimes \mathbf{E}_j)$$

Note that **F** is not symmetric, so that it might have only one real eigenvalue (in fact here it does have complex eigenvalues), and the eigenvectors may not be orthonormal. **C**, on the other hand, by its very definition, *is* symmetric; it is in fact positive definite and so has positive real eigenvalues forming an orthonormal set.

To determine the principal axes of **C**, it is necessary to evaluate the eigenvalues/eigenvectors of the tensor. The eigenvalues are the roots of the characteristic equation 1.11.5,

$$\alpha^3 - \mathrm{I}_{\mathbf{C}}\alpha^2 + \mathrm{II}_{\mathbf{C}}\alpha - \mathrm{III}_{\mathbf{C}} = 0$$

and the first, second and third invariants of the tensor are given by 1.11.6 so that $\alpha^3 - 11\alpha^2 + 26\alpha - 16 = 0$, with roots $\alpha = 8, 2, 1$. The three corresponding eigenvectors are found from 1.11.8,

$$\begin{aligned} (C_{11}-\alpha)\hat{N}_1 + C_{12}\hat{N}_2 + C_{13}\hat{N}_3 &= 0 & \qquad (5-\alpha)\hat{N}_1 - 3\hat{N}_2 &= 0 \\ C_{21}\hat{N}_1 + (C_{22}-\alpha)\hat{N}_2 + C_{23}\hat{N}_3 &= 0 \quad \rightarrow & -3\hat{N}_1 + (5-\alpha)\hat{N}_2 &= 0 \\ C_{31}\hat{N}_1 + C_{32}\hat{N}_2 + (C_{33}-\alpha)\hat{N}_3 &= 0 & (1-\alpha)\hat{N}_3 &= 0 \end{aligned}$$

Thus (normalizing the eigenvectors so that they are unit vectors, and form a right-handed set, Fig. 2.2.6):

(i) for $\alpha = 8$, $-3\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 - 3\hat{N}_2 = 0, -7\hat{N}_3 = 0$, $\quad \hat{\mathbf{N}}_1 = \frac{1}{\sqrt{2}}\mathbf{E}_1 - \frac{1}{\sqrt{2}}\mathbf{E}_2$

(ii) for $\alpha = 2$, $3\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 + 3\hat{N}_2 = 0, -\hat{N}_3 = 0$, $\quad \hat{\mathbf{N}}_2 = \frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$

(iii) for $\alpha = 1$, $4\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 + 4\hat{N}_2 = 0, 0\hat{N}_3 = 0$, $\quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$

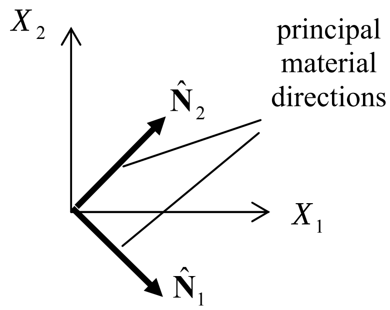

Figure 2.2.6: deformation of a square

Thus the right Cauchy-Green strain tensor **C**, with respect to coordinates with base vectors $\mathbf{E}_1' = \hat{\mathbf{N}}_1$, $\mathbf{E}_2' = \hat{\mathbf{N}}_2$ and $\mathbf{E}_3' = \hat{\mathbf{N}}_3$, that is, in terms of principal coordinates, is

$$[\mathbf{C}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_j$$

This result can be checked using the tensor transformation formulae 1.13.6, $[\mathbf{C}'] = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{C}][\mathbf{Q}]$, where Q is the transformation matrix of direction cosines (see also the example at the end of §1.5.2),

$$Q_{ij} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1' & \mathbf{e}_1 \cdot \mathbf{e}_2' & \mathbf{e}_1 \cdot \mathbf{e}_3' \\ \mathbf{e}_2 \cdot \mathbf{e}_1' & \mathbf{e}_2 \cdot \mathbf{e}_2' & \mathbf{e}_2 \cdot \mathbf{e}_3' \\ \mathbf{e}_3 \cdot \mathbf{e}_1' & \mathbf{e}_3 \cdot \mathbf{e}_2' & \mathbf{e}_3 \cdot \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 & \hat{\mathbf{N}}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The stretch tensor **U**, with respect to the principal directions is

$$[\mathbf{U}] = \left[\sqrt{\mathbf{C}}\right] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{basis}: \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_j$$

These eigenvalues of **U** (which are the square root of those of **C**) are the principal stretches and, as before, they are labeled $\lambda_1, \lambda_2, \lambda_3$.

In the original coordinate system, using the inverse tensor transformation rule 1.13.6, $[\mathbf{U}] = [\mathbf{Q}][\mathbf{U}'][\mathbf{Q}]^{\mathrm{T}}$,

$$[\mathbf{U}] = \begin{bmatrix} 3/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 3/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: \mathbf{E}_i \otimes \mathbf{E}_j$$

so that

$$[\mathbf{R}] = \left[\mathbf{F}\mathbf{U}^{-1}\right] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: \mathbf{e}_i \otimes \mathbf{E}_j$$

and it can be verified that **R** is a rotation tensor, i.e. is proper orthogonal.

Returning to the deformation of the unit square, the stretch and rotation are as illustrated in Fig. 2.2.7 – the action of **U** is indicated by the arrows, deforming the unit square to the dotted parallelogram, whereas **R** rotates the parallelogram through 45° as a rigid body to its final position.

Note that the line elements along the diagonals (indicated by the heavy lines) lie along the principal directions of **U** and therefore undergo a pure stretch; the diagonal in the $\hat{\mathbf{N}}_1$ direction has stretched but has also moved with a rigid translation.

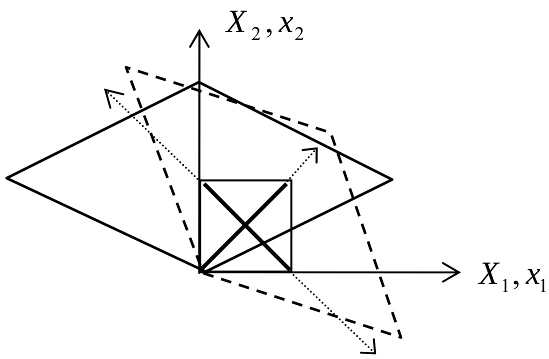

Figure 2.2.7: stretch and rotation of a square

■

Spatial Description

A polar decomposition can be made in the spatial description. In that case,

$$\boxed{\mathbf{F} = \mathbf{vR}} \qquad \textbf{Polar Decomposition (Spatial)} \tag{2.2.32}$$

Here **v** is a symmetric, positive definite second order tensor called the **left stretch tensor**, and $\mathbf{vv} = \mathbf{b}$, where **b** is the left Cauchy-Green tensor. **R** is the same rotation tensor as appears in the material description. Thus an elemental sphere can be regarded as first stretching into an ellipsoid, whose axes are the principal material axes (the principal axes of **U**), and then rotating; or first rotating, and then stretching into an ellipsoid whose axes are the **principal spatial axes** (the principal axes of **v**). The end result is the same.

The development in the spatial description is similar to that given above for the material description, and one finds by analogy with 2.2.30,

$$\lambda^{-2} = d\hat{\mathbf{x}}\mathbf{v}^{-1} \cdot \mathbf{v}^{-1} d\hat{\mathbf{x}} \tag{2.2.33}$$

In the above example, it turns out that **v** takes the simple diagonal form

$$[\mathbf{v}] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \mathbf{e}_i \otimes \mathbf{e}_j \, .$$

so the unit square rotates first and then undergoes a pure stretch along the coordinate axes, which are the principal spatial axes, and the sequence is now as shown in Fig. 2.2.9.

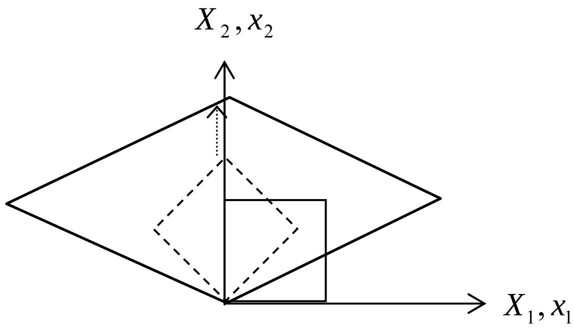

Figure 2.2.8: stretch and rotation of a square in spatial description

Relationship between the Material and Spatial Decompositions

Comparing the two decompositions, one sees that the material and spatial tensors involved are related through

$$\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{b} = \mathbf{R}\mathbf{C}\mathbf{R}^{\mathrm{T}} \tag{2.2.34}$$

Further, suppose that $\mathbf{U}$ has an eigenvalue λ and an eigenvector $\hat{\mathbf{N}}$. Then $\mathbf{U}\hat{\mathbf{N}} = \lambda\hat{\mathbf{N}}$, so that $\mathbf{RUN} = \lambda\mathbf{RN}$. But $\mathbf{RU} = \mathbf{vR}$, so $\mathbf{v}\left(\mathbf{R}\hat{\mathbf{N}}\right) = \lambda\left(\mathbf{R}\hat{\mathbf{N}}\right)$. Thus $\mathbf{v}$ also has an eigenvalue λ, but an eigenvector $\hat{\mathbf{n}} = \mathbf{R}\hat{\mathbf{N}}$. From this, it is seen that the rotation tensor $\mathbf{R}$ maps the principal material axes into the principal spatial axes. It also follows that $\mathbf{R}$ and $\mathbf{F}$ can be written explicitly in terms of the material and spatial principal axes (compare the first of these with 1.10.25)[6]:

$$\mathbf{R} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i, \qquad \mathbf{F} = \mathbf{RU} = \mathbf{R}\sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i \tag{2.2.35}$$

and the deformation gradient acts on the principal axes base vectors according to {▲Problem 4}

$$\mathbf{F}\hat{\mathbf{N}}_i = \lambda_i \hat{\mathbf{n}}_i, \qquad \mathbf{F}^{-\mathrm{T}}\hat{\mathbf{N}}_i = \frac{1}{\lambda_i}\hat{\mathbf{n}}_i, \qquad \mathbf{F}^{-1}\hat{\mathbf{n}}_i = \frac{1}{\lambda_i}\hat{\mathbf{N}}_i, \qquad \mathbf{F}^{\mathrm{T}}\hat{\mathbf{n}}_i = \lambda_i \hat{\mathbf{N}}_i \tag{2.2.36}$$

The representation of $\mathbf{F}$ and $\mathbf{R}$ in terms of both material and spatial principal base vectors in 2.3.35 highlights their two-point character.

Other Strain Measures

Some other useful measures of strain are

The **Hencky strain** measure: $\mathbf{H} \equiv \ln \mathbf{U}$ (material) or $\mathbf{h} = \ln \mathbf{v}$ (spatial)

---

[6] this is not a spectral decomposition of **F** (unless **F** happens to be symmetric, which it must be in order to have a spectral decomposition)

The **Biot strain** measure: $\overline{\mathbf{B}} = \mathbf{U} - \mathbf{I}$ (material) or $\overline{\mathbf{b}} = \mathbf{v} - \mathbf{I}$ (spatial)

The Hencky strain is evaluated by first evaluating **U** along the principal axes, so that the logarithm can be taken of the diagonal elements.

The material tensors **H**, $\overline{\mathbf{B}}$, **C**, **U** and **E** are coaxial tensors, with the same eigenvectors $\hat{\mathbf{N}}_i$. Similarly, the spatial tensors **h**, $\overline{\mathbf{b}}$, **b**, **v** and **e** are coaxial with the same eigenvectors $\hat{\mathbf{n}}_i$. From the definitions, the spectral decompositions of these tensors are

$$\begin{aligned}
\mathbf{U} &= \sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{v} &= \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{C} &= \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{b} &= \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{E} &= \sum_{i=1}^{3} \tfrac{1}{2}\left(\lambda_i^2 - 1\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{e} &= \sum_{i=1}^{3} \tfrac{1}{2}\left(1 - 1/\lambda_i^2\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{H} &= \sum_{i=1}^{3} \left(\ln \lambda_i\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{h} &= \sum_{i=1}^{3} \left(\ln \lambda_i\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\overline{\mathbf{B}} &= \sum_{i=1}^{3} \left(\lambda_i - 1\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \overline{\mathbf{b}} &= \sum_{i=1}^{3} \left(\lambda_i - 1\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i
\end{aligned} \tag{2.2.37}$$

Deformation of a Circular Material Element

A circular material element will deform into an ellipse, as indicated in Figs. 2.2.2 and 2.2.4. This can be shown as follows. With respect to the principal axes, an undeformed line element $d\mathbf{X} = dX_1\mathbf{N}_1 + dX_2\mathbf{N}_2$ has magnitude squared $\left(dX_1\right)^2 + \left(dX_2\right)^2 = c^2$, where c is the radius of the circle, Fig. 2.2.9. The deformed element is $d\mathbf{x} = \mathbf{U}d\mathbf{X}$, or $d\mathbf{x} = \lambda_1 dX_1\mathbf{N}_1 + \lambda_2 dX_2\mathbf{N}_2 \equiv dx_1\mathbf{n}_1 + dx_2\mathbf{n}_2$. Thus $dx_1/\lambda_1 = dX_1$, $dx_2/\lambda_2 = dX_2$, which leads to the standard equation of an ellipse with major and minor axes $\lambda_1 c, \lambda_2 c$: $\left(dx_1/\lambda_1 c\right)^2 + \left(dx_2/\lambda_2 c\right)^2 = 1$.

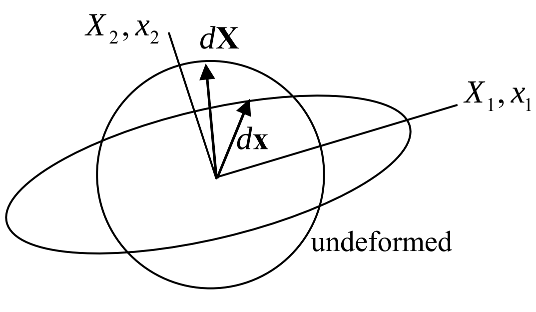

Figure 2.2.9: a circular element deforming into an ellipse

2.2.6 Some Simple Deformations

In this section, some elementary deformations are considered.

Pure Stretch

This deformation has already been seen, but now it can be viewed as a special case of the polar decomposition. The motion is

$$\boxed{x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3} \qquad \textbf{Pure Stretch} \qquad (2.2.38)$$

and the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Here, $\mathbf{R} = \mathbf{I}$ and there is no rotation. $\mathbf{U} = \mathbf{F}$ and the principal material axes are coincident with the material coordinate axes. $\lambda_1, \lambda_2, \lambda_3$, the eigenvalues of **U**, are the principal stretches.

Stretch with rotation

Consider the motion

$$x_1 = X_1 - kX_2, \quad x_2 = kX_1 + X_2, \quad x_3 = X_3$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & -k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sec\theta & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $k = \tan\theta$. This decomposition shows that the deformation consists of material stretching by $\sec\theta \, (= \sqrt{1+k^2})$, the principal stretches, along each of the axes, followed by a rigid body rotation through an angle θ about the $X_3 = 0$ axis, Fig. 2.2.10. The deformation is relatively simple because the principal material axes are aligned with the material coordinate axes (so that **U** is diagonal). The deformation of the unit square is as shown in Fig. 2.2.10.

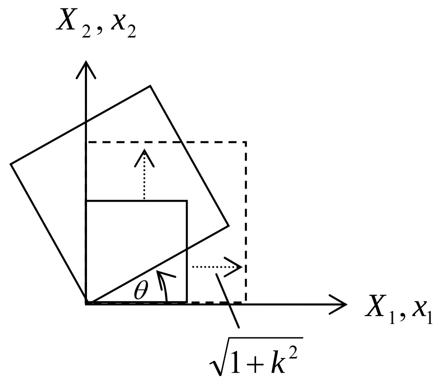

Figure 2.2.10: stretch with rotation

Pure Shear

Consider the motion

$$\boxed{x_1 = X_1 + kX_2, \quad x_2 = kX_1 + X_2, \quad x_3 = X_3} \quad \textbf{Pure Shear} \qquad (2.2.39)$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where, since $\mathbf{F}$ is symmetric, there is no rotation, and $\mathbf{F} = \mathbf{U}$. Since the rotation is zero, one can work directly with $\mathbf{U}$ and not have to consider $\mathbf{C}$. The eigenvalues of $\mathbf{U}$, the principal stretches, are $1+k, 1-k, 1$, with corresponding principal directions $\hat{\mathbf{N}}_1 = \frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$, $\hat{\mathbf{N}}_2 = -\frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$ and $\hat{\mathbf{N}}_3 = \mathbf{E}_3$.

The deformation of the unit square is as shown in Fig. 2.2.11. The diagonal indicated by the heavy line stretches by an amount $1+k$ whereas the other diagonal contracts by an amount $1-k$. An element of material along the diagonal will undergo a pure stretch as indicated by the stretching of the dotted box.

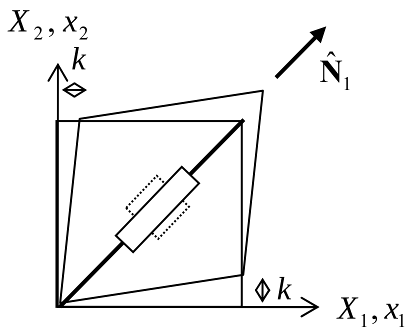

Figure 2.2.11: pure shear

Simple Shear

Consider the motion

$$\boxed{x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3} \qquad \textbf{Simple Shear} \qquad (2.2.40)$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The invariants of **C** are $\mathrm{I}_{\mathbf{C}} = 3 + k^2$, $\mathrm{II}_{\mathbf{C}} = 3 + k^2$, $\mathrm{III}_{\mathbf{C}} = 1$ and the characteristic equation is $\lambda^3 + (3 + k^2)\lambda(1-\lambda) - 1 = 0$, so the principal values of **C** are $\lambda = 1 + \frac{1}{2}k^2 \pm \frac{1}{2}k\sqrt{4+k^2}, 1$. The principal values of **U** are the (positive) square-roots of these: $\lambda = \frac{1}{2}\sqrt{4+k^2} \pm \frac{1}{2}k, 1$. These can be written as $\lambda = \sec\theta \pm \tan\theta, 1$ by letting $\tan\theta = \frac{1}{2}k$. The corresponding eigenvectors of **C** are

$$\hat{\mathbf{N}}_1 = \frac{k}{\frac{1}{2}k^2 + \frac{1}{2}k\sqrt{4+k^2}}\mathbf{E}_1 + \mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = \frac{k}{\frac{1}{2}k^2 - \frac{1}{2}k\sqrt{4+k^2}}\mathbf{E}_1 + \mathbf{E}_2, \quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$$

or, normalizing so that they are of unit size, and writing in terms of θ,

$$\hat{\mathbf{N}}_1 = \sqrt{\frac{1-\sin\theta}{2}}\mathbf{E}_1 + \sqrt{\frac{1+\sin\theta}{2}}\mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = -\sqrt{\frac{1+\sin\theta}{2}}\mathbf{E}_1 + \sqrt{\frac{1-\sin\theta}{2}}\mathbf{E}_2, \quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$$

The transformation matrix of direction cosines is then

$$[\mathbf{Q}] = \begin{bmatrix} \sqrt{(1-\sin\theta)/2} & -\sqrt{(1+\sin\theta)/2} & 0 \\ \sqrt{(1+\sin\theta)/2} & \sqrt{(1-\sin\theta)/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that, using the inverse transformation formula, $[\mathbf{U}] = [\mathbf{Q}][\mathbf{U}'][\mathbf{Q}]^{\mathrm{T}}$, one obtains **U** in terms of the original coordinates, and hence

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & (1+\sin^2\theta)/\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The deformation of the unit square is shown in Fig. 2.2.12 (for $k = 0.2, \theta = 5.7^{\circ}$). The square first undergoes a pure stretch/contraction along the principal axes, and is then brought to its final position by a negative (clockwise) rotation of θ.

For this deformation, $\det \mathbf{F} = 1$ and, as will be shown below, this means that the simple shear deformation is volume-preserving.

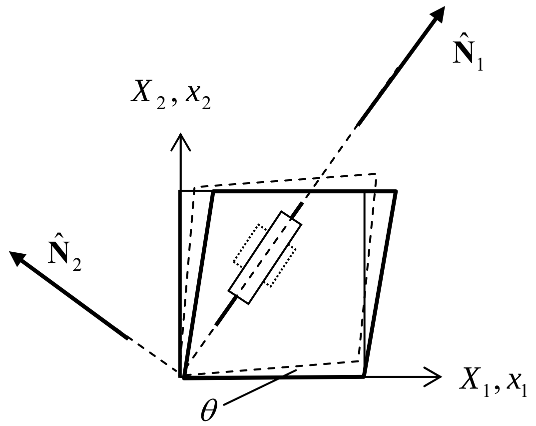

Figure 2.2.12: simple shear

2.2.7 Displacement & Displacement Gradients

The displacement of a material particle[7] is the movement it undergoes in the transition from the reference configuration to the current configuration. Thus, Fig. 2.2.13,[8]

$$\boxed{\mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}} \quad \textbf{Displacement (Material Description)} \qquad (2.2.41)$$

$$\boxed{\mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t)} \quad \textbf{Displacement (Spatial Description)} \qquad (2.2.42)$$

Note that **U** and **u** have the same values, they just have different arguments.

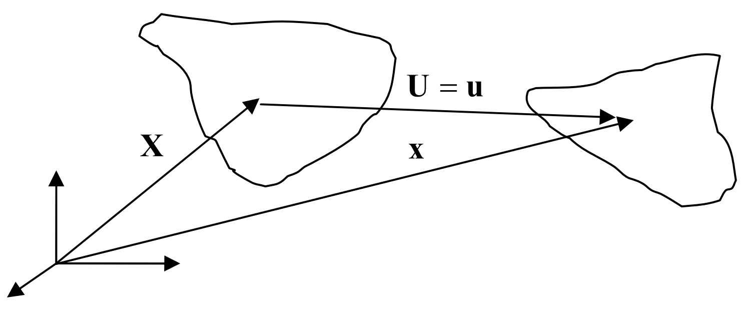

Figure 2.2.13: the displacement

[7] In solid mechanics, the motion and deformation are often described in terms of the displacement **u**. In fluid mechanics, however, the primary field quantity describing the kinematic properties is the velocity **v** (and the acceleration $\mathbf{a} = \dot{\mathbf{v}}$) – see later.
[8] The material displacement **U** here is not to be confused with the right stretch tensor discussed earlier.

Displacement Gradients

The displacement gradient in the material and spatial descriptions, $\partial \mathbf{U}(\mathbf{X},t)/\partial \mathbf{X}$ and $\partial \mathbf{u}(\mathbf{x},t)/\partial \mathbf{x}$, are related to the deformation gradient and the inverse deformation gradient through

$$
\begin{aligned}
\mathrm{Grad}\mathbf{U} &= \frac{\partial \mathbf{U}}{\partial \mathbf{X}} = \frac{\partial(\mathbf{x}-\mathbf{X})}{\partial \mathbf{X}} = \mathbf{F}-\mathbf{I} \qquad & \frac{\partial U_i}{\partial X_j} &= \frac{\partial x_i}{\partial X_j} - \delta_{ij} \\
\mathrm{grad}\mathbf{u} &= \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial(\mathbf{x}-\mathbf{X})}{\partial \mathbf{x}} = \mathbf{I}-\mathbf{F}^{-1} \qquad & \frac{\partial u_i}{\partial x_j} &= \delta_{ij} - \frac{\partial X_i}{\partial x_j}
\end{aligned}
\tag{2.2.43}
$$

and it is clear that the displacement gradients are related through (see Eqn. 2.2.8)

$$
\mathrm{grad}\mathbf{u} = \mathrm{Grad}\,\mathbf{U}\,\mathbf{F}^{-1} \tag{2.2.44}
$$

The deformation can now be written in terms of either the material or spatial displacement gradients:

$$
\begin{aligned}
d\mathbf{x} &= d\mathbf{X} + d\mathbf{U}(\mathbf{X}) = d\mathbf{X} + \mathrm{Grad}\mathbf{U}\,d\mathbf{X} \\
d\mathbf{x} &= d\mathbf{X} + d\mathbf{u}(\mathbf{x}) \;= d\mathbf{X} + \mathrm{grad}\mathbf{u}\,d\mathbf{x}
\end{aligned}
\tag{2.2.45}
$$

Example

Consider again the extension of the bar shown in Fig. 2.1.5. The displacement is

$$
\mathbf{U}(\mathbf{X}) = \left(t + 3X_1 t\right)\mathbf{E}_1, \qquad \mathbf{u}(\mathbf{x}) = \left(\frac{t+3x_1 t}{1+3t}\right)\mathbf{e}_1
$$

and the displacement gradients are

$$
\mathrm{Grad}\mathbf{U} = 3t\mathbf{E}_1, \qquad \mathrm{grad}\mathbf{u} = \left(\frac{3t}{1+3t}\right)\mathbf{e}_1
$$

The displacement is plotted in Fig. 2.2.14 for $t=1$. The two gradients $\partial U_1/\partial X_1$ and $\partial u_1/\partial x_1$ have different values (see the horizontal axes on Fig. 2.2.14). In this example, $\partial U_1/\partial X_1 > \partial u_1/\partial x_1$ – the change in displacement is not as large when "seen" from the spatial coordinates.

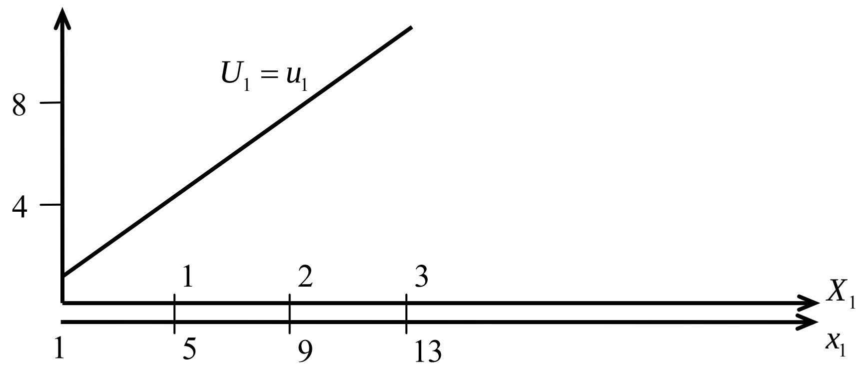

Figure 2.1.14: displacement and displacement gradient

■

Strains in terms of Displacement Gradients

The strains can be written in terms of the displacement gradients. Using 1.10.3b,

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) \\
&= \frac{1}{2}\left(\left(\mathrm{Grad}\mathbf{U} + \mathbf{I}\right)^{\mathrm{T}}\left(\mathrm{Grad}\mathbf{U} + \mathbf{I}\right) - \mathbf{I}\right) \\
&= \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + \left(\mathrm{Grad}\mathbf{U}\right)^{\mathrm{T}} + \left(\mathrm{Grad}\mathbf{U}\right)^{\mathrm{T}}\mathrm{Grad}\mathbf{U}\right), \quad E_{IJ} = \frac{1}{2}\left\{\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} + \frac{\partial U_K}{\partial X_I}\frac{\partial U_K}{\partial X_J}\right\}
\end{aligned} \tag{2.2.46a}
$$

$$
\begin{aligned}
\mathbf{e} &= \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right) \\
&= \frac{1}{2}\left(\mathbf{I} - \left(\mathbf{I} - \mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\left(\mathbf{I} - \mathrm{grad}\mathbf{u}\right)\right) \\
&= \frac{1}{2}\left(\mathrm{grad}\mathbf{u} + \left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}} - \left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\mathrm{grad}\mathbf{u}\right), \qquad e_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\}
\end{aligned} \tag{2.2.46b}
$$

Small Strain

If the displacement gradients are small, then the quadratic terms, their products, are small relative to the gradients themselves, and may be neglected. With this assumption, the Green-Lagrange strain **E** (and the Euler-Almansi strain) reduces to the **small-strain tensor**,

$$
\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + \left(\mathrm{Grad}\mathbf{U}\right)^{\mathrm{T}}\right), \qquad \varepsilon_{IJ} = \frac{1}{2}\left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I}\right) \tag{2.2.47}
$$

Since in this case the displacement gradients are small, it does not matter whether one refers the strains to the reference or current configurations – the error is of the same order as the quadratic terms already neglected[9], so the small strain tensor can equally well be written as

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^{\mathrm{T}}\right), \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \qquad \textbf{Small Strain Tensor} \qquad (2.2.48)$$

2.2.8 The Deformation of Area and Volume Elements

Line elements transform between the reference and current configurations through the deformation gradient. Here, the transformation of area and volume elements is examined.

The Jacobian Determinant

The **Jacobian determinant** of the deformation is defined as the determinant of the deformation gradient,

$$\boxed{J(\mathbf{X},t) = \det \mathbf{F}} \qquad \det\mathbf{F} = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} \qquad \textbf{The Jacobian Determinant} \qquad (2.2.49)$$

Equivalently, it can be considered to be the Jacobian of the transformation from material to spatial coordinates (see Appendix 1.B.2).

From Eqn. 1.3.17, the Jacobian can also be written in the form of the triple scalar product

$$J = \frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3}\right) \qquad (2.2.50)$$

Consider now a volume element in the reference configuration, a parallelepiped bounded by the three line-elements $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$. The volume of the parallelepiped[10] is given by the triple scalar product (Eqns. 1.1.4):

$$dV = d\mathbf{X}^{(1)} \cdot \left(d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)}\right) \qquad (2.2.51)$$

After deformation, the volume element is bounded by the three vectors $d\mathbf{x}^{(i)}$, so that the volume of the deformed element is, using 1.10.16f,

[9] although large rigid body rotations must not be allowed – see §2.7 .
[10] the vectors should form a right-handed set so that the volume is positive.

$$\begin{aligned}
dv &= d\mathbf{x}^{(1)} \cdot \left(d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)}\right) \\
&= \mathbf{F}d\mathbf{X}^{(1)} \cdot \left(\mathbf{F}d\mathbf{X}^{(2)} \times \mathbf{F}d\mathbf{X}^{(3)}\right) \\
&= \det\mathbf{F}\left(d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)}\right) \\
&= \det\mathbf{F}\; dV
\end{aligned} \tag{2.2.52}$$

Thus the scalar J is a measure of how the volume of a material element has changed with the deformation and for this reason is often called the **volume ratio**.

$$\boxed{dv = J\, dV} \qquad \textbf{Volume Ratio} \tag{2.2.53}$$

Since volumes cannot be negative, one must insist on physical grounds that $J > 0$. Also, since $\mathbf{F}$ has an inverse, $J \neq 0$. Thus one has the restriction

$$J > 0 \tag{2.2.54}$$

Note that a rigid body rotation does not alter the volume, so the volume change is completely characterised by the stretching tensor $\mathbf{U}$. Three line elements lying along the principal directions of $\mathbf{U}$ form an element with volume dV, and then undergo pure stretch into new line elements defining an element of volume $dv = \lambda_1\lambda_2\lambda_3 dV$, where λ_i are the principal stretches, Fig. 2.2.15. The unit change in volume is therefore also

$$\frac{dv - dV}{dV} = \lambda_1\lambda_2\lambda_3 - 1 \tag{2.2.55}$$

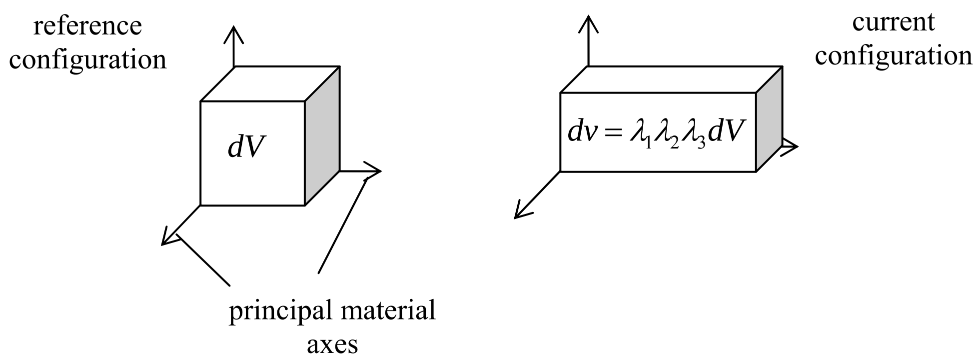

Figure 2.2.15: change in volume

For example, the volume change for pure shear is $-k^2$ (volume decreasing) and, for simple shear, is zero (*cf*. Eqn. 2.2.40 *et seq*., $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)(1) - 1 = 0$).

An **incompressible** material is one for which the volume change is zero, i.e. the deformation is isochoric. For such a material, $J = 1$, and the three principal stretches are not independent, but are constrained by

$$\boxed{\lambda_1\lambda_2\lambda_3 = 1} \qquad \textbf{Incompressibility Constraint} \tag{2.2.56}$$

Nanson's Formula

Consider an area element in the reference configuration, with area dS , unit normal $\hat{\mathbf{N}}$, and bounded by the vectors $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$, Fig. 2.2.16. Then

$$\hat{\mathbf{N}} dS = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \tag{2.2.57}$$

The volume of the element bounded by the vectors $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and some arbitrary line element $d\mathbf{X}$ is $dV = \hat{\mathbf{N}} dS \cdot d\mathbf{X}$. The area element is now deformed into an element of area ds with normal $\hat{\mathbf{n}}$ and bounded by the line elements $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$. The volume of the new element bounded by the area element and $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ is then

$$dv = \hat{\mathbf{n}} ds \cdot d\mathbf{x} = \hat{\mathbf{n}} ds \cdot \mathbf{F} d\mathbf{X} \equiv J \hat{\mathbf{N}} dS \cdot d\mathbf{X} \tag{2.2.58}$$

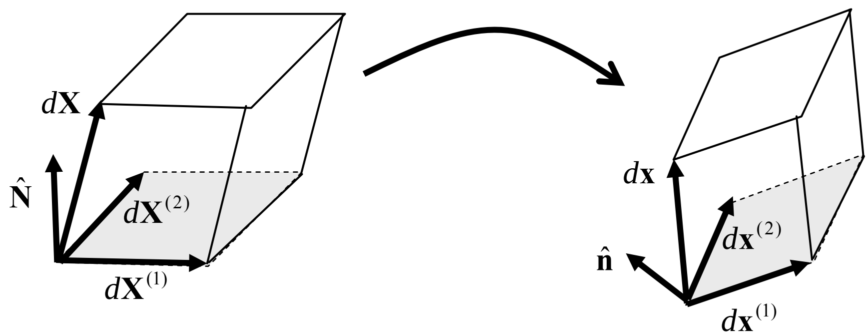

Figure 2.2.16: change of surface area

Thus, since $d\mathbf{X}$ is arbitrary, and using 1.10.3d,

$$\boxed{\hat{\mathbf{n}} ds = J\, \mathbf{F}^{-\mathrm{T}} \hat{\mathbf{N}} dS} \qquad \textbf{Nanson's Formula} \tag{2.2.59}$$

Nanson's formula shows how the vector element of area $\hat{\mathbf{n}} ds$ in the current configuration is related to the vector element of area $\hat{\mathbf{N}} dS$ in the reference configuration.

2.2.9 Inextensibility and Orientation Constraints

A constraint on the principal stretches was introduced for an incompressible material, 2.2.56. Other constraints arise in practice. For example, consider a material which is inextensible in a certain direction, defined by a unit vector $\hat{\mathbf{A}}$ in the reference configuration. It follows that $\left|\mathbf{F}\hat{\mathbf{A}}\right| = 1$ and the constraint can be expressed as 2.2.17,

$$\boxed{\hat{\mathbf{A}} \mathbf{C} \hat{\mathbf{A}} = 1} \qquad \textbf{Inextensibility Constraint} \tag{2.2.60}$$

If there are two such directions in a plane, defined by $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, making angles θ and ϕ respectively with the principal material axes $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2$, then

$$1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and $\left(\lambda_1^2 - \lambda_2^2\right)\cos^2\theta = 1 - \lambda_2^2 = \left(\lambda_1^2 - \lambda_2^2\right)\cos^2\phi$. It follows that $\phi = \theta$, $\phi = \theta + \pi$, $\theta + \phi = \pi$ or $\theta + \phi = 2\pi$ (or $\lambda_1 = \lambda_2 = 1$, i.e. no deformation).

Similarly, one can have orientation constraints. For example, suppose that the direction associated with the vector $\hat{\mathbf{A}}$ maintains that direction. Then

$$\boxed{\mathbf{F}\hat{\mathbf{A}} = \mu\hat{\mathbf{A}}} \qquad \textbf{Orientation Constraint} \qquad (2.2.61)$$

for some scalar $\mu > 0$.

2.2.10 Problems

1. In equations 2.2.8, one has from the chain rule
$$\mathrm{grad}\phi = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i = \frac{\partial \phi}{\partial X_m}\frac{\partial X_m}{\partial x_i}\mathbf{e}_i = \left(\frac{\partial \phi}{\partial X_j}\mathbf{E}_j\right)\left(\frac{\partial X_m}{\partial x_i}\mathbf{E}_m \otimes \mathbf{e}_i\right) = \mathrm{Grad}\phi\,\mathbf{F}^{-1}$$
Derive the other two relations.
2. Take the dot product $(\lambda d\hat{\mathbf{x}}) \cdot (\lambda d\hat{\mathbf{x}})$ in Eqn. 2.2.29. Then use $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, $\mathbf{U}^{\mathrm{T}} = \mathbf{U}$, and 1.10.3e to show that
$$\lambda^2 = \frac{d\mathbf{X}}{|d\mathbf{X}|}\mathbf{U} \cdot \mathbf{U}\frac{d\mathbf{X}}{|d\mathbf{X}|}$$
3. For the deformation
$$x_1 = X_1 + 2X_3, \quad x_2 = X_2 - 2X_3, \quad x_3 = -2X_1 + 2X_2 + X_3$$
   (a) Determine the Deformation Gradient and the Right Cauchy-Green tensors
   (b) Consider the two line elements $d\mathbf{X}^{(1)} = \mathbf{e}_1$, $d\mathbf{X}^{(2)} = \mathbf{e}_2$ (emanating from (0,0,0)). Use the Right Cauchy Green tensor to determine whether these elements in the current configuration ($d\mathbf{x}^{(1)}$, $d\mathbf{x}^{(2)}$) are perpendicular.
   (c) Use the right Cauchy Green tensor to evaluate the stretch of the line element $d\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_2$, and hence determine whether the element contracts, stretches, or stays the same length after deformation.
   (d) Determine the Green-Lagrange and Eulerian strain tensors
   (e) Decompose the deformation into a stretching and rotation (check that **U** is symmetric and **R** is orthogonal). What are the principal stretches?
4. Derive Equations 2.2.36.
5. For the deformation
$$x_1 = X_1, \quad x_2 = X_2 + X_3, \quad x_3 = aX_2 + X_3$$

(a) Determine the displacement vector in both the material and spatial forms
(b) Determine the displaced location of the particles in the undeformed state which originally comprise
 (i) the plane circular surface $X_1 = 0, \quad X_2^2 + X_3^2 = 1/(1-a^2)$
 (ii) the infinitesimal cube with edges along the coordinate axes of length $dX_i = \varepsilon$

 Sketch the displaced configurations if $a = 1/2$

6. For the deformation
$$x_1 = X_1 + aX_2, \quad x_2 = X_2 + aX_3, \quad x_3 = aX_1 + X_3$$
(a) Determine the displacement vector in both the material and spatial forms
(b) Calculate the full material (Green-Lagrange) strain tensor and the full spatial strain tensor
(c) Calculate the infinitesimal strain tensor as derived from the material and spatial tensors, and compare them for the case of very small a.

7. In the example given above on the polar decomposition, §2.2.5, check that the relations $\mathbf{C}\mathbf{n}_i = \lambda \mathbf{n}_i, i = 1,2,3$ are satisfied (with respect to the original axes). Check also that the relations $\mathbf{C}\mathbf{n}'_i = \lambda \mathbf{n}'_i, i = 1,2,3$ are satisfied (here, the eigenvectors are the unit vectors in the second coordinate system, the principal directions of **C**, and **C** is with respect to these axes, i.e. it is diagonal).

2.3 Deformation and Strain: Further Topics

2.3.1 Volumetric and Isochoric Deformations

When analysing materials which are only slightly incompressible, it is useful to decompose the deformation gradient multiplicatively, according to

$$\mathbf{F} = \left(J^{1/3}\mathbf{I}\right)\overline{\mathbf{F}} = J^{1/3}\overline{\mathbf{F}} \tag{2.3.1}$$

From this definition {▲Problem 1},

$$\det \overline{\mathbf{F}} = 1 \tag{2.3.2}$$

and so $\overline{\mathbf{F}}$ characterises a volume preserving (**distortional** or **isochoric**) deformation. The tensor $J^{1/3}\mathbf{I}$ characterises the volume-changing (**dilational** or **volumetric**) component of the deformation, with $\det\left(J^{1/3}\mathbf{I}\right) = \det \mathbf{F} = J$.

This concept can be carried on to other kinematic tensors. For example, with $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$,

$$\mathbf{C} = J^{2/3}\overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}} \equiv J^{2/3}\overline{\mathbf{C}}. \tag{2.3.3}$$

$\overline{\mathbf{F}}$ and $\overline{\mathbf{C}}$ are called the **modified deformation gradient** and the **modified right Cauchy-Green tensor**, respectively. The square of the stretch is given by

$$\lambda^2 = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}} = J^{2/3}\left\{d\hat{\mathbf{X}}\overline{\mathbf{C}}d\hat{\mathbf{X}}\right\} \tag{2.3.4}$$

so that $\lambda = J^{1/3}\overline{\lambda}$, where $\overline{\lambda}$ is the **modified stretch**, due to the action of $\overline{\mathbf{C}}$. Similarly, the **modified principal stretches** are

$$\overline{\lambda}_i = J^{-1/3}\lambda_i, \qquad i = 1,2,3 \tag{2.3.5}$$

with

$$\det \overline{\mathbf{F}} = \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3 = 1 \tag{2.3.6}$$

The case of simple shear discussed earlier is an example of an isochoric deformation, in which the deformation gradient and the modified deformation gradient coincide, $J^{1/3}\mathbf{I} = \mathbf{I}$.

2.3.2 Relative Deformation

It is usual to use the configuration at $(\mathbf{X}, t = 0)$ as the reference configuration, and define quantities such as the deformation gradient relative to this reference configuration. As mentioned, any configuration can be taken to be the reference configuration, and a new deformation gradient can be constructed with respect to this new reference configuration. Further, the reference configuration does not have to be fixed, but could be moving also.

In many cases, it is useful to choose the *current* configuration $(\mathbf{x}, t)$ to be the reference configuration, for example when evaluating rates of change of kinematic quantities (see later). To this end, introduce a third configuration: this is the configuration at some time $t = \tau$ and the position of a material particle $\mathbf{X}$ here is denoted by $\hat{\mathbf{x}} = \boldsymbol{\chi}(\mathbf{X}, \tau)$, where $\boldsymbol{\chi}$ is the motion function. The deformation at this time τ relative to the *current* configuration is called the **relative deformation**, and is denoted by $\hat{\mathbf{x}} = \boldsymbol{\chi}_{(t)}(\mathbf{x}, \tau)$, as illustrated in Fig. 2.3.1.

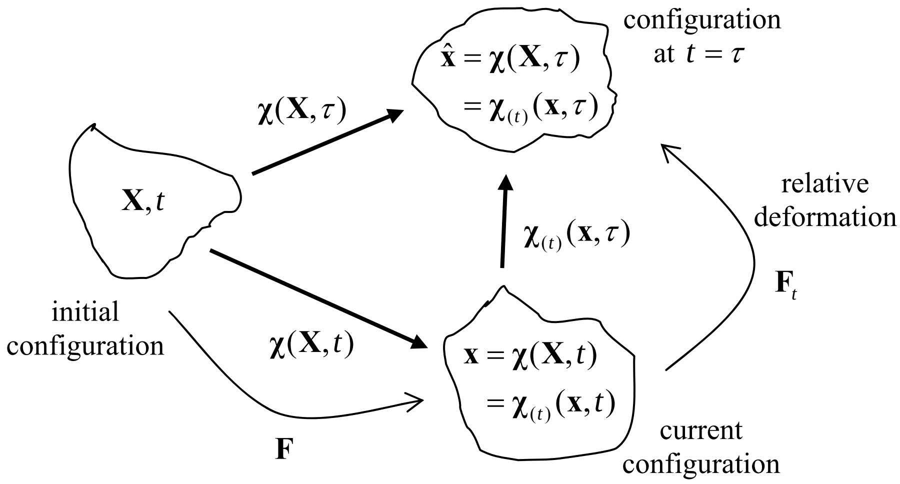

Figure 2.3.1: the relative deformation

The **relative deformation gradient** $\mathbf{F}_t$ is defined through

$$d\hat{\mathbf{x}} = \mathbf{F}_t(\mathbf{x}, \tau)\, d\mathbf{x}, \qquad \mathbf{F}_t = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \tag{2.3.7}$$

Also, since $d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}$ and $d\hat{\mathbf{x}} = \mathbf{F}(\mathbf{X}, \tau)\, d\mathbf{X}$, one has the relation

$$\mathbf{F}(\mathbf{X}, \tau) = \mathbf{F}_t(\mathbf{x}, \tau)\, \mathbf{F}(\mathbf{X}, t) \tag{2.3.8}$$

Similarly, relative strain measures can be defined, for example the relative right Cauchy-Green strain tensor is

$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau)^{\mathrm{T}}\, \mathbf{F}_t(\tau) \tag{2.3.9}$$

Example

Consider the two-dimensional motion

$$x_1 = X_1 e^t, \quad x_2 = X_2 (t+1)$$

Inverting these gives the spatial description $X_1 = x_1 e^{-t}$, $X_2 = x_2/(t+1)$, and the relative deformation is

$$\begin{aligned}
\hat{x}_1(\mathbf{x}, \tau) &= X_1 e^{\tau} = x_1 e^{\tau - t} \\
\hat{x}_2(\mathbf{x}, \tau) &= X_2(\tau+1) = x_2(\tau+1)/(t+1)
\end{aligned}$$

The deformation gradients are

$$\begin{aligned}
\mathbf{F}(\mathbf{X}, t) &= \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j = e^t \mathbf{e}_1 \otimes \mathbf{E}_1 + (t+1)\mathbf{e}_2 \otimes \mathbf{E}_2 \\
\mathbf{F}_t(\mathbf{x}, \tau) &= \frac{\partial \hat{x}_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j = e^{\tau - t}\mathbf{e}_1 \otimes \mathbf{e}_1 + (\tau+1)/(t+1)\mathbf{e}_2 \otimes \mathbf{e}_2
\end{aligned}$$

■

2.3.3 Derivatives of the Stretch

In this section, some useful formulae involving the derivatives of the stretches with respect to the Cauchy-Green strain tensors are derived.

Derivatives with respect to b

First, take the stretches to be functions of the left Cauchy-Green strain **b**. Write **b** using the spatial principal directions $\hat{\mathbf{n}}_i$ as a basis, 2.2.37, so that the total differential can be expressed as

$$d\mathbf{b} = \sum_{i=1}^{3} 2\lambda_i d\lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \left[d\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \hat{\mathbf{n}}_i \otimes d\hat{\mathbf{n}}_i \right] \qquad (2.3.10)$$

Since $\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = \delta_{ij}$, then

$$\hat{\mathbf{n}}_i d\mathbf{b} \hat{\mathbf{n}}_i = 2\lambda_i d\lambda_i + \lambda_i^2 \left[\hat{\mathbf{n}}_i \cdot d\hat{\mathbf{n}}_i + d\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_i \right] = 2\lambda_i d\lambda_i \quad \text{(no sum over } i\text{)} \qquad (2.3.11)$$

This last follows since the change in a vector of constant length is always orthogonal to the vector itself (as in the curvature analysis of §1.6.2). Using the property $\mathbf{uTv} = \mathbf{T} : (\mathbf{u} \otimes \mathbf{v})$, one has (summing over the *k* but not over the *i*; here $d\lambda_k / d\lambda_i = \delta_{ik}$)

$$d\mathbf{b} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) \equiv \frac{\partial \mathbf{b}}{\partial \lambda_k} d\lambda_k : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = 2\lambda_i d\lambda_i \quad \to \quad \frac{1}{2\lambda_i}\frac{\partial \mathbf{b}}{\partial \lambda_i} : (\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = 1 \qquad (2.3.12)$$

Then, since $\partial \mathbf{b} / \partial \lambda_i : \partial \lambda_i / \partial \mathbf{b}$ is also equal to 1, one has

$$\frac{1}{2\lambda_i}\frac{\partial \mathbf{b}}{\partial \lambda_i}:(\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) = \frac{\partial \mathbf{b}}{\partial \lambda_i}:\frac{\partial \lambda_i}{\partial \mathbf{b}} \quad \to \quad \frac{\partial \lambda_i}{\partial \mathbf{b}} = \frac{1}{2\lambda_i}(\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i) \tag{2.3.13}$$

The chain rule then gives the second derivative.

The above analysis is for distinct principal stretches. When $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$, then $\mathbf{b} = \lambda^2 \mathbf{I}$, $d\mathbf{b} = 2\lambda d\lambda \mathbf{I}$. Also, $d\mathbf{b} = 3(\partial \mathbf{b}/\partial \lambda)d\lambda$, so $3(\partial \mathbf{b}/\partial \lambda) = 2\lambda \mathbf{I}$, or

$$3\frac{\partial \mathbf{b}}{\partial \lambda}:\frac{\partial \lambda}{\partial \mathbf{b}} = 2\lambda \mathbf{I}:\frac{\partial \lambda}{\partial \mathbf{b}} \tag{2.3.14}$$

But $\partial \mathbf{b}/\partial \lambda : \partial \lambda/\partial \mathbf{b} = 1$ and $3 = \mathbf{I}:\mathbf{I}$, and so in this case, $\partial \lambda/\partial \mathbf{b} = \mathbf{I}/2\lambda$.

A similar calculation can be carried out for two equal eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$. In summary,

$$\boxed{\begin{array}{lr}
\dfrac{\partial \lambda_i}{\partial \mathbf{b}} = \dfrac{1}{2\lambda_i}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \qquad \text{(no sum over } i\text{)} & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \\
\dfrac{\partial \lambda}{\partial \mathbf{b}} = \dfrac{1}{2\lambda}(\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2) & \\
\dfrac{\partial \lambda_3}{\partial \mathbf{b}} = \dfrac{1}{2\lambda_3}(\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3) & \lambda_1 = \lambda_2 = \lambda \neq \lambda_3 \\
\dfrac{\partial \lambda}{\partial \mathbf{b}} = \dfrac{1}{2\lambda}\sum_{i=1}^{3}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i = \dfrac{1}{2\lambda}\mathbf{I} & \lambda_1 = \lambda_2 = \lambda_3 = \lambda \\
 & \\
\dfrac{\partial^2 \lambda_i}{\partial \mathbf{b}^2} = -\dfrac{1}{4\lambda_i^3}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \quad \text{(no sum over } i\text{)} & \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1
\end{array}} \tag{2.3.15}$$

Derivatives with respect to C

The stretch can also be considered to be a function of the right Cauchy-Green strain **C**. The derivatives of the stretches with respect to **C** can be found in exactly the same way as for the left Cauchy-Green strain. The results are the same as given in 2.3.15 except that, referring to 2.2.37, **b** is replaced by **C** and $\hat{\mathbf{n}}$ is replaced by $\hat{\mathbf{N}}$.

2.3.4 The Directional Derivative of Kinematic Quantities

The directional derivative of vectors and tensors was introduced in §1.6.11 and §1.15.4. Taking directional derivatives of kinematic quantities is often very useful, for example in linearising equations in order to apply numerical solution algorithms

The Deformation Gradient

First, consider the deformation gradient as a function of the current position **x** (or motion $\boldsymbol{\chi}$) and examine its value at $\mathbf{x} + \mathbf{a}$:

$$\mathbf{F}(\mathbf{x}+\mathbf{a}) = \mathbf{F}(\mathbf{x}) + \partial_{\mathbf{x}}\mathbf{F}[\mathbf{a}] + o(|\mathbf{a}|) \tag{2.3.16}$$

The directional derivative $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{a}] = (\partial\mathbf{F}/\partial\mathbf{x})\mathbf{a}$ can be expressed as

$$\begin{aligned} \partial_{\mathbf{x}}\mathbf{F}[\mathbf{a}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{F}(\mathbf{x}+\varepsilon\mathbf{a}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial(\mathbf{x}+\varepsilon\mathbf{a})}{\partial\mathbf{X}} \\ &= \mathrm{Grad}\mathbf{a} \\ &= (\mathrm{grad}\mathbf{a})\mathbf{F} \end{aligned} \tag{2.3.17}$$

the last line resulting from 2.2.8b. It follows that the directional derivative of the deformation gradient in the direction of a displacement vector $\mathbf{u}$ from the *current* configuration is

$$\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = (\mathrm{grad}\mathbf{u})\mathbf{F} \tag{2.3.18}$$

On the other hand, consider the deformation gradient as a function of $\mathbf{X}$ and examine its value at $\mathbf{X}+\mathbf{A}$:

$$\mathbf{F}(\mathbf{X}+\mathbf{A}) = \mathbf{F}(\mathbf{X}) + \partial_{\mathbf{X}}\mathbf{F}[\mathbf{A}] \tag{2.3.19}$$

and now

$$\begin{aligned} \partial_{\mathbf{X}}\mathbf{F}[\mathbf{A}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{F}(\mathbf{X}+\varepsilon\mathbf{A}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial\mathbf{X}}\mathbf{x}(\mathbf{X}+\varepsilon\mathbf{A}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial\mathbf{X}}(\mathbf{x}+\mathbf{F}\varepsilon\mathbf{A}) \\ &= \mathrm{Grad}(\mathbf{F}\mathbf{A}) \\ &= \mathrm{Grad}\mathbf{a} \end{aligned} \tag{2.3.20}$$

where $\mathbf{a} = \mathbf{F}\mathbf{A}$.

Other Kinematic Quantities

The directional derivative of the Green-Lagrange strain, the right and left Cauchy-Green tensors and the Jacobian in the direction of a displacement $\mathbf{u}$ from the current configuration are {▲Problem 2}

$$
\begin{aligned}
\partial_{\mathbf{x}}\mathbf{E}[\mathbf{u}] &= \mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F} \\
\partial_{\mathbf{x}}\mathbf{C}[\mathbf{u}] &= 2\mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F} \\
\partial_{\mathbf{x}}\mathbf{b}[\mathbf{u}] &= (\mathrm{grad}\mathbf{u})\mathbf{b} + \mathbf{b}(\mathrm{grad}\mathbf{u})^{\mathrm{T}} \\
\partial_{\mathbf{x}}J[\mathbf{u}] &= J\mathrm{div}\mathbf{u}
\end{aligned}
\qquad (2.3.21)
$$

where $\boldsymbol{\varepsilon}$ is the small-strain tensor, 2.2.48.

The directional derivative is also useful for deriving various relations between the kinematic variables. For example, for an arbitrary vector **a**, using the chain rule 1.15.28, 2.3.20, 1.15.24, the trace relations 1.10.10e and 1.10.10b, and 2.2.8b, 1.14.9,

$$
\begin{aligned}
(\mathrm{Grad}J)\cdot\mathbf{a} &= \partial_{\mathbf{X}}J[\mathbf{a}] \\
&= \partial_{\mathbf{F}}J[\partial_{\mathbf{X}}\mathbf{F}[\mathbf{a}]] \\
&= \partial_{\mathbf{F}}J[\mathrm{Grad}(\mathbf{F}\mathbf{a})] \\
&= J\mathbf{F}^{-\mathrm{T}} : \mathrm{Grad}(\mathbf{F}\mathbf{a}) \\
&= J\mathrm{tr}\left(\mathbf{F}^{-1}\mathrm{Grad}(\mathbf{F}\mathbf{a})\right) \\
&= J\mathrm{tr}\left(\mathrm{Grad}(\mathbf{F}\mathbf{a})\mathbf{F}^{-1}\right) \\
&= J\mathrm{tr}(\mathrm{grad}(\mathbf{F}\mathbf{a})) \\
&= J\,\mathrm{div}(\mathbf{F}\mathbf{a})
\end{aligned}
\qquad (2.3.22)
$$

so that, from 1.14.16b with **a** constant,

$$
\boxed{\mathrm{Grad}J = J\mathrm{div}\mathbf{F}^{\mathrm{T}}} \qquad (2.3.23)
$$

2.3.5 Problems

1. Use 1.10.16c to show that $\det\overline{\mathbf{F}} = 1$.
2. (a) use the relation $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})$, Eqn. 2.3.18, $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = (\mathrm{grad}\mathbf{u})\mathbf{F}$, and the product rule of differentiation to derive 2.3.21a, $\partial_{\mathbf{x}}\mathbf{E}[\mathbf{u}] = \mathbf{F}^{\mathrm{T}}\boldsymbol{\varepsilon}\mathbf{F}$, where $\boldsymbol{\varepsilon}$ is the small strain tensor.
   (b) evaluate $\partial_{\mathbf{x}}\mathbf{C}[\mathbf{u}]$ (in terms of $\mathbf{F}$ and $\boldsymbol{\varepsilon}$, the small strain tensor)
   (c) evaluate $\partial_{\mathbf{x}}\mathbf{b}[\mathbf{u}]$ (in terms of $\mathrm{grad}\mathbf{u}$ and **b**)
   (d) evaluate $\partial_{\mathbf{x}}J[\mathbf{u}]$ (in terms of J and $\mathrm{div}\mathbf{u}$; use the chain rule $\partial_{\mathbf{x}}J[\mathbf{u}] = \partial_{\mathbf{F}}\hat{J}[\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}]]$, with $\hat{J}(\mathbf{F}) = \det\mathbf{F}$, $\partial_{\mathbf{x}}\mathbf{F}[\mathbf{u}] = \mathrm{Grad}\mathbf{u}$)

2.4 Material Time Derivatives

The motion is now allowed to be a function of time, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, and attention is given to time derivatives, both the **material time derivative** and the **local time derivative**.

2.4.1 Velocity & Acceleration

The velocity of a moving particle is the time rate of change of the position of the particle. From 2.1.3, by definition,

$$\mathbf{V}(\mathbf{X}, t) \equiv \frac{d\boldsymbol{\chi}(\mathbf{X}, t)}{dt} \tag{2.4.1}$$

In the motion expression $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X}$ and t are independent variables and $\mathbf{X}$ is independent of time, denoting the particle for which the velocity is being calculated. The velocity can thus be written as $\partial\boldsymbol{\chi}(\mathbf{X}, t)/\partial t$ or, denoting the motion by $\mathbf{x}(\mathbf{X}, t)$, as $d\mathbf{x}(\mathbf{X}, t)/dt$ or $\partial\mathbf{x}(\mathbf{X}, t)/\partial t$.

The spatial description of the velocity field may be obtained from the material description by simply replacing $\mathbf{X}$ with $\mathbf{x}$, i.e.

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right) \tag{2.4.2}$$

As with displacements in both descriptions, there is only *one* velocity, $\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$ – they are just given in terms of different coordinates.

The velocity is most often expressed in the spatial description, as

$$\boxed{\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}} \qquad \textbf{velocity} \tag{2.4.3}$$

To be precise, the right hand side here involves $\mathbf{x}$ which is a function of the material coordinates, but it is understood that the substitution back to spatial coordinates, as in 2.4.2, is made (see example below).

Similarly, the acceleration is defined to be

$$\mathbf{A}(\mathbf{X}, t) = \frac{d^2\boldsymbol{\chi}(\mathbf{X}, t)}{dt^2} = \frac{d^2\mathbf{x}}{dt^2} = \frac{d\mathbf{V}}{dt} = \frac{\partial^2\boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2} \tag{2.4.4}$$

Example

Consider the motion

$$x_1 = X_1 + t^2 X_2, \quad x_2 = X_2 + t^2 X_1, \quad x_3 = X_3$$

The velocity and acceleration can be evaluated through

$$\mathbf{V}(\mathbf{X},t) = \frac{d\mathbf{x}}{dt} = 2tX_2\mathbf{e}_1 + 2tX_1\mathbf{e}_2, \qquad \mathbf{A}(\mathbf{X},t) = \frac{d^2\mathbf{x}}{dt^2} = 2X_2\mathbf{e}_1 + 2X_1\mathbf{e}_2$$

One can write the motion in the spatial description by inverting the material description:

$$X_1 = \frac{x_1 - t^2 x_2}{1-t^4}, \quad X_2 = \frac{x_2 - t^2 x_1}{1-t^4}, \quad X_3 = x_3$$

Substituting in these equations then gives the spatial description of the velocity and acceleration:

$$\begin{aligned}
\mathbf{v}(\mathbf{x},t) &= \mathbf{V}\left(\chi^{-1}(\mathbf{x},t),t\right) = 2t\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2t\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2 \\
\mathbf{a}(\mathbf{x},t) &= \mathbf{A}\left(\chi^{-1}(\mathbf{x},t),t\right) = 2\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2
\end{aligned}$$

■

2.4.2 The Material Derivative

One can analyse deformation by examining the current configuration only, discounting the reference configuration. This is the viewpoint taken in Fluid Mechanics – one focuses on material as it flows at the *current time*, and does not consider "where the fluid was". In order to do this, quantities must be cast in terms of the velocity. Suppose that the velocity in terms of spatial coordinates, $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ is known; for example, one could have a measuring instrument which records the velocity at a specific location, but the motion $\boldsymbol{\chi}$ itself is unknown. In that case, to evaluate the acceleration, the chain rule of differentiation must be applied:

$$\dot{\mathbf{v}} \equiv \frac{d}{dt}\mathbf{v}\left(\mathbf{x}(t),t\right) = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt}$$

or

$$\boxed{\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \left(\operatorname{grad}\mathbf{v}\right)\mathbf{v}} \qquad \textbf{acceleration (spatial description)} \qquad (2.4.5)$$

The acceleration can now be determined, because the derivatives can be determined (measured) without knowing the motion.

In the above, the **material derivative**, or **total derivative**, of the particle's velocity was taken to obtain the acceleration. In general, one can take the time derivative of any physical or kinematic property $(\bullet)$ expressed in the spatial description:

$$\boxed{\frac{d}{dt}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \operatorname{grad}(\bullet)\mathbf{v}} \qquad \textbf{Material Time Derivative} \tag{2.4.6}$$

For example, the rate of change of the density $\rho = \rho(\mathbf{x}, t)$ of a particle instantaneously at $\mathbf{x}$ is

$$\dot{\rho} \equiv \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \operatorname{grad} \rho \cdot \mathbf{v} \tag{2.4.7}$$

The Local Rate of Change

The first term, $\partial \rho / \partial t$, gives the **local rate of change** of density at **x** whereas the second term $\mathbf{v} \cdot \operatorname{grad} \rho$ gives the change due to the particle's motion, and is called the **convective rate of change**.

Note the difference between the material derivative and the local derivative. For example, the material derivative of the velocity, 2.4.5 (or, equivalently, $d\mathbf{V}(\mathbf{X}, t)/dt$ in 2.4.4, with **X** fixed) is not the same as the derivative $\partial \mathbf{v}(\mathbf{x}, t)/\partial t$ (with **x** fixed). The former is the acceleration of a material particle **X**. The latter is the time rate of change of the velocity of particles *at a fixed location* in space; in general, *different* material particles will occupy position **x** at different times.

The material derivative d/dt can be applied to any scalar, vector or tensor:

$$\begin{aligned}
\dot{\alpha} &\equiv \frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + \operatorname{grad} \alpha \cdot \mathbf{v} \\
\dot{\mathbf{a}} &\equiv \frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + (\operatorname{grad} \mathbf{a})\mathbf{v} \\
\dot{\mathbf{A}} &\equiv \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\operatorname{grad} \mathbf{A})\mathbf{v}
\end{aligned} \tag{2.4.8}$$

Another notation often used for the material derivative is D/Dt:

$$\frac{Df}{Dt} \equiv \frac{df}{dt} \equiv \dot{f} \tag{2.4.9}$$

Steady and Uniform Flows

In a **steady flow**, quantities are independent of time, so the local rate of change is zero and, for example, $\dot{\rho} = \operatorname{grad} \rho \cdot \mathbf{v}$. In a **uniform flow**, quantities are independent of position so that, for example, $\dot{\rho} = \partial \rho / \partial t$

Example

Consider again the previous example. This time, with only the velocity $\mathbf{v}(\mathbf{x}, t)$ known, the acceleration can be obtained through the material derivative:

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{v}$$

$$= \frac{\partial}{\partial t}\left(2t\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2t\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2\right) + \begin{bmatrix} -\dfrac{2t^3}{1-t^4} & \dfrac{2t}{1-t^4} & 0 \\ \dfrac{2t}{1-t^4} & -\dfrac{2t^3}{1-t^4} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2t\dfrac{x_2 - t^2 x_1}{1-t^4} \\ 2t\dfrac{x_1 - t^2 x_2}{1-t^4} \\ 0 \end{bmatrix}$$

$$= 2\frac{x_2 - t^2 x_1}{1-t^4}\mathbf{e}_1 + 2\frac{x_1 - t^2 x_2}{1-t^4}\mathbf{e}_2$$

as before.

■

The Relationship between the Displacement and Velocity

The velocity can be derived directly from the displacement 2.2.42:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{u}+\mathbf{X})}{dt} = \frac{d\mathbf{u}}{dt}, \tag{2.4.10}$$

or

$$\mathbf{v} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad}\mathbf{u})\mathbf{v} \tag{2.4.11}$$

When the displacement field is given in material form one has

$$\mathbf{V} = \frac{d\mathbf{U}}{dt} \tag{2.4.12}$$

2.4.3 Problems

1. The density of a material is given by

$$\rho = \frac{e^{-2t}}{\mathbf{x}\cdot\mathbf{x}}$$

The velocity field is given by

$$v_1 = x_2 + 2x_3, \quad v_2 = x_3 - 2x_1, \quad v_3 = x_1 + 2x_2$$

Determine the time derivative of the density (a) at a certain position $\mathbf{x}$ in space, and (b) of a material particle instantaneously occupying position **x**.

2.5 Deformation Rates

In this section, rates of change of the deformation tensors introduced earlier, **F**, **C**, **E**, etc., are evaluated, and special tensors used to measure deformation rates are discussed, for example the velocity gradient **l**, the rate of deformation **d** and the spin tensor **w**.

2.5.1 The Velocity Gradient

The **velocity gradient** is used as a measure of the rate at which a material is deforming.

Consider two fixed neighbouring points, $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$, Fig. 2.5.1. The velocities of the material particles at these points at any given time instant are $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}+d\mathbf{x})$, and

$$\mathbf{v}(\mathbf{x}+d\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x},$$

The relative velocity between the points is

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x} \equiv \mathbf{l} d\mathbf{x} \tag{2.5.1}$$

with **l** defined to be the (spatial) velocity gradient,

$$\boxed{\mathbf{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \operatorname{grad}\mathbf{v}, \qquad l_{ij} = \frac{\partial v_i}{\partial x_j}} \qquad \textbf{Spatial Velocity Gradient} \tag{2.5.2}$$

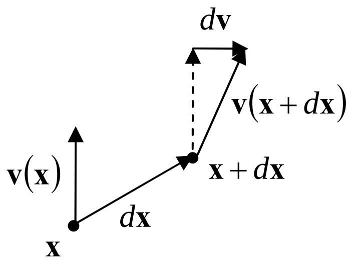

Figure 2.5.1: velocity gradient

Expression 2.5.1 emphasises the tensorial character of the spatial velocity gradient, mapping as it does one vector into another. Its physical meaning will become clear when it is decomposed into its symmetric and skew-symmetric parts below.

The spatial velocity gradient is commonly used in both solid and fluid mechanics. Less commonly used is the material velocity gradient, which is related to the rate of change of the deformation gradient:

$$\operatorname{Grad}\mathbf{V} = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial \mathbf{X}}\right) = \dot{\mathbf{F}} \tag{2.5.3}$$

and use has been made of the fact that, since **X** and *t* are independent variables, material time derivatives and material gradients commute.

2.5.2 Material Derivatives of the Deformation Gradient

The spatial velocity gradient may be written as

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}}{\partial t}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}}$$

or $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ so that the material derivative of **F** can be expressed as

$$\boxed{\dot{\mathbf{F}} = \mathbf{l}\,\mathbf{F}} \qquad \textbf{Material Time Derivative of the Deformation Gradient} \qquad (2.5.4)$$

Also, it can be shown that {▲Problem 1}

$$\boxed{\begin{aligned} \overline{\mathbf{F}^{\mathrm{T}}}^{\,\cdot} &= \dot{\mathbf{F}}^{\mathrm{T}} \\ \overline{\mathbf{F}^{-1}}^{\,\cdot} &= -\mathbf{F}^{-1}\mathbf{l} \\ \overline{\mathbf{F}^{-\mathrm{T}}}^{\,\cdot} &= -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} \end{aligned}} \qquad (2.5.5)$$

2.5.3 The Rate of Deformation and Spin Tensors

The velocity gradient can be decomposed into a symmetric tensor and a skew-symmetric tensor as follows (see §1.10.10):

$$\boxed{\mathbf{l} = \mathbf{d} + \mathbf{w}} \qquad (2.5.6)$$

where **d** is the **rate of deformation tensor** (or **rate of stretching tensor**) and **w** is the **spin tensor** (or **rate of rotation**, or **vorticity tensor**), defined by

$$\boxed{\begin{aligned} \mathbf{d} &= \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \qquad & d_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ \mathbf{w} &= \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right), \qquad & w_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \end{aligned}} \qquad \textbf{Rate of Deformation and Spin Tensors}$$

(2.5.7)

The physical meaning of these tensors is next examined.

The Rate of Deformation

Consider first the rate of deformation tensor **d** and note that

$$\mathbf{l}d\mathbf{x} = d\mathbf{v} = \frac{d}{dt}(d\mathbf{x}) \tag{2.5.8}$$

The rate at which the square of the length of $d\mathbf{x}$ is changing is then

$$\begin{aligned} \frac{d}{dt}\left(|d\mathbf{x}|^2\right) &= 2|d\mathbf{x}|\frac{d}{dt}\left(|d\mathbf{x}|\right), \\ \frac{d}{dt}\left(|d\mathbf{x}|^2\right) &= \frac{d}{dt}(d\mathbf{x}\cdot d\mathbf{x}) = 2d\mathbf{x}\cdot\frac{d}{dt}(d\mathbf{x}) = 2d\mathbf{x}\,\mathbf{l}\,d\mathbf{x} = 2d\mathbf{x}\,\mathbf{d}\,d\mathbf{x} \end{aligned} \tag{2.5.9}$$

the last equality following from 2.5.6 and 1.10.31e. Dividing across by $2|d\mathbf{x}|^2$, then leads to

$$\boxed{\frac{\dot{\lambda}}{\lambda} = \hat{\mathbf{n}}\,\mathbf{d}\hat{\mathbf{n}}} \qquad \textbf{Rate of stretching per unit stretch in the direction } \hat{\mathbf{n}} \tag{2.5.10}$$

where $\lambda = |d\mathbf{x}|/|d\mathbf{X}|$ is the stretch and $\hat{\mathbf{n}} = d\mathbf{x}/|d\mathbf{x}|$ is a unit normal in the direction of $d\mathbf{x}$. Thus the rate of deformation **d** gives the rate of stretching of line elements. The diagonal components of **d**, for example

$$d_{11} = \mathbf{e}_1\mathbf{d}\mathbf{e}_1,$$

represent unit rates of extension in the coordinate directions.

Note that these are *instantaneous* rates of extension, in other words, they are rates of extensions of elements in the current configuration at the current time; they are not a measure of the rate at which a line element in the original configuration changed into the corresponding line element in the current configuration.

Note:
- Eqn. 2.5.10 can also be derived as follows: let $\hat{\mathbf{N}}$ be a unit normal in the direction of $d\mathbf{X}$, and $\hat{\mathbf{n}}$ be the corresponding unit normal in the direction of $d\mathbf{x}$. Then $\hat{\mathbf{n}}|d\mathbf{x}| = \mathbf{F}\hat{\mathbf{N}}|d\mathbf{X}|$, or $\hat{\mathbf{n}}\lambda = \mathbf{F}\hat{\mathbf{N}}$. Differentiating gives $\dot{\hat{\mathbf{n}}}\lambda + \hat{\mathbf{n}}\dot{\lambda} = \dot{\mathbf{F}}\hat{\mathbf{N}} = \mathbf{l}\mathbf{F}\hat{\mathbf{N}}$ or $\dot{\hat{\mathbf{n}}}\lambda + \hat{\mathbf{n}}\dot{\lambda} = \mathbf{l}\hat{\mathbf{n}}\lambda$. Contracting both sides with $\hat{\mathbf{n}}$ leads to $\hat{\mathbf{n}}\cdot\dot{\hat{\mathbf{n}}} + \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\left(\dot{\lambda}/\lambda\right) = \hat{\mathbf{n}}\mathbf{l}\hat{\mathbf{n}}$. But $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1 \rightarrow d(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})dt = 0$ so, by the chain rule, $\hat{\mathbf{n}}\cdot\dot{\hat{\mathbf{n}}} = 0$ (confirming that a vector $\hat{\mathbf{n}}$ of constant length is orthogonal to a change in that vector $d\hat{\mathbf{n}}$), and the result follows

Consider now the rate of change of the angle θ between two vectors $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$. Using 2.5.8 and 1.10.3d,

$$\begin{aligned}\frac{d}{dt}\left(d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}\right) &= \frac{d}{dt}\left(d\mathbf{x}^{(1)}\right)\cdot d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)}\cdot\frac{d}{dt}\left(d\mathbf{x}^{(2)}\right)\\ &= \mathbf{l}d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)}\cdot\mathbf{l}d\mathbf{x}^{(2)}\\ &= \left(\mathbf{l}+\mathbf{l}^{\mathrm{T}}\right)d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}\\ &= 2\, d\mathbf{x}^{(1)}\mathbf{d}d\mathbf{x}^{(2)}\end{aligned} \tag{2.5.11}$$

which reduces to 2.5.9 when $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)}$. An alternative expression for this dot product is

$$\begin{aligned}\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\cos\theta\right) &= \frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\right)\left|d\mathbf{x}^{(2)}\right|\cos\theta + \frac{d}{dt}\left(\left|d\mathbf{x}^{(2)}\right|\right)\left|d\mathbf{x}^{(1)}\right|\cos\theta - \sin\theta\,\dot{\theta}\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\\ &= \left(\frac{\frac{d}{dt}\left(\left|d\mathbf{x}^{(1)}\right|\right)}{\left|d\mathbf{x}^{(1)}\right|}\cos\theta + \frac{\frac{d}{dt}\left(\left|d\mathbf{x}^{(2)}\right|\right)}{\left|d\mathbf{x}^{(2)}\right|}\cos\theta - \sin\theta\,\dot{\theta}\right)\left|d\mathbf{x}^{(1)}\right|\left|d\mathbf{x}^{(2)}\right|\end{aligned}$$

(2.5.12)

Equating 2.5.11 and 2.5.12 leads to

$$2\,\hat{\mathbf{n}}_1\,\mathbf{d}\,\hat{\mathbf{n}}_2 = \left(\frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2}\right)\cos\theta - \sin\theta\,\dot{\theta} \tag{2.5.13}$$

where $\lambda_i = \left|d\mathbf{x}^{(i)}\right| / \left|d\mathbf{X}^{(i)}\right|$ is the stretch and $\hat{\mathbf{n}}_i = d\mathbf{x}^{(i)} / \left|d\mathbf{x}^{(i)}\right|$ is a unit normal in the direction of $d\mathbf{x}^{(i)}$.

It follows from 2.5.13 that the off-diagonal terms of the rate of deformation tensor represent **shear rates**: the rate of change of the right angle between line elements aligned with the coordinate directions. For example, taking the base vectors $\mathbf{e}_1 = \hat{\mathbf{n}}_1$, $\mathbf{e}_2 = \hat{\mathbf{n}}_2$, 2.5.13 reduces to

$$d_{12} = -\frac{1}{2}\dot{\theta}_{12} \tag{2.5.14}$$

where θ_{12} is the instantaneous right angle between the axes in the current configuration.

The Spin

Consider now the spin tensor $\mathbf{w}$; since it is skew-symmetric, it can be written in terms of its axial vector $\boldsymbol{\omega}$ (Eqn. 1.10.34), called the **angular velocity vector**:

$$\begin{aligned}\boldsymbol{\omega} &= -w_{23}\mathbf{e}_1 + w_{13}\mathbf{e}_2 - w_{12}\mathbf{e}_3 \\ &= \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 \\ &= \frac{1}{2}\operatorname{curl}\mathbf{v}\end{aligned} \tag{2.5.15}$$

(The vector $2\boldsymbol{\omega}$ is called the **vorticity** (or **spin**) **vector**.) Thus when $\mathbf{d}$ is zero, the motion consists of a rotation about some axis at angular velocity $\omega = |\boldsymbol{\omega}|$ (*cf*. the end of §1.10.11), with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, $\mathbf{r}$ measured from a point on the axis, and $\mathbf{wr} = \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}$.

On the other hand, when $\mathbf{l} = \mathbf{d}$, $\mathbf{w} = \mathbf{0}$, one has $\boldsymbol{\omega} = \mathbf{o}$, and the motion is called **irrotational**.

Example (Shear Flow)

Consider a **simple shear flow** in which the velocity profile is "triangular" as shown in Fig. 2.5.2. This type of flow can be generated (at least approximately) in many fluids by confining the fluid between plates a distance *h* apart, and by sliding the upper plate over the lower one at constant velocity *V*. If the material particles adjacent to the upper plate have velocity $V\mathbf{e}_1$, then the velocity field is $\mathbf{v} = \dot{\gamma} x_2 \mathbf{e}_1$, where $\dot{\gamma} = V/h$. This is a steady flow ($\partial \mathbf{v}/\partial t = \mathbf{0}$); at any given point, there is no change over time. The velocity gradient is $\mathbf{l} = \dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_2$ and the acceleration of material particles is zero: $\mathbf{a} = \mathbf{lv} = \mathbf{0}$. The rate of deformation and spin are

$$\mathbf{d} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{w} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and, from 2.5.14, $\dot{\gamma} = -\dot{\theta}_{12}$, the rate of change of the angle shown in Fig. 2.5.2.

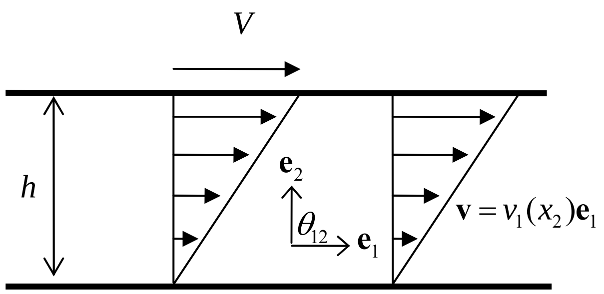

Figure 2.5.2: shear flow

The eigenvalues of $\mathbf{d}$ are $\lambda = 0, \pm\dot{\gamma}/2$ ($\det\mathbf{d} = 0$) and the principal invariants, Eqn. 1.11.17, are $\mathrm{I}_{\mathbf{d}} = 0$, $\mathrm{II}_{\mathbf{d}} = -\frac{1}{4}\dot{\gamma}^2$, $\mathrm{III}_{\mathbf{d}} = 0$. For $\lambda = +\dot{\gamma}/2$, the eigenvector is $\mathbf{n}_1 = [1 \quad 1 \quad 0]^T$ and for $\lambda = -\dot{\gamma}/2$, it is $\mathbf{n}_2 = [-1 \quad 1 \quad 0]^T$ (for $\lambda = 0$ it is $\mathbf{e}_3$). (The eigenvalues and eigenvectors of $\mathbf{w}$ are complex.) Relative to the basis of eigenvectors,

$$\mathbf{d} = \begin{bmatrix} \dot{\gamma}/2 & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so at 45° there is an instantaneous pure rate of stretching/contraction of material.

■

2.5.4 Other Rates of Strain Tensors

From 2.2.9, 2.2.22,

$$\frac{1}{2}\frac{d}{dt}\left(d\mathbf{x} \cdot d\mathbf{x}\right) = d\mathbf{X}\frac{1}{2}\dot{\mathbf{C}}d\mathbf{X} = d\mathbf{X}\dot{\mathbf{E}}d\mathbf{X} \tag{2.5.16}$$

This can also be written in terms of spatial line elements:

$$d\mathbf{X}\dot{\mathbf{E}}d\mathbf{X} = d\mathbf{x}\left[\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}\mathbf{F}^{-1}\right]d\mathbf{x} \tag{2.5.17}$$

But from 2.5.9, these also equal $d\mathbf{x}\mathbf{d}d\mathbf{x}$, which leads to expressions for the material time derivatives of the right Cauchy-Green and Green-Lagrange strain tensors (also given here are expressions for the time derivatives of the left Cauchy-Green and Euler-Almansi tensors {▲Problem 3})

$$\boxed{\begin{aligned} &\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F} \\ &\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F} \\ &\dot{\mathbf{b}} = \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}} \\ &\dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^{\mathrm{T}}\mathbf{e} - \mathbf{e}\mathbf{l} \end{aligned}} \tag{2.5.18}$$

Note that

$$\int \dot{\mathbf{E}}\, dt = \int d\mathbf{E}$$

so that the integral of the rate of Green-Lagrange strain is path independent and, in particular, the integral of $\dot{\mathbf{E}}$ around any closed loop (so that the final configuration is the same as the initial configuration) is zero. However, in general, the integral of the rate of deformation,

$$\int \mathbf{d}dt$$

is not independent of the path – there is no universal function $\mathbf{h}$ such that $\mathbf{d} = d\mathbf{h}/dt$ with $\int \mathbf{d}dt = \int d\mathbf{h}$. Thus the integral $\int \mathbf{d}dt$ over a closed path may be non-zero, and hence the integral of the rate of deformation is not a good measure of the total strain.

The Hencky Strain

The Hencky strain is, Eqn. 2.2.37, $\mathbf{h} = \sum_{i=1}^{3} (\ln \lambda_i) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$, where $\mathbf{n}_i$ are the principal spatial axes. Thus, if the principal spatial axes do not change with time, $\dot{\mathbf{h}} = \sum_{i=1}^{3} (\dot{\lambda}_i / \lambda_i) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$. With the left stretch $\mathbf{v} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$, it follows that (and similarly for the corresponding material tensors), $\dot{\mathbf{H}} \equiv \dot{\overline{\ln \mathbf{U}}} = \dot{\mathbf{U}}\mathbf{U}^{-1}$, $\dot{\mathbf{h}} \equiv \dot{\overline{\ln \mathbf{v}}} = \dot{\mathbf{v}}\mathbf{v}^{-1}$.

For example, consider an extension in the coordinate directions, so $\mathbf{F} = \mathbf{U} = \mathbf{v} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i$. The motion and velocity are

$$x_i = \lambda_i X_i, \quad \dot{x}_i = \dot{\lambda}_i X_i = \frac{\dot{\lambda}_i}{\lambda_i} x_i \qquad (\text{no sum})$$

so $d_i = \dot{\lambda}_i / \lambda_i$ (no sum), and $\mathbf{d} = \dot{\mathbf{h}}$. Further, $\mathbf{h} = \int \mathbf{d} dt$. Note that, as mentioned above, this expression does not hold in general, but does in this case of uniform extension.

2.5.5 Material Derivatives of Line, Area and Volume Elements

The material derivative of a line element $d(d\mathbf{x})/dt$ has been derived (defined) through 2.4.8. For area and volume elements, it is necessary first to evaluate the material derivative of the Jacobian determinant *J*. From the chain rule, one has (see Eqns 1.15.11, 1.15.7)

$$\dot{J} = \frac{d}{dt}\big(J(\mathbf{F})\big) = \frac{\partial J}{\partial \mathbf{F}} : \dot{\mathbf{F}} = J\mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} \tag{2.5.19}$$

Hence {▲Problem 4}

$$\boxed{\begin{aligned} \dot{J} &= J\,\mathrm{tr}(\mathbf{l}) \\ &= J\,\mathrm{tr}(\mathrm{grad}\mathbf{v}) \\ &= J\mathrm{div}\mathbf{v} \end{aligned}} \tag{2.5.20}$$

Since $\mathbf{l} = \mathbf{d} + \mathbf{w}$ and $\mathrm{tr}\mathbf{w} = 0$, it also follows that $\dot{J} = J\mathrm{tr}\mathbf{d}$.

As mentioned earlier, an isochoric motion is one for which the volume is constant – thus any of the following statements characterise the necessary and sufficient conditions for an isochoric motion:

$$J = 1, \quad \dot{J} = 0, \quad \mathrm{div}\mathbf{v} = 0, \quad \mathrm{tr}\,\mathbf{d} = 0, \quad \mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} = 0 \tag{2.5.21}$$

Applying Nanson's formula 2.2.59, the material derivative of an area vector element is {▲Problem 6}

$$\boxed{\frac{d}{dt}(\hat{\mathbf{n}}ds) = \left(\mathrm{div}\mathbf{v} - \mathbf{l}^{\mathrm{T}}\right)\hat{\mathbf{n}}ds} \tag{2.5.22}$$

Finally, from 2.2.53, the material time derivative of a volume element is

$$\boxed{\frac{d}{dt}(dv) = \frac{d}{dt}(JdV) = \dot{J}dV = \mathrm{div}\mathbf{v}\,dv} \tag{2.5.23}$$

Example (Shear and Stretch)

Consider a sample of material undergoing the following motion, Fig. 2.4.3.

$$\begin{aligned} x_1 &= X_1 + k\lambda X_2 \\ x_2 &= \lambda X_2 \\ x_3 &= X_3 \end{aligned} \quad , \qquad \begin{aligned} X_1 &= x_1 - kx_2 \\ X_2 &= \frac{1}{\lambda}x_2 \\ X_3 &= x_3 \end{aligned}$$

with $\lambda = \lambda(t)$, $k = k(t)$.

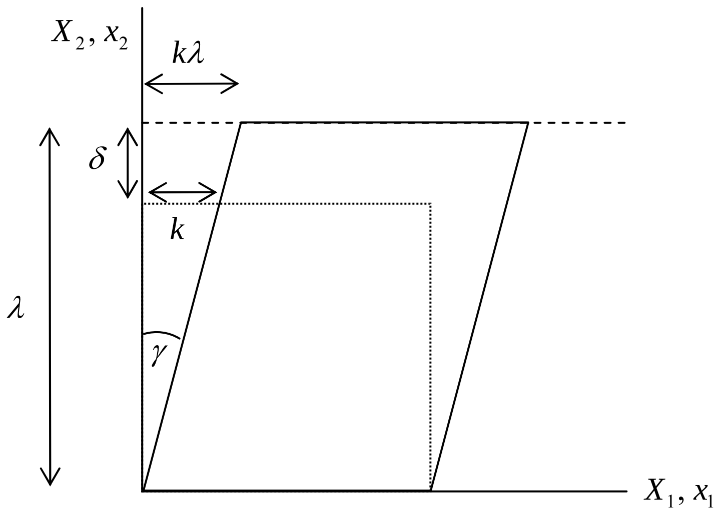

Figure 2.4.3: shear and stretch

The deformation gradient and material strain tensors are

$$\mathbf{F} = \begin{bmatrix} 1 & k\lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & k\lambda & 0 \\ k\lambda & (1+k^2)\lambda^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & \frac{1}{2}k\lambda & 0 \\ \frac{1}{2}k\lambda & \frac{1}{2}\left(\lambda^2(1+k^2)-1\right) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the Jacobian $J = \det\mathbf{F} = \lambda$, and the spatial strain tensors are

$$\mathbf{b}=\begin{bmatrix}1+k^2\lambda^2 & k\lambda^2 & 0\\ k\lambda^2 & \lambda^2 & 0\\ 0 & 0 & 1\end{bmatrix},\qquad \mathbf{e}=\begin{bmatrix}0 & \frac{1}{2}k & 0\\ \frac{1}{2}k & \frac{1}{2}\dfrac{\left(1-k^2\right)\lambda^2-1}{\lambda^2} & 0\\ 0 & 0 & 0\end{bmatrix}$$

This deformation can also be expressed as a stretch followed by a simple shear:

$$\mathbf{F}=\begin{bmatrix}1 & k & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & 1\end{bmatrix}$$

The velocity is

$$\mathbf{V}=\frac{d\mathbf{x}}{dt}=\begin{bmatrix}\left(\dot{k}\lambda+k\dot{\lambda}\right)X_2\\ \dot{\lambda}X_2\\ 0\end{bmatrix},\qquad \mathbf{v}=\begin{bmatrix}\left(\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right)x_2\\ \left(\dot{\lambda}/\lambda\right)x_2\\ 0\end{bmatrix}$$

The velocity gradient is

$$\mathbf{l}=\frac{d\mathbf{v}}{d\mathbf{x}}=\begin{bmatrix}0 & \dot{k}+k\left(\dot{\lambda}/\lambda\right) & 0\\ 0 & \dot{\lambda}/\lambda & 0\\ 0 & 0 & 0\end{bmatrix}$$

and the rate of deformation and spin are

$$\mathbf{d}=\begin{bmatrix}0 & \frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & 0\\ \frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & \dot{\lambda}/\lambda & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}0 & \frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & 0\\ -\frac{1}{2}\left[\dot{k}+k\left(\dot{\lambda}/\lambda\right)\right] & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Also

$$\dot{\mathbf{C}}=2\mathbf{F}^T\mathbf{d}\mathbf{F}=\begin{bmatrix}0 & \lambda\dot{k}+k\dot{\lambda} & 0\\ \lambda\dot{k}+k\dot{\lambda} & 2\lambda\left(k\lambda\dot{k}+\left(k^2+1\right)\dot{\lambda}\right) & 0\\ 0 & 0 & 0\end{bmatrix}$$

As expected, from 2.5.20,

$$\dot{J}=J\mathrm{tr}(\mathbf{d})=J\left(\dot{\lambda}/\lambda\right)=\dot{\lambda}$$

■

2.5.6 Problems

1. (a) Differentiate the relation $\mathbf{I} = \mathbf{F}\mathbf{F}^{-1}$ and use 2.5.4, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, to derive 2.5.5b,

   $\dot{\overline{\mathbf{F}^{-1}}} = -\mathbf{F}^{-1}\mathbf{l}$.

   (b) Differentiate the relation $\mathbf{I} = \mathbf{F}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$ and use 2.5.4, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, and 1.10.3e to derive 2.5.5c, $\dot{\overline{\mathbf{F}^{-\mathrm{T}}}} = -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$.

2. For the velocity field

$$v_1 = x_1^2 x_2, \quad v_2 = 2x_2^2 x_3, \quad v_3 = 3x_1 x_2 x_3$$

   determine the rate of stretching per unit stretch at (2,0,1) in the direction of the unit vector

$$(4\mathbf{e}_1 - 3\mathbf{e}_2)/5$$

   And in the direction of $\mathbf{e}_1$?

3. (a) Derive the relation 2.5.18a, $\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{d}\mathbf{F}$ directly from $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$

   (b) Use the definitions $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{e} = (\mathbf{I} - \mathbf{b}^{-1})/2$ to derive the relations 2.5.18c,d: $\dot{\mathbf{b}} = \mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}}$, $\dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^{\mathrm{T}}\mathbf{e} - \mathbf{e}\mathbf{l}$

4. Use 2.5.4, 2.5.19, 1.10.3h, 1.10.6, to derive 2.5.20.

5. For the motion $x_1 = 3X_1 t - t^2, x_2 = X_1 + X_2 t, x_3 = tX_3$, verify that $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$. What is the ratio of the volume element currently occupying $(1,1,1)$ to its volume in the undeformed configuration? And what is the rate of change of this volume element, per unit current volume?

6. Use Nanson's formula 2.2.59, the product rule of differentiation, and 2.5.20, 2.5.5c, to derive the material time derivative of a vector area element, 2.5.22 (note that $\hat{\mathbf{N}}$, a unit normal in the undeformed configuration, is constant).

2.6 Deformation Rates: Further Topics

2.6.1 Relationship between l, d, w and the rate of change of R and U

Consider the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$. Since $\mathbf{R}$ is orthogonal, $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$, and a differentiation of this equation leads to

$$\boldsymbol{\Omega}_{\mathbf{R}} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} \tag{2.6.1}$$

with $\boldsymbol{\Omega}_{\mathbf{R}}$ skew-symmetric (see Eqn. 1.14.2). Using this relation, the expression $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, and the definitions of **d** and **w**, Eqn. 2.5.7, one finds that {▲Problem 1}

$$\begin{aligned}
\mathbf{l} &= \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}} \\
\mathbf{w} &= \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}} \\
&= \mathbf{R}\,\mathrm{skew}\left[\dot{\mathbf{U}}\mathbf{U}^{-1}\right]\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}} \\
\mathbf{d} &= \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} \\
&= \mathbf{R}\,\mathrm{sym}\left[\dot{\mathbf{U}}\mathbf{U}^{-1}\right]\mathbf{R}^{\mathrm{T}}
\end{aligned} \tag{2.6.2}$$

Note that $\boldsymbol{\Omega}_{\mathbf{R}}$ being skew-symmetric is consistent with **w** being skew-symmetric, and that both **w** and **d** involve **R**, and the rate of change of **U**.

When the motion is a rigid body rotation, then $\dot{\mathbf{U}} = \mathbf{0}$, and

$$\mathbf{w} = \boldsymbol{\Omega}_{\mathbf{R}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{2.6.3}$$

2.6.2 Deformation Rate Tensors and the Principal Material and Spatial Bases

The rate of change of the stretch tensor in terms of the principal material base vectors is

$$\dot{\mathbf{U}} = \sum_{i=1}^{3}\left\{\dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \lambda_i\dot{\hat{\mathbf{N}}}_i \otimes \hat{\mathbf{N}}_i + \lambda_i\hat{\mathbf{N}}_i \otimes \dot{\hat{\mathbf{N}}}_i\right\} \tag{2.6.4}$$

Consider the case when the principal material axes stay constant, as can happen in some simple deformations. In that case, $\dot{\mathbf{U}}$ and $\mathbf{U}^{-1}$ are coaxial (see §1.11.5):

$$\dot{\mathbf{U}} = \sum_{i=1}^{3}\dot{\lambda}_i\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \quad \text{and} \quad \mathbf{U}^{-1} = \sum_{i=1}^{3}\frac{1}{\lambda_i}\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \tag{2.6.5}$$

with $\dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{U}^{-1}\dot{\mathbf{U}}$ and, as expected, from 2.5.25b, $\mathbf{w} = \mathbf{\Omega_R} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, that is, any spin is due to rigid body rotation.

Similarly, from 2.2.37, and differentiating $\hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i = \mathbf{I}$,

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \left\{ \lambda_i \dot{\lambda}_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \tfrac{1}{2}\lambda_i^2 \dot{\hat{\mathbf{N}}}_i \otimes \hat{\mathbf{N}}_i + \tfrac{1}{2}\lambda_i^2 \hat{\mathbf{N}}_i \otimes \dot{\hat{\mathbf{N}}}_i \right\}. \tag{2.6.6}$$

Also, differentiating $\hat{\mathbf{N}}_i \cdot \hat{\mathbf{N}}_j = \delta_{ij}$ leads to $\dot{\hat{\mathbf{N}}}_i \cdot \hat{\mathbf{N}}_j = -\hat{\mathbf{N}}_i \cdot \dot{\hat{\mathbf{N}}}_j$ and so the expression

$$\dot{\hat{\mathbf{N}}}_i = \sum_{m=1}^{3} W_{im} \hat{\mathbf{N}}_m \tag{2.6.7}$$

is valid provided W_{ij} are the components of a skew-symmetric tensor, $W_{ij} = -W_{ji}$. This leads to an alternative expression for the Green-Lagrange tensor:

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \lambda_i \dot{\lambda}_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i + \sum_{\substack{m,n=1 \\ m \neq n}}^{3} \tfrac{1}{2} W_{mn} \left(\lambda_m^2 - \lambda_n^2 \right) \hat{\mathbf{N}}_m \otimes \hat{\mathbf{N}}_n \tag{2.6.8}$$

Similarly, from 2.2.37, the left Cauchy-Green tensor can be expressed in terms of the principal spatial base vectors:

$$\mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \quad \dot{\mathbf{b}} = \sum_{i=1}^{3} \left\{ 2\lambda_i \dot{\lambda}_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \dot{\hat{\mathbf{n}}}_i \otimes \hat{\mathbf{n}}_i + \lambda_i^2 \hat{\mathbf{n}}_i \otimes \dot{\hat{\mathbf{n}}}_i \right\} \tag{2.6.9}$$

Then, from inspection of 2.5.18c, $\dot{\mathbf{b}} = \mathbf{lb} + \mathbf{bl}^{\mathrm{T}}$, the velocity gradient can be expressed as {▲Problem 2}

$$\mathbf{l} = \sum_{i=1}^{3} \left\{ \frac{\dot{\lambda}_i}{\lambda_i} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i + \dot{\hat{\mathbf{n}}}_i \otimes \hat{\mathbf{n}}_i \right\} = \sum_{i=1}^{3} \left\{ \frac{\dot{\lambda}_i}{\lambda_i} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i - \hat{\mathbf{n}}_i \otimes \dot{\hat{\mathbf{n}}}_i \right\} \tag{2.6.7}$$

2.6.3 Rates of Change and the Relative Deformation

Just as the material time derivative of the deformation gradient is defined as

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t} \mathbf{F}(\mathbf{X}, t) = \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right)$$

one can define the material time derivative of the relative deformation gradient, *cf.* §2.3.2, the rate of change *relative to the current configuration*:

$$\dot{\mathbf{F}}_t(\mathbf{x}, t) = \frac{\partial}{\partial \tau} \mathbf{F}_t(\mathbf{x}, \tau) \Big|_{\tau = t} \tag{2.6.8}$$

From 2.3.8, $\mathbf{F}_t(\mathbf{x},\tau) = \mathbf{F}(\mathbf{X},\tau)\mathbf{F}(\mathbf{X},t)^{-1}$, so taking the derivative with respect to τ (t is now fixed) and setting $\tau = t$ gives

$$\dot{\mathbf{F}}_t(\mathbf{x},t) = \dot{\mathbf{F}}(\mathbf{X},t)\mathbf{F}(\mathbf{X},t)^{-1}$$

Then, from 2.5.4,

$$\mathbf{l} = \dot{\mathbf{F}}_t(\mathbf{x},t) \tag{2.6.9}$$

as expected – the velocity gradient is the rate of change of deformation relative to the current configuration. Further, using the polar decomposition,

$$\mathbf{F}_t(\mathbf{x},\tau) = \mathbf{R}_t(\mathbf{x},\tau)\mathbf{U}_t(\mathbf{x},\tau)$$

Differentiating with respect to τ and setting $\tau = t$ then gives

$$\dot{\mathbf{F}}_t(\mathbf{x},t) = \mathbf{R}_t(\mathbf{x},t)\dot{\mathbf{U}}_t(\mathbf{x},t) + \dot{\mathbf{R}}_t(\mathbf{x},t)\mathbf{U}_t(\mathbf{x},t)$$

Relative to the current configuration, $\mathbf{R}_t(\mathbf{x},t) = \mathbf{U}_t(\mathbf{x},t) = \mathbf{I}$, so, from 2.4.34,

$$\mathbf{l} = \dot{\mathbf{U}}_t(\mathbf{x},t) + \dot{\mathbf{R}}_t(\mathbf{x},t) \tag{2.6.10}$$

With **U** symmetric and **R** skew-symmetric, $\dot{\mathbf{U}}_t(\mathbf{x},t)$, $\dot{\mathbf{R}}_t(\mathbf{x},t)$ are, respectively, symmetric and skew-symmetric, and it follows that

$$\begin{aligned} \mathbf{d} &= \dot{\mathbf{U}}_t(\mathbf{x},t) \\ \mathbf{w} &= \dot{\mathbf{R}}_t(\mathbf{x},t) \end{aligned} \tag{2.6.11}$$

again, as expected – the rate of deformation is the instantaneous rate of stretching and the spin is the instantaneous rate of rotation.

The Corotational Derivative

The **corotational derivative** of a vector **a** is $\overset{\circ}{\mathbf{a}} \equiv \dot{\mathbf{a}} - \mathbf{w}\mathbf{a}$. Formally, it is defined through

$$\begin{aligned} \overset{\circ}{\mathbf{a}} &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \mathbf{R}_t(t+\Delta t)\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \left[\mathbf{R}_t(t) + \Delta t\dot{\mathbf{R}}_t(t) + \cdots\right]\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \left[\mathbf{I} + \Delta t\mathbf{w}(t) + \cdots\right]\mathbf{a}(t)\right\} \\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left\{\mathbf{a}(t+\Delta t) - \mathbf{a}(t)\right\} - \mathbf{w}(t)\mathbf{a}(t) \\ &= \dot{\mathbf{a}} - \mathbf{w}\mathbf{a} \end{aligned} \tag{2.6.12}$$

The definition shows that the corotational derivative involves taking a vector **a** in the current configuration and rotating it with the rigid body rotation part of the motion, Fig. 2.6.1. It is this new, rotated, vector which is compared with the vector $\mathbf{a}(t+\Delta t)$, which has undergone rotation and stretch.

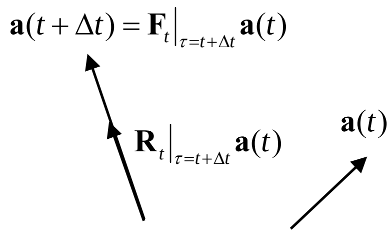

Figure 2.6.1: rotation and stretch of a vector

2.6.4 Rivlin-Ericksen Tensors

The n-th **Rivlin-Ericksen tensor** is defined as

$$\mathbf{A}_n(t) = \frac{d^n}{d\tau^n}\mathbf{C}_t(\tau)\bigg|_{\tau=t}, \qquad n = 0, 1, 2, \cdots \tag{2.6.13}$$

where $\mathbf{C}_t(\tau)$ is the relative right Cauchy-Green strain. Since $\mathbf{C}_t(\tau)\big|_{\tau=t} = \mathbf{I}$, $\mathbf{A}_0 = \mathbf{I}$. To evaluate the next Rivlin-Ericksen tensor, one needs the derivatives of the relative deformation gradient; from 2.5.4, 2.3.8,

$$\frac{d}{d\tau}\mathbf{F}_t(\tau) = \frac{d}{d\tau}\left[\mathbf{F}(\tau)\mathbf{F}(t)^{-1}\right] = \mathbf{l}(\tau)\mathbf{F}(\tau)\mathbf{F}(t)^{-1} = \mathbf{l}(\tau)\mathbf{F}_t(\tau) \tag{2.6.14}$$

Then, with 2.5.5a, $d\left(\mathbf{F}_t(\tau)^{\mathrm{T}}\right)/d\tau = \mathbf{F}_t(\tau)^{\mathrm{T}}\mathbf{l}(\tau)^{\mathrm{T}}$, and

$$\begin{aligned}\mathbf{A}_1(t) &= \left[\mathbf{F}_t(\tau)^{\mathrm{T}}\left(\mathbf{l}(\tau)+\mathbf{l}(\tau)^{\mathrm{T}}\right)\mathbf{F}_t(\tau)\right]_{\tau=t} \\ &= \left(\mathbf{l}(t)+\mathbf{l}(t)^{\mathrm{T}}\right) \\ &= 2\mathbf{d}\end{aligned}$$

Thus the tensor $\mathbf{A}_1$ gives a measure of the rate of stretching of material line elements (see Eqn. 2.5.10). Similarly, higher Rivlin-Ericksen tensors give a measure of higher order stretch rates, $\ddot{\lambda}, \dddot{\lambda}$, and so on.

2.6.5 The Directional Derivative and the Material Time Derivative

The directional derivative of a function $\mathbf{T}(t)$ in the direction of an increment in t is, by definition (see, for example, Eqn. 1.15.27),

$$\partial_t \mathbf{T}[\Delta t] = \mathbf{T}(t + \Delta t) - \mathbf{T}(t) \tag{2.6.15}$$

or

$$\partial_t \mathbf{T}[\Delta t] = \frac{d\mathbf{T}}{dt}\Delta t \tag{2.6.16}$$

Setting $\Delta t = 1$, and using the chain rule 1.15.28,

$$\begin{aligned}\dot{\mathbf{T}} &= \partial_t \mathbf{T}[1] \\ &= \partial_{\mathbf{x}} \mathbf{T}\left[\partial_t \mathbf{x}[1]\right] \\ &= \partial_{\mathbf{x}} \mathbf{T}\left[\mathbf{v}\right]\end{aligned} \tag{2.6.17}$$

The material time derivative is thus equivalent to the directional derivative in the direction of the velocity vector.

2.6.6 Problems

1. Derive the relations 2.6.2.
2. Use 2.6.9 to verify 2.5.18, $\dot{\mathbf{b}} = \mathbf{lb} + \mathbf{bl}^{\mathrm{T}}$.

2.7 Small Strain Theory

When the deformation is small, from 2.2.43-4,

$$\begin{aligned}\mathbf{F} &= \mathbf{I} + \operatorname{Grad}\mathbf{U} \\ &= \mathbf{I} + (\operatorname{grad}\mathbf{u})\mathbf{F} \\ &\approx \mathbf{I} + \operatorname{grad}\mathbf{u}\end{aligned} \tag{2.7.1}$$

neglecting the product of $\operatorname{grad}\mathbf{u}$ with $\operatorname{Grad}\mathbf{U}$, since these are small quantities. Thus one can take $\operatorname{Grad}\mathbf{U} = \operatorname{grad}\mathbf{u}$ and there is no distinction to be made between the undeformed and deformed configurations. The deformation gradient is of the form $\mathbf{F} = \mathbf{I} + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is small.

2.7.1 Decomposition of Strain

Any second order tensor can be decomposed into its symmetric and antisymmetric part according to 1.10.28, so that

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right) + \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right) = \boldsymbol{\varepsilon} + \boldsymbol{\Omega} \\ \frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij} + \Omega_{ij}\end{aligned} \tag{2.7.2}$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor 2.2.48 and $\boldsymbol{\Omega}$, the anti-symmetric part of the displacement gradient, is the **small rotation tensor**, so that **F** can be written as

$$\boxed{\mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}} \quad \textbf{Small Strain Decomposition of the Deformation Gradient} \tag{2.7.3}$$

It follows that (for the calculation of **e**, one can use the relation $(\mathbf{I} + \boldsymbol{\delta})^{-1} \approx \mathbf{I} - \boldsymbol{\delta}$ for small $\boldsymbol{\delta}$)

$$\begin{aligned}\mathbf{C} &= \mathbf{b} = \mathbf{I} + 2\boldsymbol{\varepsilon} \\ \mathbf{E} &= \mathbf{e} = \boldsymbol{\varepsilon}\end{aligned} \tag{2.7.4}$$

Rotation

Since $\boldsymbol{\Omega}$ is antisymmetric, it can be written in terms of an axial vector $\boldsymbol{\omega}$, *cf.* §1.10.11, so that for any vector **a**,

$$\boldsymbol{\Omega}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}, \qquad \boldsymbol{\omega} = -\Omega_{23}\mathbf{e}_1 + \Omega_{13}\mathbf{e}_1 - \Omega_{12}\mathbf{e}_3 \tag{2.7.5}$$

The relative displacement can now be written as

$$\begin{aligned} d\mathbf{u} &= (\mathrm{grad}\mathbf{u})d\mathbf{X} \\ &= \boldsymbol{\varepsilon}d\mathbf{X} + \boldsymbol{\omega} \times d\mathbf{X} \end{aligned} \tag{2.7.6}$$

The component of relative displacement given by $\boldsymbol{\omega} \times d\mathbf{X}$ is perpendicular to $d\mathbf{X}$, and so represents a pure rotation of the material line element, Fig. 2.7.1.

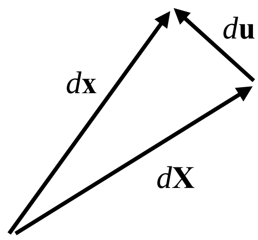

Figure 2.7.1: a pure rotation

Principal Strains

Since $\boldsymbol{\varepsilon}$ is symmetric, it must have three mutually orthogonal eigenvectors, the **principal axes of strain**, and three corresponding real eigenvalues, the **principal strains,** e_1, e_2, e_3), which can be positive or negative, *cf.* §1.11. The effect of $\boldsymbol{\varepsilon}$ is therefore to deform an elemental unit sphere into an elemental ellipsoid, whose axes are the principal axes, and whose lengths are $1+e_1, 1+e_2, 1+e_3$. Material fibres in these principal directions are stretched only, in which case the deformation is called a **pure deformation**; fibres in other directions will be stretched and rotated.

The term $\boldsymbol{\varepsilon}d\mathbf{X}$ in 2.7.6 therefore corresponds to a pure stretch along the principal axes. The total deformation is the sum of a pure deformation, represented by $\boldsymbol{\varepsilon}$, and a rigid body rotation, represented by $\mathbf{\Omega}$. This result is similar to that obtained for the exact finite strain theory, but here the decomposition is *additive* rather than *multiplicative*. Indeed, here the corresponding small strain stretch and rotation tensors are $\mathbf{U} = \mathbf{I} + \boldsymbol{\varepsilon}$ and $\mathbf{R} = \mathbf{I} + \mathbf{\Omega}$, so that

$$\mathbf{F} = \mathbf{RU} = \mathbf{I} + \boldsymbol{\varepsilon} + \mathbf{\Omega} \tag{2.7.7}$$

Example

Consider the simple shear (*c.f.* Eqn. 2.2.40)

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3$$

where k is small. The displacement vector is $\mathbf{u} = kx_2\mathbf{e}_1$ so that

$$\mathrm{grad}\,\mathbf{u} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The deformation can be written as the additive decomposition

$$d\mathbf{u} = \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\Omega} d\mathbf{X} \qquad \text{or} \qquad d\mathbf{u} = \boldsymbol{\varepsilon} d\mathbf{X} + \boldsymbol{\omega} \times d\mathbf{X}$$

with

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\boldsymbol{\omega} = -(k/2)\mathbf{e}_3$. For the rotation component, one can write

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\Omega} = \begin{bmatrix} 1 & k/2 & 0 \\ -k/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, since for small θ, $\cos\theta \approx 1, \sin\theta \approx \theta$, can be seen to be a rotation through an angle $\theta = -k/2$ (a clockwise rotation).

The principal values of $\boldsymbol{\varepsilon}$ are $\pm k/2, 0$ with corresponding principal directions $\mathbf{n}_1 = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2$, $\mathbf{n}_2 = -(1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2$ and $\mathbf{n}_3 = \mathbf{e}_3$.

Thus the simple shear with small displacements consists of a rotation through an angle $k/2$ superimposed upon a pure shear with angle $k/2$, Fig. 2.6.2.

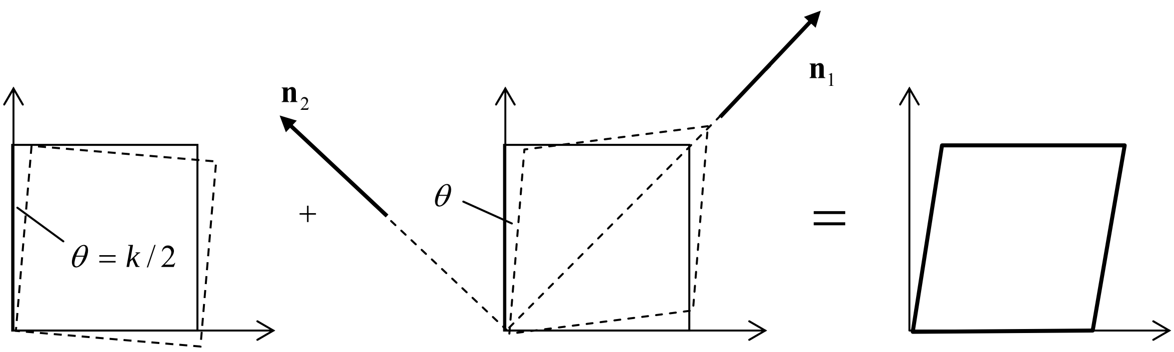

Figure 2.6.2: simple shear

■

2.7.2 Rotations and Small Strain

Consider now a pure rotation about the X_3 axis (within the exact finite strain theory), $d\mathbf{x} = \mathbf{R} d\mathbf{X}$, with

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.7.8}$$

This rotation does not change the length of line elements $d\mathbf{X}$. According to the small strain theory, however,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which does predict line element length changes, but which can be neglected if θ is small. For example, if the rotation is of the order $10^{-2}\,\text{rad}$, then $\varepsilon_{11} = \varepsilon_{22} = 10^{-4}$. However, if the rotation is large, the errors will be appreciable; in that case, rigid body rotation introduces geometrical non-linearities which must be dealt with using the finite deformation theory.

Thus the small strain theory is restricted to not only the case of small displacement gradients, but also small rigid body rotations.

2.7.3 Volume Change

An elemental cube with edges of unit length in the directions of the principal axes deforms into a cube with edges of lengths $1+e_1, 1+e_2, 1+e_3$, so the unit change in volume of the cube is

$$\frac{dv - dV}{dV} = (1+e_1)(1+e_2)(1+e_3) - 1 = e_1 + e_2 + e_3 + O(2) \tag{2.7.9}$$

Since second order quantities have already been neglected in introducing the small strain tensor, they must be neglected here. Hence the increase in volume per unit volume, called the **dilatation** (or **dilation**) is

$$\boxed{\frac{\delta V}{V} = e_1 + e_2 + e_3 = e_{ii} = \mathrm{tr}\boldsymbol{\varepsilon} = \mathrm{div}\mathbf{u}} \quad \textbf{Dilatation} \tag{2.7.10}$$

Since any elemental volume can be constructed out of an infinite number of such elemental cubes, this result holds for any elemental volume irrespective of shape.

2.7.4 Rate of Deformation, Strain Rate and Spin Tensors

Take now the expressions 2.4.7 for the rate of deformation and spin tensors. Replacing $\mathbf{v}$ in these expressions by $\dot{\mathbf{u}}$, one has

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \qquad d_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right)$$
$$\mathbf{w} = \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^{\mathrm{T}}\right), \qquad w_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \frac{\partial \dot{u}_j}{\partial x_i}\right) \tag{2.7.11}$$

For small strains, one can take the time derivative outside (by considering the x_i to be material coordinates independent of time):

$$d_{ij} = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}$$
$$w_{ij} = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right\} \tag{2.7.12}$$

The rate of deformation in this context is seen to be the **rate of strain**, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$, and the spin is seen to be the **rate of rotation**, $\mathbf{w} = \dot{\boldsymbol{\Omega}}$.

The instantaneous motion of a material particle can hence be regarded as the sum of three effects:

(i) a translation given by $\dot{\mathbf{u}}$ (so in the time interval Δt the particle has been displaced by $\dot{\mathbf{u}}\Delta t$)
(ii) a pure deformation given by $\dot{\boldsymbol{\varepsilon}}$
(iii) a rigid body rotation given by $\dot{\boldsymbol{\Omega}}$

2.7.5 Compatibility Conditions

Suppose that the strains ε_{ij} in a body are known. If the displacements are to be determined, then the strain-displacement partial differential equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.7.13}$$

need to be integrated. However, there are six independent strain components but only three displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**, and it can be shown that they are given by

$$\varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0 \tag{2.7.14}$$

These are 81 equations, but only six of them are distinct, and these six equations are necessary and sufficient to evaluate the displacement field.

2.9 Rigid Body Rotations of Configurations

In this section are discussed rigid body rotations to the current and reference configurations.

2.9.1 A Rigid Body Rotation of the Current Configuration

As mentioned in §2.8.1, the circumstance of two observers, moving relative to each other and examining a fixed configuration (the current configuration) is equivalent to one observer taking measurements of two different configurations, moving relative to each other[1]. The objectivity requirements of the various kinematic objects discussed in the previous section can thus also be examined by considering rigid body rotations and translations of the current configuration.

Any rigid body rotation and translation of the current configuration can be expressed in the form

$$\mathbf{x}^*(\mathbf{X},t) = \mathbf{Q}(t)\mathbf{x}(\mathbf{X},t) + \mathbf{c}(t) \tag{2.9.1}$$

where $\mathbf{Q}$ is a rotation tensor. This is illustrated in Fig. 2.9.5. The current configuration is denoted by S and the rotated configuration by S^*.

Just as $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, the deformation gradient for the configuration S^* relative to the reference configuration S_0 is defined through $d\mathbf{x}^* = \mathbf{F}^* d\mathbf{X}$. From 2.9.1, as in §2.8.5 (see Eqn. 2.8.23), and similarly for the right and left Cauchy-Green tensors,

$$\begin{aligned}
\mathbf{F}^* &= \mathbf{QF} \\
\mathbf{C}^* &= \mathbf{F}^{*\mathrm{T}}\mathbf{F}^* = \mathbf{C} \\
\mathbf{b}^* &= \mathbf{F}^*\mathbf{F}^{*\mathrm{T}} = \mathbf{QbQ}^{\mathrm{T}}
\end{aligned} \tag{2.9.2}$$

Thus in the deformations $\mathbf{F}: S_0 \to S$ and $\mathbf{F}^*: S_0 \to S^*$, the right Cauchy Green tensors, $\mathbf{C}$ and $\mathbf{C}^*$, are the same, but the left Cauchy Green tensors are different, and related through $\mathbf{b}^* = \mathbf{QbQ}^{\mathrm{T}}$.

All the other results obtained in the last section in the context of observer transformations, for example for the Jacobian, stretch tensors, etc., hold also for the case of rotations to the current configuration.

[1] Although equivalent, there is a difference: in one, there are two observers who record one event (a material particle say) as at two different points, in the other there is one observer who records two different events (the place where the one material particle is in two different configurations)

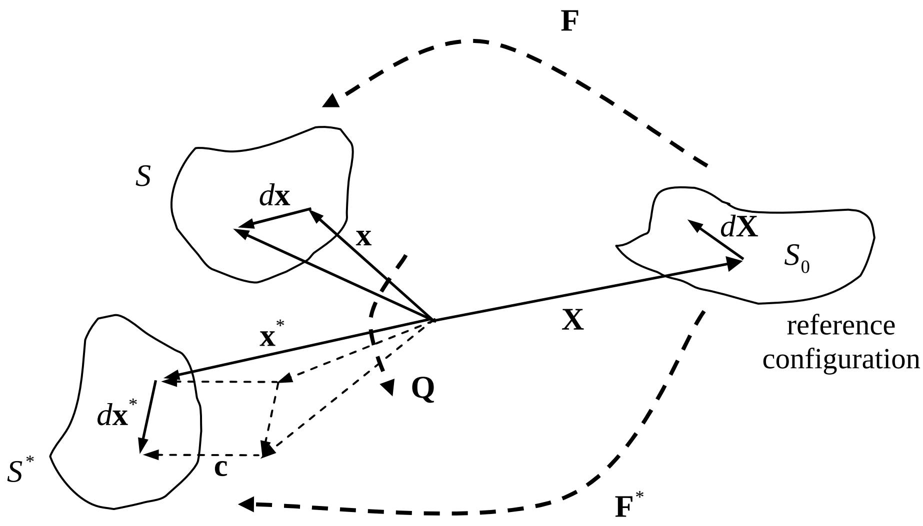

Figure 2.9.1: a rigid body rotation and translation of the current configuration

2.9.2 A Rigid Body Rotation of the Reference Configuration

Consider now a rigid-body rotation to the *reference* configuration. Such rotations play an important role in the notion of material symmetry (see Chapter 5).

The reference configuration is denoted by S_0 and the rotated/translated configuration by $S^\lozenge$, Fig. 2.9.2. The deformation gradient for the current configuration S relative to $S^\lozenge$ is defined through $d\mathbf{x} = \mathbf{F}^\lozenge d\mathbf{X}^\lozenge = \mathbf{F}^\lozenge \mathbf{Q} d\mathbf{X}$. But $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ and so (and similarly for the right and left Cauchy-Green tensors)

$$
\begin{aligned}
\mathbf{F}^\lozenge &= \mathbf{F}\mathbf{Q}^{\mathrm{T}} \\
\mathbf{C}^\lozenge &= \mathbf{F}^{\lozenge\mathrm{T}}\mathbf{F}^\lozenge = \mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}} \\
\mathbf{b}^\lozenge &= \mathbf{F}^\lozenge\mathbf{F}^{\lozenge\mathrm{T}} = \mathbf{b}
\end{aligned}
\qquad (2.9.3)
$$

Thus the change to the right (left) Cauchy-Green strain tensor under a rotation to the reference configuration is the same as the change to the left (right) Cauchy-Green strain tensor under a rotation of the current configuration.

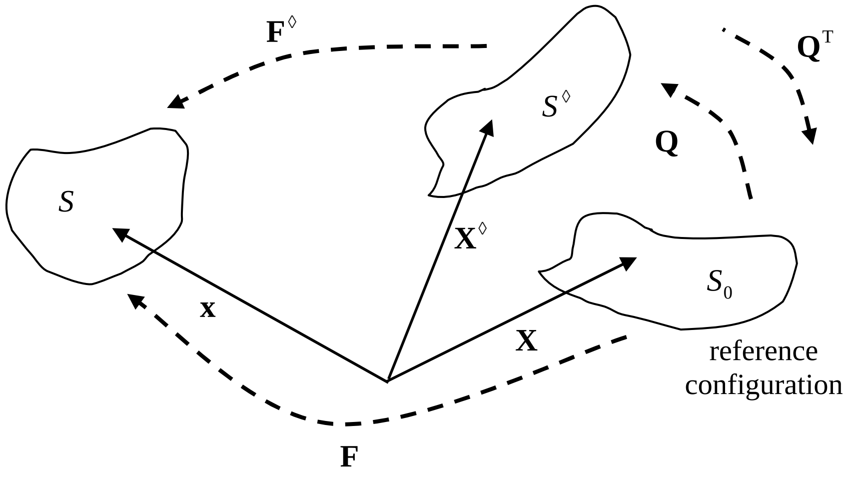

Figure 2.9.2: a rigid body rotation of the reference configuration

3 Stress and the Balance Principles

Three basic laws of physics are discussed in this Chapter:

(1) The Law of Conservation of Mass
(2) The Balance of Linear Momentum
(3) The Balance of Angular Momentum

together with the conservation of mechanical energy and the principle of virtual work, which are different versions of (2).

(2) and (3) involve the concept of stress, which allows one to describe the action of forces in materials.

3.1 Conservation of Mass

3.1.1 Mass and Density

Mass is a non-negative scalar measure of a body's tendency to resist a change in motion.

Consider a small volume element Δv whose mass is Δm. Define the average **density** of this volume element by the ratio

$$\rho_{\text{AVE}} = \frac{\Delta m}{\Delta v} \tag{3.1.1}$$

If p is some point within the volume element, then define the **spatial mass density** at p to be the limiting value of this ratio as the volume shrinks down to the point,

$$\boxed{\rho(\mathbf{x}, t) = \lim_{\Delta v \to 0} \frac{\Delta m}{\Delta v}} \quad \textbf{Spatial Density} \tag{3.1.2}$$

In a real material, the incremental volume element Δv must not actually get too small since then the limit ρ would depend on the atomistic structure of the material; the volume is only allowed to decrease to some minimum value which contains a large number of molecules. The spatial mass density is a representative average obtained by having Δv large compared to the atomic scale, but small compared to a typical length scale of the problem under consideration.

The density, as with displacement, velocity, and other quantities, is defined for *specific particles* of a continuum, and is a continuous function of coordinates and time, $\rho = \rho(\mathbf{x}, t)$. However, the mass is not defined this way – one writes for the mass of an infinitesimal volume of material – a **mass element**,

$$dm = \rho(\mathbf{x}, t) dv \tag{3.1.3}$$

or, for the mass of a volume v of material at time t,

$$m = \int_v \rho(\mathbf{x}, t) dv \tag{3.1.4}$$

3.1.2 Conservation of Mass

The law of conservation of mass states that mass can neither be created nor destroyed.

Consider a collection of matter located somewhere in space. This quantity of matter with well-defined boundaries is termed a **system**. The law of conservation of mass then implies that the mass of this given system remains constant,

$$\boxed{\frac{Dm}{Dt} = 0} \qquad \textbf{Conservation of Mass} \tag{3.1.5}$$

The volume occupied by the matter may be changing and the density of the matter within the system may be changing, but the mass remains constant.

Considering a differential mass element at position **X** in the reference configuration and at **x** in the current configuration, Eqn. 3.1.5 can be rewritten as

$$dm(\mathbf{X}) = dm(\mathbf{x}, t) \tag{3.1.6}$$

The conservation of mass equation can be expressed in terms of densities. First, introduce ρ_0, the **reference mass density** (or simply the **density**), defined through

$$\boxed{\rho_0(\mathbf{X}) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}} \qquad \textbf{Density} \tag{3.1.7}$$

Note that the density ρ_0 and the spatial mass density ρ are *not* the same quantities[1].

Thus the **local** (or **differential**) **form** of the conservation of mass can be expressed as (see Fig. 3.1.1)

$$dm = \rho_0(\mathbf{X})dV = \rho(\mathbf{x}, t)dv = \text{const} \tag{3.1.8}$$

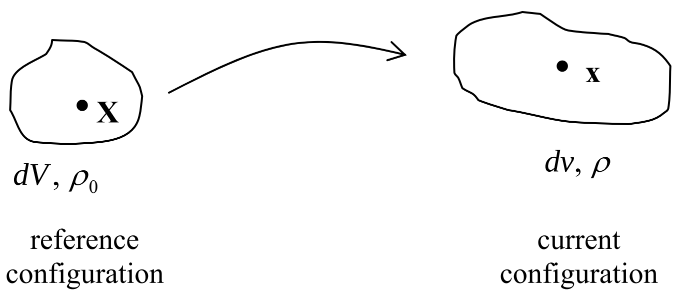

Figure 3.1.1: Conservation of Mass for a deforming mass element

Integration over a finite region of material gives the **global** (or **integral**) **form**,

$$m = \int_V \rho_0(\mathbf{X})dV = \int_v \rho(\mathbf{x}, t)dv = \text{const} \tag{3.1.9}$$

or

$$\dot{m} = \frac{dm}{dt} = \frac{d}{dt}\int_v \rho(\mathbf{x}, t)dv = 0 \tag{3.1.10}$$

[1] they not only are functions of different variables, but also have different values; they are not different representations of the same thing, as were, for example, the velocities **v** and **V**. One could introduce a material mass density, $\mathrm{P}(X, t) = \rho(x(X, t), t)$, but such a quantity is not useful in analysis

3.1.3 Control Mass and Control Volume

A **control mass** is a *fixed mass* of material whose volume and density may change, and which may move through space, Fig. 3.1.2. There is no mass transport through the moving surface of the control mass. For such a system, Eqn. 3.1.10 holds.

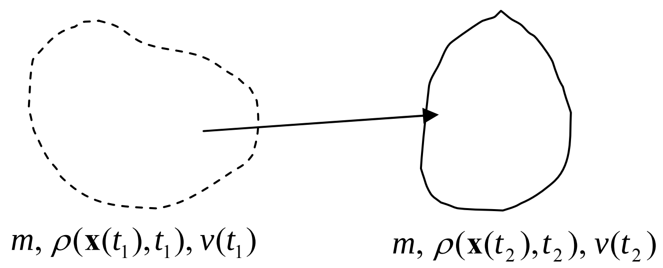

Figure 3.1.2: Control Mass

By definition, the derivative in 3.1.10 is the time derivative of a property (in this case mass) of a collection of material particles as they move through space, and when they instantaneously occupy the volume *v*, Fig. 3.1.3, or

$$\frac{d}{dt}\int_{dv} \rho\, dv = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{ \int_{v(t+\Delta t)} \rho(\mathbf{x}, t+\Delta t) dv - \int_{v(t)} \rho(\mathbf{x},t) dv \right\} = 0 \qquad (3.1.11)$$

Alternatively, one can take the material derivative inside the integral sign:

$$\frac{dm}{dt} = \int_v \frac{d}{dt}\left[\rho(\mathbf{x},t) dv\right] = 0 \qquad (3.1.12)$$

This is now equivalent to the sum of the rates of change of mass of the mass elements occupying the volume *v*.

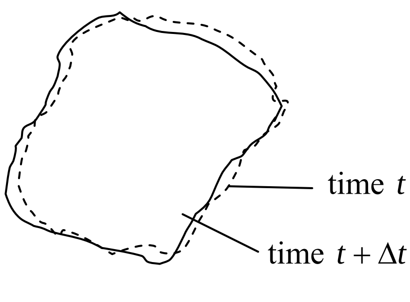

Figure 3.1.3: Control Mass occupying different volumes at different times

A **control volume**, on the other hand, is a *fixed volume* (region) of space through which material may flow, Fig. 3.1.4, and for which *the mass may change*. For such a system, one has

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\int_v \rho(\mathbf{x},t)dv = \int_v \frac{\partial}{\partial t}[\rho(\mathbf{x},t)]dv \neq 0 \tag{3.1.13}$$

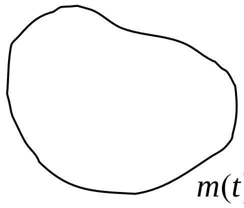

Figure 3.1.4: Control Volume

3.1.4 The Continuity Equation (Spatial Form)

A consequence of the law of conservation of mass is the **continuity equation**, which (in the spatial form) relates the density and velocity of any material particle during motion. This equation can be derived in a number of ways:

Derivation of the Continuity Equation using a Control Volume (Global Form)

The continuity equation can be derived directly by considering a control volume - this is the derivation appropriate to fluid mechanics. Mass inside this fixed volume cannot be created or destroyed, so that the rate of increase of mass in the volume must equal the rate at which mass is flowing into the volume through its bounding surface.

The rate of increase of mass inside the fixed volume v is

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\int_v \rho(\mathbf{x},t)\,dv = \int_v \frac{\partial \rho}{\partial t}dv \tag{3.1.14}$$

The **mass flux** (rate of flow of mass) out through the surface is given by Eqn. 1.7.9,

$$\int_s \rho\mathbf{v}\cdot\mathbf{n}ds, \qquad \int_s \rho v_i n_i\,ds$$

where **n** is the unit outward normal to the surface and **v** is the velocity. It follows that

$$\int_v \frac{\partial \rho}{\partial t}dv + \int_s \rho\mathbf{v}\cdot\mathbf{n}ds = 0, \qquad \int_v \frac{\partial \rho}{\partial t}dv + \int_s \rho v_i n_i\,ds = 0 \tag{3.1.15}$$

Use of the divergence theorem 1.7.12 leads to

$$\int_v \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v})\right]dv = 0, \qquad \int_v \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i}\right]dv = 0 \tag{3.1.16}$$

leading to the continuity equation,

$$
\boxed{
\begin{array}{rr}
\dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 & \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho v_i)}{\partial x_i} = 0 \\
\dfrac{d\rho}{dt} + \rho \operatorname{div}\mathbf{v} = 0 & \dfrac{d\rho}{dt} + \rho \dfrac{\partial v_i}{\partial x_i} = 0 \\
\dfrac{\partial \rho}{\partial t} + \operatorname{grad}\rho \cdot \mathbf{v} + \rho \operatorname{div}\mathbf{v} = 0 & \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho}{\partial x_i} v_i + \rho \dfrac{\partial v_i}{\partial x_i} = 0
\end{array}
}
\quad \textbf{Continuity Equation}
$$

(3.1.17)

This is (these are) the continuity equation in spatial form. The second and third forms of the equation are obtained by re-writing the local derivative in terms of the material derivative 2.4.7 (see also 1.6.23b).

If the material is incompressible, so the density remains constant in the neighbourhood of a particle as it moves, then the continuity equation reduces to

$$
\boxed{\operatorname{div}\mathbf{v} = 0, \quad \frac{\partial v_i}{\partial x_i} = 0} \quad \textbf{Continuity Eqn. for Incompressible Material} \tag{3.1.18}
$$

Derivation of the Continuity Equation using a Control Mass

Here follow two ways to derive the continuity equation using a control mass.

1. Derivation using the Formal Definition

From 3.1.11, adding and subtracting a term:

$$
\begin{aligned}
\frac{d}{dt}\int_{dv} \rho \, dv = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Bigg\{ & \left[\int_{v(t+\Delta t)} \rho(\mathbf{x}, t+\Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t+\Delta t) dv \right] \\
& + \left[\int_{v(t)} \rho(\mathbf{x}, t+\Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t) dv \right] \Bigg\}
\end{aligned} \tag{3.1.19}
$$

The terms in the second square bracket correspond to holding the volume v fixed and evidently equals the local rate of change:

$$
\frac{d}{dt}\int_{dv} \rho \, dv = \int_{v} \frac{\partial \rho}{\partial t} dv + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{v(t+\Delta t) - v(t)} \rho(\mathbf{x}, t+\Delta t) dv \tag{3.1.20}
$$

The region $v(t+\Delta t) - v(t)$ is swept out in time Δt. Superimposing the volumes $v(t)$ and $v(t+\Delta t)$, Fig. 3.1.5, it can be seen that a small element Δv of $v(t+\Delta t) - v(t)$ is given by (see the example associated with Fig. 1.7.7)

$$
\Delta v = \Delta t \mathbf{v} \cdot \mathbf{n} \Delta s \tag{3.1.21}
$$

where s is the surface. Thus

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{v(t+\Delta t)-v(t)} \rho(\mathbf{x}, t+\Delta t) dv = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_s \Delta t \rho(\mathbf{x}, t+\Delta t) \mathbf{v} \cdot \mathbf{n} ds = \int_s \rho(\mathbf{x}, t) \mathbf{v} \cdot \mathbf{n} ds \quad (3.1.22)$$

and 3.1.15 is again obtained, from which the continuity equation results from use of the divergence theorem.

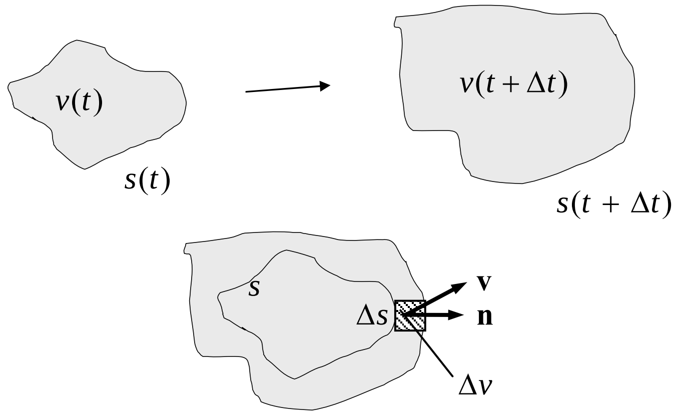

Figure 3.1.5: Evaluation of Eqn. 3.1.22

2. Derivation by Converting to Mass Elements

This derivation requires the kinematic relation for the material time derivative of a volume element, 2.5.23: $d(dv)/dt = \mathrm{div}\mathbf{v}\, dv$. One has

$$\frac{dm}{dt} = \frac{d}{dt} \int_v \rho(\mathbf{x}, t) dv = \int_v \frac{d}{dt} (\rho dv) = \int_v \left(\dot{\rho} dv + \rho \dot{\overline{dv}} \right) = \int_v (\dot{\rho} + \mathrm{div}\mathbf{v} \rho) dv \equiv 0 \quad (3.1.23)$$

The continuity equation then follows, since this must hold for any arbitrary region of the volume v.

Derivation of the Continuity Equation using a Control Volume (Local Form)

The continuity equation can also be derived using a differential control volume element. This calculation is similar to that given in §1.6.6, with the velocity $\mathbf{v}$ replaced by $\rho\mathbf{v}$.

3.1.5 The Continuity Equation (Material Form)

From 3.1.9, and using 2.2.53, $dv = JdV$,

$$\int_V [\rho_0(\mathbf{X}) - \rho(\chi(\mathbf{X}, t), t) J(\mathbf{X}, t)] dV = 0 \quad (3.1.24)$$

Since V is an arbitrary region, the integrand must vanish everywhere, so that

$$\boxed{\rho_0(\mathbf{X}) = \rho(\boldsymbol{\chi}(\mathbf{X},t),t)J(\mathbf{X},t)} \quad \textbf{Continuity Equation (Material Form)} \qquad (3.1.25)$$

This is known as the continuity (mass) equation in the material description. Since $\dot{\rho}_0 = 0$, the rate form of this equation is simply

$$\frac{d}{dt}(\rho J) = 0 \qquad (3.1.26)$$

The material form of the continuity equation, $\rho_0 = \rho J$, is an algebraic equation, unlike the partial differential equation in the spatial form. However, the two must be equivalent, and indeed the spatial form can be derived directly from this material form: using 2.5.20, $dJ / dt = J\mathrm{div}\mathbf{v}$,

$$\begin{aligned} \frac{d}{dt}(\rho J) &= \dot{\rho} J + \rho \dot{J} \\ &= J\left(\dot{\rho} + \rho \,\mathrm{div}\mathbf{v}\right) \end{aligned} \qquad (3.1.27)$$

This is zero, and $J > 0$, and the spatial continuity equation follows.

Example (of Conservation of Mass)

Consider a bar of material of length l_0, with density in the undeformed configuration ρ_0 and spatial mass density $\rho(x,t)$, undergoing the 1-D motion $\mathbf{X} = \mathbf{x}/(1+At)$, $\mathbf{x} = \mathbf{X} + At\mathbf{X}$. The volume ratio (taking unit cross-sectional area) is $J = 1 + At$. The continuity equation in the material form 3.1.25 specifies that

$$\rho_0 = \rho(1 + At)$$

Suppose now that

$$\rho_o(\mathbf{X}) = \frac{2m}{l_0^2}\mathbf{X}$$

so that the total mass of the bar is $\int_0^{l_o} \rho_0(\mathbf{X}) d\mathbf{X} = m$. It follows that the spatial mass density is

$$\rho = \frac{\rho_0}{(1+At)} = \frac{2m}{l_0^2}\frac{\mathbf{X}}{1+At} = \frac{2m}{l_0^2}\frac{\mathbf{x}}{(1+At)^2}$$

Evaluating the total mass of the bar at time *t* leads to

$$\int_0^{l_o(1+At)} \rho(\mathbf{x},t)\, d\mathbf{x} = \frac{2m}{l_0^2}\frac{1}{(1+At)^2}\int_0^{l_o(1+At)} \mathbf{x}\, d\mathbf{x}$$

which is again *m*, as required.

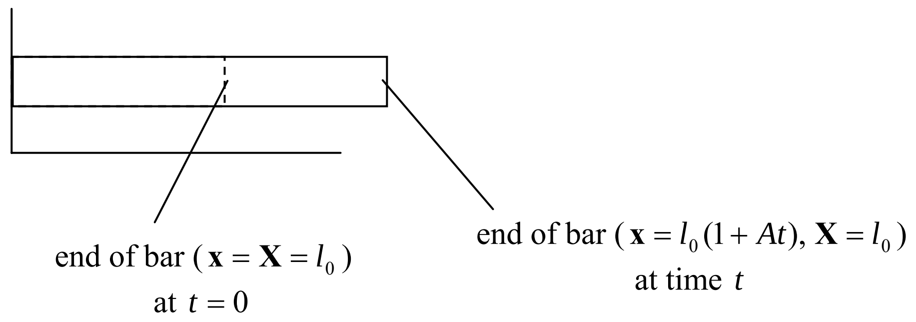

Figure 3.1.6: a stretching bar

The density could have been derived from the equation of continuity in the spatial form: since the velocity is

$$\mathbf{V}(\mathbf{X},t) = \frac{d\mathbf{x}(\mathbf{X},t)}{dt} = A\mathbf{X}, \qquad \mathbf{v}(\mathbf{x},t) = \mathbf{V}(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t) = \frac{A\mathbf{x}}{1+At}$$

one has

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial \mathbf{x}} + \rho\frac{\partial v}{\partial \mathbf{x}} = \frac{\partial \rho}{\partial t} + \frac{A\mathbf{x}}{1+At}\frac{\partial \rho}{\partial \mathbf{x}} + \rho\frac{A}{1+At} = 0$$

Without attempting to solve this first order partial differential equation, it can be seen by substitution that the value for ρ obtained previously satisfies the equation.

■

3.1.6 Material Derivatives of Integrals

Reynold's Transport Theorem

In the above, the material derivative of the total mass carried by a control mass,

$$\frac{d}{dt}\int_v \rho(\mathbf{x},t)dv,$$

was considered. It is quite often that one needs to evaluate material time derivatives of similar volume (and line and surface) integrals, involving other properties, for example momentum or energy. Thus, suppose that $\mathbf{A}(\mathbf{x},t)$ is the distribution of some property (per unit volume) throughout a volume *v* (**A** is taken to be a second order tensor, but what follows applies also to vectors and scalars). Then the rate of change of the total amount of the property carried by the mass system is

$$\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv$$

Again, this integral can be evaluated in a number of ways. For example, one could evaluate it using the formal definition of the material derivative, as done above for $\mathbf{A}=\rho$. Alternatively, one can evaluate it using the relation 2.5.23, $d(dv)/dt = \operatorname{div}\mathbf{v}\,dv$, through

$$\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv = \int_v \frac{d}{dt}\left[\mathbf{A}(\mathbf{x},t)dv\right] = \int_v \left[\dot{\mathbf{A}}dv + \mathbf{A}\,\dot{\overline{dv}}\right] = \int_v \left[\dot{\mathbf{A}} + \operatorname{div}\mathbf{v}\mathbf{A}\right]dv \qquad (3.1.28)$$

Thus one arrives at **Reynold's transport theorem**

$$\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv = \begin{cases} \displaystyle\int_v \left[\frac{d\mathbf{A}}{dt} + \operatorname{div}\mathbf{v}\mathbf{A}\right]dv & \displaystyle\int_v \left[\frac{dA_{ij}}{dt} + \frac{\partial v_k}{\partial x_k}A_{ij}\right]dv \\ \displaystyle\int_v \left[\frac{\partial \mathbf{A}}{\partial t} + \operatorname{grad}\mathbf{A}\cdot\mathbf{v} + \operatorname{div}\mathbf{v}\mathbf{A}\right]dv & \displaystyle\int_v \left[\frac{\partial A_{ij}}{\partial t} + \frac{\partial A_{ij}}{\partial x_k}v_k + \frac{\partial v_k}{\partial x_k}A_{ij}\right]dv \\ \displaystyle\int_v \left[\frac{\partial \mathbf{A}}{\partial t} + \operatorname{div}(\mathbf{A}\otimes\mathbf{v})\right]dv & \displaystyle\int_v \left[\frac{\partial A_{ij}}{\partial t} + \frac{\partial (A_{ij}v_k)}{\partial x_k}\right]dv \\ \displaystyle\int_v \frac{\partial \mathbf{A}}{\partial t}dv + \int_s \mathbf{A}(\mathbf{v}\cdot\mathbf{n})ds & \displaystyle\int_v \frac{\partial A_{ij}}{\partial t}dv + \int_s A_{ij}v_k n_k ds \end{cases}$$

Reynold's Transport Theorem (3.1.29)

The index notation is shown for the case when **A** is a second order tensor. In the last of these forms[2] (obtained by application of the divergence theorem), the first term represents the amount (of **A**) created within the volume v whereas the second term (the flux term) represents the (volume) rate of flow of the property through the surface. In the last three versions, Reynold's transport theorem gives the material derivative of the moving control mass in terms of the derivative of the instantaneous fixed volume in space (the first term).

Of course when $\mathbf{A}=\rho$, the continuity equation is recovered.

Another way to derive this result is to first convert to the reference configuration, so that integration and differentiation commute (since dV is independent of time):

$$\begin{aligned}\frac{d}{dt}\int_v \mathbf{A}(\mathbf{x},t)dv &= \frac{d}{dt}\int_V \mathbf{A}(\mathbf{X},t)JdV = \int_V \frac{d}{dt}\left(\mathbf{A}(\mathbf{X},t)J\right)dV \\ &= \int_V \left(\dot{\mathbf{A}}J + \mathbf{A}\dot{J}\right)dV = \int_V \left(\dot{\mathbf{A}} + \operatorname{div}\mathbf{v}\mathbf{A}\right)JdV \\ &= \int_v \left(\dot{\mathbf{A}}(\mathbf{x},t) + \operatorname{div}\mathbf{v}\mathbf{A}(\mathbf{x},t)\right)dv\end{aligned} \qquad (3.1.30)$$

[2] also known as the **Leibniz formula**

Reynold's Transport Theorem for Specific Properties

A property that is given per unit mass is called a **specific property**. For example, specific heat is the heat per unit mass. Consider then a property **B**, a scalar, vector or tensor, which is defined per unit mass through a volume. Then the rate of change of the total amount of the property carried by the mass system is simply

$$\frac{d}{dt}\int_v \rho\mathbf{B}(\mathbf{x},t)dv = \int_v \frac{d}{dt}\left[\mathbf{B}\rho dv\right] = \int_v \frac{d}{dt}\left[\mathbf{B}dm\right] = \int_v \frac{d\mathbf{B}}{dt}dm = \int_v \rho\frac{d\mathbf{B}}{dt}dv \qquad (3.1.31)$$

Material Derivatives of Line and Surface Integrals

Material derivatives of line and surface integrals can also be evaluated. From 2.5.8, $d(d\mathbf{x})/dt = \mathbf{l}d\mathbf{x}$,

$$\frac{d}{dt}\int \mathbf{A}(\mathbf{x},t)d\mathbf{x} = \int\left[\dot{\mathbf{A}} + \mathbf{A}\mathbf{l}\right]d\mathbf{x} \qquad (3.1.32)$$

and, using 2.5.22, $d\left(\hat{\mathbf{n}}ds\right)/dt = \left(\mathrm{div}\mathbf{v} - \mathbf{l}^{\mathrm{T}}\right)\hat{\mathbf{n}}ds$,

$$\frac{d}{dt}\int_s \mathbf{A}(\mathbf{x},t)\hat{\mathbf{n}}ds = \int_s\left[\dot{\mathbf{A}} + \mathbf{A}\left(\mathrm{div}\mathbf{v} - \mathbf{l}^{\mathrm{T}}\right)\right]\hat{\mathbf{n}}ds \qquad (3.1.33)$$

3.1.7 Problems

1. A motion is given by the equations

$$x_1 = X_1 + 3X_2 t, \quad x_2 = -X_1 t^2 + X_2(t+1), \quad x_3 = X_3$$

   (a) Calculate the spatial mass density ρ in terms of the density ρ_0

   (b) Derive a first order ordinary differential equation for the density ρ (in terms of **x** and *t* only) assuming that it is independent of position **x**

3.2 The Momentum Principles

In Parts I and II, the basic dynamics principles used were Newton's Laws, and these are equivalent to force equilibrium and moment equilibrium. For example, they were used to derive the stress transformation equations in Part I, §3.4 and the Equations of Motion in Part II, §1.1. Newton's laws there were applied to differential material elements.

An alternative but completely equivalent set of dynamics laws are **Euler's Laws**; these are more appropriate for finite-sized collections of moving particles, and can be used to express the force and moment equilibrium in terms of integrals. Euler's Laws are also called the **Momentum Principles**: the **principle of linear momentum** (Euler's first law) and the **principle of angular momentum** (Euler's second law).

3.2.1 The Principle of Linear Momentum

Momentum is a measure of the tendency of an object to keep moving once it is set in motion. Consider first the particle of rigid body dynamics: the (linear) momentum $\mathbf{p}$ is defined to be its mass times velocity, $\mathbf{p} = m\mathbf{v}$. The rate of change of momentum $\dot{\mathbf{p}}$ is

$$\frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{3.2.1}$$

and use has been made of the fact that $dm/dt = 0$. Thus Newton's second law, $\mathbf{F} = m\mathbf{a}$, can be rewritten as

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{3.2.2}$$

This equation, formulated by Euler, states that *the rate of change of momentum is equal to the applied force*. It is called the **principle of linear momentum**, or **balance of linear momentum**. If there are no forces applied to a system, the total momentum of the system remains constant; the law in this case is known as the **law of conservation of (linear) momentum**.

Eqn. 3.2.2 as applied to a particle can be generalized to the mechanics of a continuum in one of two ways. One could consider a differential element of material, of mass dm and velocity $\mathbf{v}$. Alternatively, one can consider a finite portion of material, a control mass in the current configuration with spatial mass density $\rho(\mathbf{x},t)$ and spatial velocity field $\mathbf{v}(\mathbf{x},t)$. The total linear momentum of this mass of material is

$$\boxed{\mathbf{L}(t) = \int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv} \quad \textbf{Linear Momentum} \tag{3.2.3}$$

The principle of linear momentum states that

$$\dot{\mathbf{L}}(t) = \frac{d}{dt}\int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv = \mathbf{F}(t) \tag{3.2.4}$$

where $\mathbf{F}(t)$ is the resultant of the forces acting on the portion of material.

Note that the volume over which the integration in Eqn. 3.2.4 takes place is not fixed; the integral is taken over a *fixed portion of material particles*, and the space occupied by this matter may change over time.

By virtue of the Transport theorem relation 3.1.31, this can be written as

$$\dot{\mathbf{L}}(t) = \int_v \rho(\mathbf{x},t)\dot{\mathbf{v}}(\mathbf{x},t)dv = \mathbf{F}(t) \tag{3.2.5}$$

The resultant force acting on a body is due to the surface tractions **t** acting over surface elements and body forces **b** acting on volume elements, Fig. 3.2.1:

$$\boxed{\mathbf{F}(t) = \int_s \mathbf{t}ds + \int_v \mathbf{b}dv, \qquad F_i = \int_s t_i\, ds + \int_v b_i\, dv} \qquad \textbf{Resultant Force} \tag{3.2.6}$$

and so the principle of linear momentum can be expressed as

$$\boxed{\int_s \mathbf{t}ds + \int_v \mathbf{b}dv = \int_v \rho\dot{\mathbf{v}}dv} \qquad \textbf{Principle of Linear Momentum} \tag{3.2.7}$$

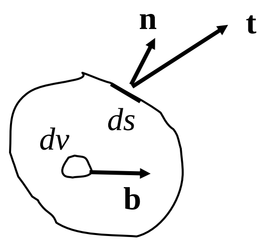

Figure 3.2.1: surface and body forces acting on a finite volume of material

The principle of linear momentum, Eqns. 3.2.7, will be used to prove Cauchy's Lemma and Cauchy's Law in the next section and, in §3.6, to derive the Equations of Motion.

3.2.2 The Principle of Angular Momentum

Considering again the mechanics of a single particle: the **angular momentum** is the moment of momentum about an axis, in other words, it is the product of the linear momentum of the particle and the perpendicular distance from the axis of its line of action. In the notation of Fig. 3.2.2, the angular momentum **h** is

$$\mathbf{h} = \mathbf{r} \times m\mathbf{v} \tag{3.2.8}$$

which is the vector with magnitude $d \times m|\mathbf{v}|$ and perpendicular to the plane shown.

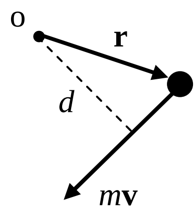

Figure 3.2.2: surface and body forces acting on a finite volume of material

Consider now a collection of particles. The **principle of angular momentum** states that the resultant moment of the external forces acting on the system of particles, $\mathbf{M}$, equals the rate of change of the total angular momentum of the particles:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt} \tag{3.2.9}$$

Generalising to a continuum, the angular momentum is

$$\boxed{\mathbf{H} = \int_v \mathbf{r} \times \rho \mathbf{v} dv} \quad \textbf{Angular Momentum} \tag{3.2.10}$$

and the principle of angular momentum is

$$\boxed{\begin{gathered}\int_s \mathbf{r} \times \mathbf{t}^{(\mathbf{n})} ds + \int_v \mathbf{r} \times \mathbf{b} dv = \frac{d}{dt} \int_v \mathbf{r} \times \rho \mathbf{v} dv \\ \int_s \varepsilon_{ijk} x_j t_k^{(\mathbf{n})} ds + \int_v \varepsilon_{ijk} x_j b_k \, dv = \frac{d}{dt} \int_V \varepsilon_{ijk} x_j \rho v_k \, dv\end{gathered}} \quad \textbf{Principle of Angular Momentum}$$

(3.2.11)

The principle of angular momentum, 3.2.11, will be used, in §3.6, to deduce the symmetry of the Cauchy stress.

3.3 The Cauchy Stress Tensor

3.3.1 The Traction Vector

The **traction vector** was introduced in Part I, §3.3. To recall, it is the limiting value of the ratio of force over area; for Force ΔF acting on a surface element of area ΔS, it is

$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \tag{3.3.1}$$

and **n** denotes the normal to the surface element. An infinite number of traction vectors act at a point, each acting on different surfaces through the point, defined by different normals.

3.3.2 Cauchy's Lemma

Cauchy's lemma states that traction vectors acting on opposite sides of a surface are equal and opposite[1]. This can be expressed in vector form:

$$\boxed{\mathbf{t}^{(\mathbf{n})} = -\mathbf{t}^{(-\mathbf{n})}} \quad \textbf{Cauchy's Lemma} \tag{3.3.2}$$

This can be proved by applying the principle of linear momentum to a collection of particles of mass Δm instantaneously occupying a small box with parallel surfaces of area Δs, thickness δ and volume $\Delta v = \delta \Delta s$, Fig. 3.3.1. The resultant *surface* force acting on this matter is $\mathbf{t}^{(\mathbf{n})}\Delta s + \mathbf{t}^{(-\mathbf{n})}\Delta s$.

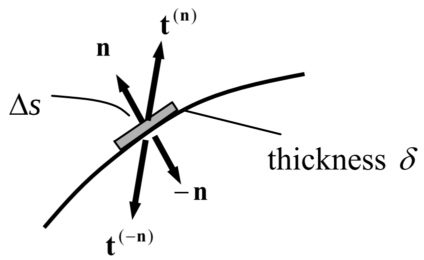

Figure 3.3.1: traction acting on a small portion of material particles

The total linear momentum of the matter is $\int_{\Delta V} \rho \mathbf{v} dv = \int_{\Delta m} \mathbf{v} dm$. By the mean value theorem (see Appendix A to Chapter 1, §1.B.1), this equals $\overline{\mathbf{v}}\Delta m$, where $\overline{\mathbf{v}}$ is the velocity at some interior point. Similarly, the body force acting on the matter is $\int_{\Delta V} \mathbf{b} dv = \overline{\mathbf{b}}\Delta v$, where $\overline{\mathbf{b}}$ is the body force (per unit volume) acting at some interior point. The total mass

[1] this is equivalent to Newton's (third) law of action and reaction – it seems like a lot of work to prove this seemingly obvious result but, to be consistent, it is supposed that the only fundamental dynamic laws available here are the principles of linear and angular momentum, and not any of Newton's laws

can also be written as $\Delta m = \int_{\Delta V} \rho dv = \bar{\rho}\Delta v$. From the principle of linear momentum, Eqn. 3.2.7, and since Δm does not change with time,

$$\mathbf{t}^{(\mathbf{n})}\Delta s + \mathbf{t}^{(-\mathbf{n})}\Delta s + \overline{\mathbf{b}}\Delta v = \frac{d}{dt}\left[\overline{\mathbf{v}}\Delta m\right] = \Delta m \frac{d\overline{\mathbf{v}}}{dt} = \bar{\rho}\Delta v \frac{d\overline{\mathbf{v}}}{dt} = \bar{\rho}\delta\Delta s \frac{d\overline{\mathbf{v}}}{dt} \tag{3.3.3}$$

Dividing through by Δs and taking the limit as $\delta \to 0$, one finds that $\mathbf{t}^{(\mathbf{n})} = -\mathbf{t}^{(-\mathbf{n})}$. Note that the values of $\mathbf{t}^{(\mathbf{n})}$, $\mathbf{t}^{(-\mathbf{n})}$ acting on the box with finite thickness are not the same as the final values, but approach the final values *at* the surface as $\delta \to 0$.

3.3.3 Stress

In Part I, the components of the traction vector were called stress components, and it was illustrated how there were nine stress components associated with each material particle. Here, the stress is defined more formally,

Cauchy's Law

Cauchy's Law states that there exists a **Cauchy stress tensor** $\boldsymbol{\sigma}$ which maps the normal to a surface to the traction vector acting on that surface, according to

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \qquad t_i = \sigma_{ij} n_j} \qquad \textbf{Cauchy's Law} \tag{3.3.4}$$

or, in full,

$$\begin{aligned} t_1 &= \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \\ t_2 &= \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 \\ t_3 &= \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{aligned} \tag{3.3.5}$$

Note:
- many authors define the stress tensor as $\mathbf{t} = \mathbf{n}\boldsymbol{\sigma}$. This amounts to the definition used here since, as mentioned in Part I, and as will be (re-)proved below, the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$, $\sigma_{ij} = \sigma_{ji}$
- the Cauchy stress refers to the *current* configuration, that is, it is a measure of force per unit area acting on a surface in the current configuration.

Stress Components

Taking Cauchy's law to be true (it is proved below), the components of the stress tensor with respect to a Cartesian coordinate system are, from 1.9.4 and 3.3.4,

$$\sigma_{ij} = \mathbf{e}_i \boldsymbol{\sigma} \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{t}^{(\mathbf{e}_j)} \tag{3.3.6}$$

which is the *i*th component of the traction vector acting on a surface with normal $\mathbf{e}_j$. Note that this definition is inconsistent with that given in Part I, §3.2 – there, the first subscript denoted the direction of the normal – but, again, the two definitions are equivalent because of the symmetry of the stress tensor.

The three traction vectors acting on the surface elements whose outward normals point in the directions of the three base vectors $\mathbf{e}_j$ are

$$\mathbf{t}^{(\mathbf{e}_j)} = \boldsymbol{\sigma}\mathbf{e}_j, \qquad \begin{aligned} \mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \end{aligned} \tag{3.3.7}$$

Eqns. 3.3.6-7 are illustrated in Fig. 3.3.2.

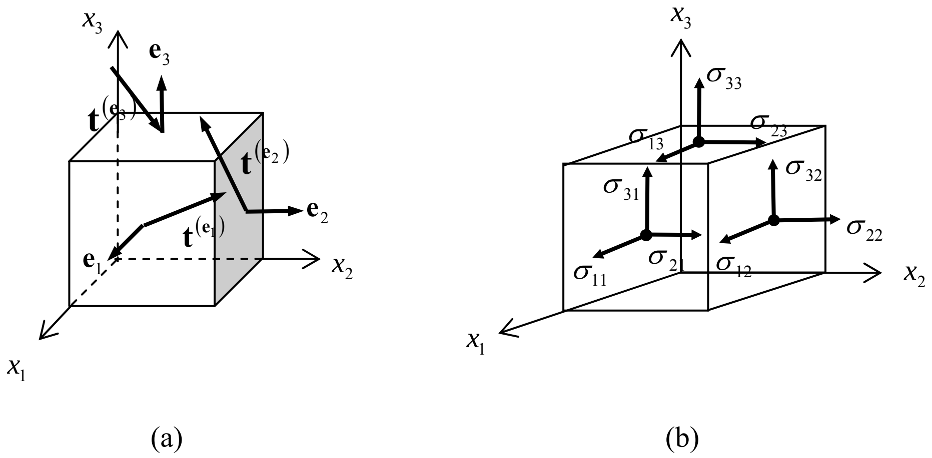

Figure 3.3.2: traction acting on surfaces with normals in the coordinate directions; (a) traction vectors, (b) stress components

Proof of Cauchy's Law

The proof of Cauchy's law essentially follows the same method as used in the proof of Cauchy's lemma.

Consider a small tetrahedral free-body, with vertex at the origin, Fig. 3.3.3. It is required to determine the traction $\mathbf{t}$ in terms of the nine stress components (which are all shown positive in the diagram).

Let the area of the base of the tetrahedran, with normal $\mathbf{n}$, be Δs. The area ds_1 is then $\Delta s \cos\alpha$, where α is the angle between the planes, as shown in Fig. 3.3.3b; this angle is the same as that between the vectors $\mathbf{n}$ and $\mathbf{e}_1$, so $\Delta s_1 = (\mathbf{n}\cdot\mathbf{e}_1)\Delta s = n_1 \Delta s$, and similarly for the other surfaces: $\Delta s_2 = n_2 \Delta s$ and $\Delta s_3 = n_3 \Delta s$.

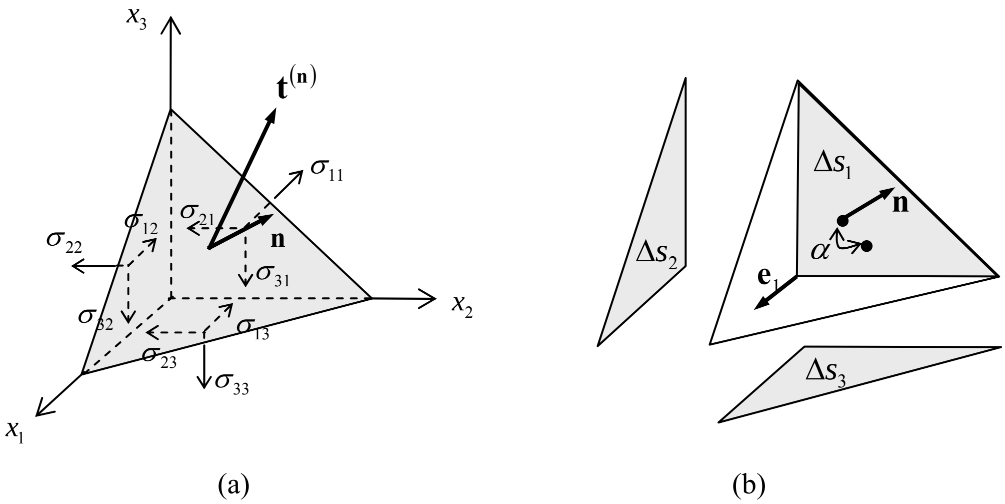

Figure 3.3.3: free body diagram of a tetrahedral portion of material; (a) traction acting on the material, (b) relationship between surface areas and normal components

The resultant surface force on the body, acting in the x_1 direction, is

$$t_1\Delta s - \sigma_{11}n_1\Delta s - \sigma_{12}n_2\Delta s - \sigma_{13}n_3\Delta s$$

Again, the momentum is $\overline{\mathbf{v}}\Delta M$, the body force is $\overline{\mathbf{b}}\Delta v$ and the mass is $\Delta m = \overline{\rho}\Delta v = \overline{\rho}(h/3)\Delta s$, where h is the perpendicular distance from the origin (vertex) to the base. The principle of linear momentum then states that

$$t_1\Delta s - \sigma_{11}n_1\Delta s - \sigma_{12}n_2\Delta s - \sigma_{13}n_3\Delta s + \overline{b}_1(h/3)\Delta s = \overline{\rho}(h/3)\Delta s\frac{d\overline{v}_1}{dt}$$

Again, the values of the traction and stress components on the faces will in general vary over the faces, so the values used in this equation are average values over the faces.

Dividing through by Δs, and taking the limit as $h \to 0$, one finds that

$$t_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3$$

and now these quantities, t_1, σ_{11}, σ_{12}, σ_{13}, are the values *at* the origin. The equations for the other two traction components can be derived in a similar way.

Normal and Shear Stress

The stress acting normal to a surface is given by

$$\sigma_N = \mathbf{n}\cdot\mathbf{t}^{(\mathbf{n})} \tag{3.3.8}$$

The shear stress acting on the surface can then be obtained from

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} \tag{3.3.9}$$

Example

The state of stress at a point is given in the matrix form

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine
(a) the traction vector acting on a plane through the point whose unit normal is $\hat{\mathbf{n}} = (1/3)\hat{\mathbf{e}}_1 + (2/3)\hat{\mathbf{e}}_2 - (2/3)\hat{\mathbf{e}}_3$
(b) the component of this traction acting perpendicular to the plane
(c) the shear component of traction.

<u>Solution</u>

(a) The traction is

$$\begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

or $\mathbf{t}^{(\hat{\mathbf{n}})} = (-2/3)\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$.

(b) The component normal to the plane is the projection of $\mathbf{t}^{(\hat{\mathbf{n}})}$ in the direction of $\hat{\mathbf{n}}$, i.e.

$$\sigma_N = \mathbf{t}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

(c) The shearing component of traction is

$$\begin{aligned} \sigma_S &= \mathbf{t}^{(\hat{\mathbf{n}})} - (22/9)\hat{\mathbf{n}} \\ &= \left[[(-2/3) - (22/27)\right]\hat{\mathbf{e}}_1 + \left[3 - (44/27)\right]\hat{\mathbf{e}}_2 + \left[-1 + (44/27)\right]\hat{\mathbf{e}}_3 \right] \\ &= \left[(-40/27)\hat{\mathbf{e}}_1 + (37/27)\hat{\mathbf{e}}_2 + (17/27)\hat{\mathbf{e}}_3\right] \end{aligned}$$

i.e. of magnitude $\sqrt{(-40/27)^2 + (37/27)^2 + (17/27)^2} \approx 2.1$, which equals $\sqrt{\left|\mathbf{t}^{(\hat{\mathbf{n}})}\right|^2 - \sigma_N^2}$.

■

3.4 Properties of the Stress Tensor

3.4.1 Stress Transformation

Let the components of the Cauchy stress tensor in a coordinate system with base vectors $\mathbf{e}_i$ be σ_{ij}. The components in a second coordinate system with base vectors $\mathbf{e}'_j$, σ'_{ij}, are given by the tensor transformation rule 1.10.5:

$$\sigma'_{ij} = Q_{pi} Q_{qj} \sigma_{pq} \tag{3.4.1}$$

where Q_{ij} are the direction cosines, $Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$.

Isotropic State of Stress

Suppose the state of stress in a body is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$$

One finds that the application of the tensor transformation rule yields the very same components no matter what the coordinate system. This is termed an **isotropic** state of stress, or a **spherical** state of stress (see §1.13.3). One example of isotropic stress is the stress arising in fluid at rest, which cannot support shear stress, in which case

$$\boldsymbol{\sigma} = -p\mathbf{I} \tag{3.4.2}$$

where the scalar p is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic** state of stress.

A note on the Transformation Formula

Using the vector transformation rule 1.5.5, the traction and normal transform according to $[\mathbf{t}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{t}]$, $[\mathbf{n}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{n}]$. Also, Cauchy's law transforms according to $[\mathbf{t}'] = [\boldsymbol{\sigma}'][\mathbf{n}']$ which can be written as $[\mathbf{Q}^{\mathrm{T}}][\mathbf{t}] = [\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{n}]$, so that, pre-multiplying by $[\mathbf{Q}]$, and since $[\mathbf{Q}]$ is orthogonal, $[\mathbf{t}] = \{[\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}]\}[\mathbf{n}]$, so $[\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}]$, which is the inverse tensor transformation rule 1.13.6a, showing the internal consistency of the theory.

In Part I, Newton's law was applied to a material element to derive the two-dimensional stress transformation equations, Eqn. 3.4.7 of Part I. Cauchy's law was proved in a similar way, using the principle of momentum. In fact, Cauchy's law and the stress transformation equations are equivalent. Given the stress components in one coordinate system, the stress transformation equations give the components in a new coordinate system; particularising this, they give the stress components, and thus the traction vector, acting on new surfaces, oriented in some way with respect to the original axes, which is what Cauchy's law does.

3.4.2 Principal Stresses

Since the stress $\boldsymbol{\sigma}$ is a symmetric tensor, it has three real eigenvalues $\sigma_1, \sigma_2, \sigma_3$, called **principal stresses**, and three corresponding orthonormal eigenvectors called **principal directions**. The eigenvalue problem can be written as

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n} \tag{3.4.3}$$

where $\mathbf{n}$ is a principal direction and σ is a scalar principal stress. Since the traction vector is a multiple of the unit normal, σ is a normal stress component. Thus a principal stress is a stress which acts on a plane of zero shear stress, Fig. 3.4.1.

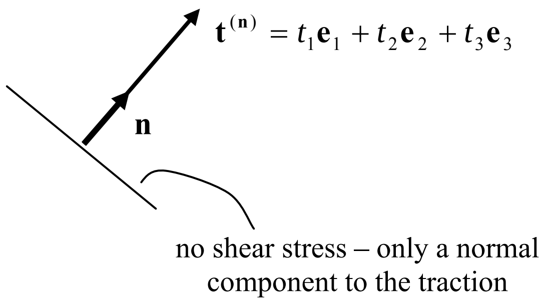

Figure 3.4.1: traction acting on a plane of zero shear stress

The principal stresses are the roots of the characteristic equation 1.11.5,

$$\sigma^3 - \mathbf{I}_1\sigma^2 + \mathbf{I}_2\sigma - \mathbf{I}_3 = 0 \tag{3.4.4}$$

where, Eqn. 1.11.6-7, 1.11.17,

$$\begin{aligned}
I_1 &= \mathrm{tr}\boldsymbol{\sigma} \\
&= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \tfrac{1}{2}\left[(\mathrm{tr}\boldsymbol{\sigma})^2 - \mathrm{tr}\boldsymbol{\sigma}^2\right] \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
&= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\
I_3 &= \tfrac{1}{3}\left[\mathrm{tr}\boldsymbol{\sigma}^3 - \tfrac{3}{2}\mathrm{tr}\boldsymbol{\sigma}\mathrm{tr}\boldsymbol{\sigma}^2 + \tfrac{1}{2}(\mathrm{tr}\boldsymbol{\sigma})^3\right] \\
&= \det\boldsymbol{\sigma} \\
&= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{32} \\
&= \sigma_1\sigma_2\sigma_3
\end{aligned} \tag{3.4.5}$$

The principal stresses and principal directions are properties of the stress tensor, and do not depend on the particular axes chosen to describe the state of stress., and the **stress invariants** I_1, I_2, I_3 are invariant under coordinate transformation. *c.f.* §1.11.1.

If one chooses a coordinate system to coincide with the three eigenvectors, one has the spectral decomposition 1.11.11 and the stress matrix takes the simple form 1.11.12,

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{3.4.6}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions.

3.4.3 Maximum Stresses

Directly from §1.11.3, the three principal stresses include the maximum and minimum normal stress components acting at a point. This result is re-derived here, together with results for the maximum shear stress

Normal Stresses

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions* and consider an arbitrary unit normal vector $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, Fig. 3.4.2. From 3.3.8 and Cauchy's law, the normal stress acting on the plane with normal **n** is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \tag{3.4.7}$$

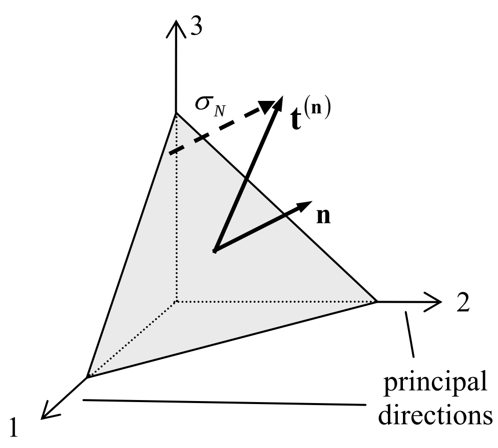

Figure 3.4.2: normal stress acting on a plane defined by the unit normal n

With respect to the principal stresses, using 3.4.6,

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n} = \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \tag{3.4.8}$$

and the normal stress is

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{3.4.9}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1\left(n_1^2 + n_2^2 + n_3^2\right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \tag{3.4.10}$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3\left(n_1^2 + n_2^2 + n_3^2\right) \geq \sigma_3 \tag{3.4.11}$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45° to the principal planes and that they have magnitude equal to half the difference between the principal stresses.

From 3.3.39, 3.4.8 and 3.4.9, the shear stress on the plane is

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \tag{3.4.12}$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right)n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right)n_2^2 + \sigma_3^2 - \left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2 + \sigma_3\right]^2 \tag{3.4.13}$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned}
\frac{\partial\left(\sigma_S^2\right)}{\partial n_1} &= n_1\left(\sigma_1 - \sigma_3\right)\left\{\sigma_1 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2\right]\right\} = 0 \\
\frac{\partial\left(\sigma_S^2\right)}{\partial n_2} &= n_2\left(\sigma_2 - \sigma_3\right)\left\{\sigma_2 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2\right]\right\} = 0
\end{aligned} \tag{3.4.14}$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 3.4.14 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned}
\mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}|\sigma_2 - \sigma_3| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}|\sigma_3 - \sigma_1| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}|\sigma_1 - \sigma_2|
\end{aligned} \tag{3.4.15}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{3.4.16}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 3.4.3.

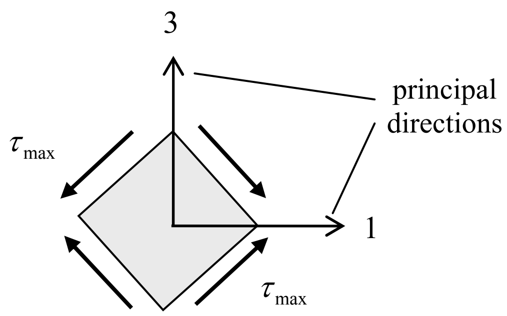

Figure 3.4.3: maximum shear stress at apoint

Example (maximum shear stress)

Consider the stress state

$$\left[\sigma_{ij}\right]=\begin{bmatrix}5 & 0 & 0\\ 0 & -6 & -12\\ 0 & -12 & 1\end{bmatrix}$$

This is the same tensor considered in the example of §1.11.1. Using the results of that example, the principal stresses are $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress at that point is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{25}{2}$$

The planes and direction upon which they act are shown in Fig. 3.4.4.

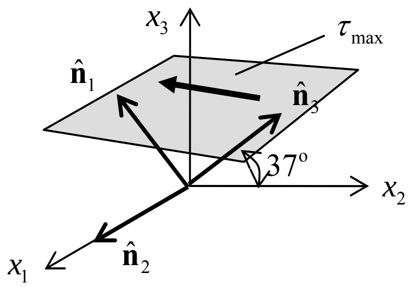

Figure 3.4.4: maximum shear stress

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3.6 The Equations of Motion and Symmetry of Stress

In Part II, §1.1, the Equations of Motion were derived using Newton's Law applied to a differential material element. Here, they are derived using the principle of linear momentum.

3.6.1 The Equations of Motion (Spatial Form)

Application of Cauchy's law $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ and the divergence theorem 1.14.21 to 3.2.7 leads directly to the global form of the equations of motion

$$\int_v [\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}]dv = \int_v \rho\,\dot{\mathbf{v}}\,dv, \qquad \int_v \left[\frac{\partial \sigma_{ij}}{\partial x_j} + b_i\right]dv = \int_v \rho\,\dot{v}_i\,dv \tag{3.6.1}$$

The corresponding local form is then

$$\boxed{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\frac{d\mathbf{v}}{dt}, \qquad \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho\frac{dv_i}{dt}} \quad \textbf{Equations of Motion} \tag{3.6.2}$$

The term on the right is called the inertial, or kinetic, term, representing the change in momentum. The material time derivative of the spatial velocity field is

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{v} \quad \text{so} \quad \frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \left(\frac{\partial v_1}{\partial x_1}v_1 + \frac{\partial v_1}{\partial x_2}v_2 + \frac{\partial v_1}{\partial x_3}v_3\right), \text{ etc.}$$

and it can be seen that the equations of motion are non-linear in the velocities.

Equations of Equilibrium

When the acceleration is zero, the equations reduce to the equations of equilibrium,

$$\boxed{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = 0} \quad \textbf{Equations of Equilibrium} \tag{3.6.3}$$

Flows

A **flow** is a set of quantities associated with the system of forces **t** and **b**, for example the quantities $\mathbf{v}, \boldsymbol{\sigma}, \rho$. A flow is **steady** if the associated spatial quantities are independent of time. A **potential flow** is one for which the velocity field can be written as the gradient of a scalar function, $\mathbf{v} = \operatorname{grad}\phi$. An **irrotational flow** is one for which $\operatorname{curl}\mathbf{v} = 0$.

3.6.2 The Equations of Motion (Material Form)

In the spatial form, the linear momentum of a mass element is $\rho \mathbf{v} dv$. In the material form it is $\rho_0 \mathbf{V} dV$. Here, **V** is the same velocity as **v**, only it is now expressed in terms of the material coordinates **X**, and $\rho dv = \rho_0 dV$. The linear momentum of a collection of material particles occupying the volume *v* in the current configuration can thus be expressed in terms of an integral over the corresponding volume *V* in the reference configuration:

$$\boxed{\mathbf{L}(t) = \int_V \rho_0(\mathbf{X}) \mathbf{V}(\mathbf{X}, t) dV} \qquad \textbf{Linear Momentum (Material Form)} \qquad (3.6.4)$$

and the principle of linear momentum is now, using 3.1.31,

$$\frac{d}{dt} \int_V \rho_0(\mathbf{X}) \mathbf{V}(\mathbf{X}, t) dV = \int_V \rho_0 \frac{d\mathbf{V}}{dt} dV \equiv \mathbf{F}(t) \qquad (3.6.5)$$

The external forces **F** to be considered are those acting on the *current* configuration. Suppose that the surface force acting on a surface element ds in the current configuration is $d\mathbf{f}_{\text{surf}} = \mathbf{t} ds = \mathbf{T} dS$, where **t** and **T** are, respectively, the Cauchy traction vector and the PK1 traction vector (Eqns. 3.5.3-4). Also, just as the PK1 stress measures the actual force in the current configuration, but per unit surface area in the reference configuration, one can introduce the **reference body force B**: this is the actual body force acting in the current configuration, per unit volume in the reference configuration. Thus if the body force acting on a volume element dv in the current configuration is $d\mathbf{f}_{\text{body}}$, then

$$d\mathbf{f}_{\text{body}} = \mathbf{b} dv = \mathbf{B} dV \qquad (3.6.6)$$

The resultant force acting on the body is then

$$\mathbf{F}(t) = \int_S \mathbf{T} dS + \int_V \mathbf{B} dV, \qquad F_i = \int_S T_i dS + \int_V B_i dV \qquad (3.6.7)$$

Using Cauchy's law, $\mathbf{T} = \mathbf{PN}$, where **P** is the PK1 stress, and the divergence theorem 1.12.21, 3.6.5 and 3.6.7 lead to

$$\int_V [\text{Div}\mathbf{P} + \mathbf{B}] dV = \int_V \rho_0 \frac{d\mathbf{V}}{dt} dV \qquad (3.6.8)$$

and the corresponding local form is

$$\boxed{\text{Div}\,\mathbf{P} + \mathbf{B} = \rho_0 \frac{d\mathbf{V}}{dt}, \qquad \frac{\partial P_{ij}}{\partial X_j} + B_i = \rho_0 \frac{dV_i}{dt}}$$

$$\textbf{Equations of Motion (Material Form)} \qquad (3.6.9)$$

Derivation from the Spatial Form

The equations of motion can also be derived directly from the spatial equations. In order to do this, one must first show that $\mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right)$ is zero. One finds that (using the divergence theorem, Nanson's formula 2.2.59 and the fact that $\mathrm{div}\mathbf{I} = 0$)

$$\begin{aligned}
&\int_V \mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right) dV = \int_S J\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS = \int_s \mathbf{n}ds = \int_s \mathbf{In}ds = \int_v \mathrm{div}\mathbf{I}\,dv = 0 \\
&\int_V \frac{\partial\left(JF_{ji}^{-1}\right)}{\partial X_j} dV = \int_S JF_{ji}^{-1}N_i dS = \int_s n_i ds = \int_s \delta_{ij} n_j ds = \int_v \frac{\partial\delta_{ij}}{\partial x_i} dv = 0
\end{aligned} \qquad (3.6.10)$$

This result is known as the **Piola identity**. Thus, with the PK1 stress related to the Cauchy stress through 3.5.8, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, and using identity 1.14.16c,

$$\begin{aligned}
\mathrm{Div}\mathbf{P} &= \mathrm{Div}\left(\boldsymbol{\sigma}\left(J\mathbf{F}^{-\mathrm{T}}\right)\right) \\
&= \boldsymbol{\sigma}\mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right) + \mathrm{Grad}\boldsymbol{\sigma} : \left(J\mathbf{F}^{-\mathrm{T}}\right) \\
&= J\mathrm{Grad}\boldsymbol{\sigma} : \mathbf{F}^{-\mathrm{T}}
\end{aligned} \qquad (3.6.11)$$

From 2.2.8c,

$$\mathrm{Div}\mathbf{P} = J\mathrm{div}\boldsymbol{\sigma} \qquad (3.6.12)$$

Then, with $dv = JdV$ and 3.6.6, the equations of motion in the spatial form can now be transformed according to

$$\int_v \left[\mathrm{div}\boldsymbol{\sigma} + \mathbf{b}\right] dv = \int_v \rho\,\dot{\mathbf{v}}\,dv \quad \to \quad \int_V \left[\mathrm{Div}\mathbf{P} + \mathbf{B}\right] dV = \int_V \rho_0 \dot{\mathbf{V}}\,dV$$

as before.

3.6.3 Symmetry of the Cauchy Stress

It will now be shown that the principle of angular momentum leads to the requirement that the Cauchy stress tensor is symmetric. Applying Cauchy's law to 3.2.11,

$$\begin{aligned}
&\int_s \mathbf{r} \times (\boldsymbol{\sigma}\mathbf{n}) ds + \int_v \mathbf{r} \times \mathbf{b} dv = \frac{d}{dt}\int_v \mathbf{r} \times \rho\mathbf{v} dv \\
&\int_s \varepsilon_{ijk} x_j \sigma_{kl} n_l dS + \int_v \varepsilon_{ijk} x_j b_k\, dv = \frac{d}{dt}\int_v \varepsilon_{ijk} x_j \rho v_k\, dv
\end{aligned} \qquad (3.6.13)$$

The surface integral can be converted into a volume integral using the divergence theorem. Using the index notation, and concentrating on the integrand of the resulting volume integral, one has, using 1.3.14 (the permutation symbol is a constant here, $\partial\varepsilon_{ijk} / \partial x_l = 0$),

$$\varepsilon_{ijk}\frac{\partial(x_j\sigma_{kl})}{\partial x_l}=\varepsilon_{ijk}\left\{x_j\frac{\partial\sigma_{kl}}{\partial x_l}+\sigma_{kl}\delta_{jl}\right\}=\varepsilon_{ijk}\left\{x_j\frac{\partial\sigma_{kl}}{\partial x_l}+\sigma_{kj}\right\}\equiv\mathbf{r}\times\operatorname{div}\boldsymbol{\sigma}+\mathsf{E}:\boldsymbol{\sigma}^{\mathrm{T}}\qquad(3.6.14)$$

where E is the third-order permutation tensor, Eqn. 1.9.6, $\mathsf{E}=\varepsilon_{ijk}(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k)$. Thus, with the Reynold's transport identity 3.1.31,

$$\int_v\{\mathbf{r}\times\operatorname{div}\boldsymbol{\sigma}+\mathsf{E}:\boldsymbol{\sigma}^{\mathrm{T}}\}dv+\int_v\mathbf{r}\times\mathbf{b}dv=\int_v\rho\frac{d}{dt}(\mathbf{r}\times\mathbf{v})dv\qquad(3.6.15)$$

The material derivative of this cross product is

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v})=\mathbf{r}\times\frac{d\mathbf{v}}{dt}+\frac{d\mathbf{r}}{dt}\times\mathbf{v}=\mathbf{r}\times\frac{d\mathbf{v}}{dt}+\mathbf{v}\times\mathbf{v}=\mathbf{r}\times\frac{d\mathbf{v}}{dt}\qquad(3.6.16)$$

and so

$$\int_v\mathsf{E}:\boldsymbol{\sigma}^{\mathrm{T}}dv+\int_v\mathbf{r}\times\left\{\operatorname{div}\boldsymbol{\sigma}+\mathbf{b}-\rho\frac{d\mathbf{v}}{dt}\right\}dv=0\qquad(3.6.17)$$

From the equations of motion 2.6.2, the term inside the brackets is zero, so that

$$\mathsf{E}:\boldsymbol{\sigma}^{\mathrm{T}}=0,\qquad\varepsilon_{ijk}\sigma_{kj}=0\qquad(3.6.18)$$

It follows, from expansion of this relation, that the matrix of stress components must be symmetric:

$$\boxed{\boldsymbol{\sigma}=\boldsymbol{\sigma}^{\mathrm{T}},\qquad\sigma_{ij}=\sigma_{ji}}\qquad\textbf{Symmetry of Stress}\qquad(3.6.19)$$

3.6.4 Consequences in the Material Form

Here, the consequences of 3.6.19 on the PK1 and PK2 stresses is examined. Using the result $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{\mathrm{T}}$ and 3.5.8, $\boldsymbol{\sigma}=J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}$,

$$J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}=\left(J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}\right)^{\mathrm{T}}=J^{-1}\mathbf{F}\mathbf{P}^{\mathrm{T}}\qquad(3.6.20)$$

so that

$$\mathbf{P}\mathbf{F}^{\mathrm{T}}=\mathbf{F}\mathbf{P}^{\mathrm{T}},\qquad P_{ik}F_{jk}=F_{ik}P_{jk}\qquad(3.6.21)$$

These equations are trivial when $i=j$, not providing any constraint on $\mathbf{P}$. On the other hand, when $i\neq j$ one has the three equations

$$\begin{aligned}
P_{11}F_{21} + P_{12}F_{22} + P_{13}F_{23} &= F_{11}P_{21} + F_{12}P_{22} + F_{13}P_{23} \\
P_{11}F_{31} + P_{12}F_{32} + P_{13}F_{33} &= F_{11}P_{31} + F_{12}P_{32} + F_{13}P_{33} \\
P_{21}F_{31} + P_{22}F_{32} + P_{23}F_{33} &= F_{21}P_{31} + F_{22}P_{32} + F_{23}P_{33}
\end{aligned} \tag{3.6.22}$$

Thus angular momentum considerations imposes these three constraints on the PK1 stress (as they imposed the three constraints $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$ on the Cauchy stress).

It has already been seen that a consequence of the symmetry of the Cauchy stress is the symmetry of the PK2 stress **S**; thus, formally, the symmetry of **S** is the result of the angular momentum principle.

3.7 Boundary Conditions and The Boundary Value Problem

In order to solve a mechanics problem, one must specify certain conditions around the boundary of the material under consideration. Such **boundary conditions** will be discussed here, together with the resulting **boundary value problem** (**BVP**). (see Part I, 3.5.1, for a discussion of stress boundary conditions.)

3.7.1 Boundary Conditions

There are two types of boundary condition, those on displacement and those on traction. Denote the body in the reference condition by B_0 and in the current configuration by *B*. Denote the boundary of the body in the reference configuration by S and in the current configuration by *s*, Fig. 3.7.1.

Displacement Boundary Conditions

The position of particles may be specified over some portion of the boundary in the current configuration. That is, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is specified to be $\bar{\mathbf{x}}$ say, over some portion $s_\mathbf{u}$ of *s*, Fig. 3.7.1, which corresponds to the portion $S_\mathbf{u}$ of S. With $\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{X}(\mathbf{x})$, or $\mathbf{U}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}$, this can be expressed as

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \bar{\mathbf{u}}(\mathbf{x}), \quad && \mathbf{x} \in s_\mathbf{u} \\ \mathbf{U}(\mathbf{X}) &= \bar{\mathbf{U}}(\mathbf{X}), \quad && \mathbf{X} \in S_\mathbf{u} \end{aligned} \tag{3.7.1}$$

These are called **displacement boundary conditions**. The most commonly encountered displacement boundary condition is where some portion of the boundary is fixed, in which case $\bar{\mathbf{u}}(\mathbf{x}) = \mathbf{o}$.

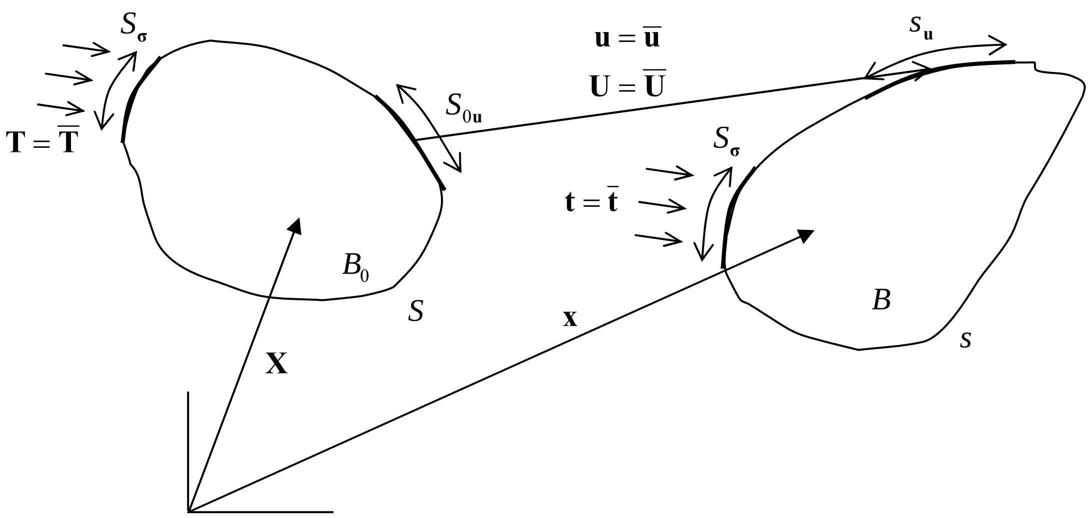

Figure 3.7.1: Boundary conditions

Traction Boundary Conditions

Traction $\mathbf{t} = \bar{\mathbf{t}}$ can be specified over a portion $s_{\boldsymbol{\sigma}}$ of the boundary, Fig. 3.7.1. These traction boundary conditions are related to the PK1 traction $\mathbf{T} = \overline{\mathbf{T}}$ over the corresponding surface $S_{\boldsymbol{\sigma}}$ in the reference configuration, through Eqns. 3.5.1-4,

$$\mathbf{T}dS = \mathbf{PN}dS = \mathbf{t}ds = \boldsymbol{\sigma}\mathbf{n}ds \tag{3.7.2}$$

One usually knows the position of the boundary S and the normal $\mathbf{N}(\mathbf{X})$ in the reference configuration. As deformation proceeds, the PK1 traction develops according to $\overline{\mathbf{T}} = \mathbf{PN}$ with, from 3.5.8, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$. The PK1 stress will in general depend on the motion $\mathbf{x}$ and the deformation gradient $\mathbf{F}$, so the traction boundary condition can be expressed in the general form

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X}, \mathbf{x}, \mathbf{F}) \tag{3.7.3}$$

Example: Fluid Pressure

Consider the case of fluid pressure p around the boundary, $\bar{\mathbf{t}} = -p\mathbf{n}$, Fig. 3.7.2. The Cauchy traction $\bar{\mathbf{t}}$ depends through the normal $\mathbf{n}$ on the new position and geometry of the surface $s_{\boldsymbol{\sigma}}$. Also, $\overline{\mathbf{T}} = -pJ\mathbf{F}^{-\mathrm{T}}\mathbf{N}$, which is of the general form 3.7.3.

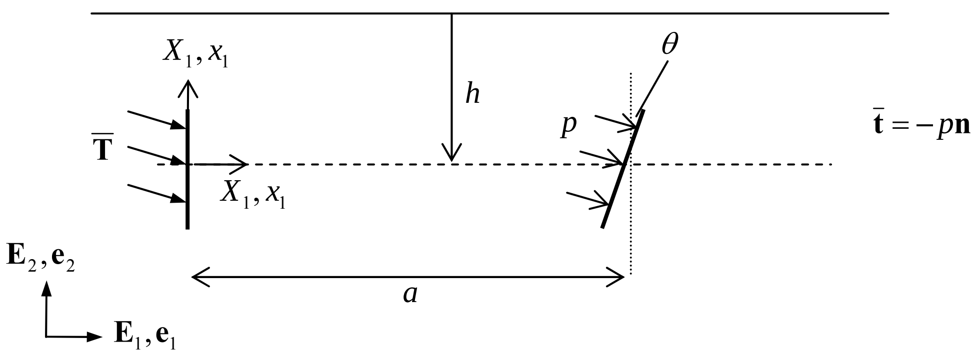

Figure 3.7.2: Fluid pressure on deforming material

Consider a material under water with part of its surface deforming as shown in Fig. 3.7.2. Referring to the figure, $\mathbf{N} = -\mathbf{E}_1$, $\mathbf{n} = -\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, $\boldsymbol{\sigma} = -p\mathbf{I}$, $p = \rho g(h - x_2)$ and

$$\begin{aligned} x_1 &= X_1 + a + X_2 \tan\theta \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned}, \quad \mathbf{F} = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \det\mathbf{F} = 1$$

The traction vectors and PK1 stress are

$$\bar{\mathbf{t}} = -\rho g(h - x_2)\begin{bmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad \overline{\mathbf{T}} = -\rho g(h - X_2)\begin{bmatrix} -1 \\ \tan\theta \\ 0 \end{bmatrix}, \quad \mathbf{P} = -\rho g(h - x_2)\begin{bmatrix} 1 & 0 & 0 \\ -\tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with (note that $dS / ds = \cos\theta$) $|\bar{\mathbf{t}}| = p$ and $|\overline{\mathbf{T}}| = p / \cos\theta$. The traction vectors clearly depend on both position, and the deformation through θ. In this example, $\operatorname{grad}\mathbf{u} = \mathbf{F} - \mathbf{I} = \operatorname{Grad}\mathbf{U} = \mathbf{I} - \mathbf{F}^{-1} = \tan\theta\mathbf{e}_1 \otimes \mathbf{e}_2$ and

$$\theta(\operatorname{grad}\mathbf{u}) = \arctan\|\operatorname{grad}\mathbf{u}\| = \arctan\sqrt{\operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u}}$$

■

Dead Loading

A special case of loading is that of **dead loading**, where

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}(\mathbf{X}) \tag{3.7.4}$$

Here, the PK1 stress on the boundary does not change with the deformation and an initially normal traction will not remain so as deformation proceeds.

For example, if one considers again the geometry of Fig. 3.7.2, this time take

$$\overline{\mathbf{T}}(\mathbf{X}) = \mathbf{P}\mathbf{N} = -p\mathbf{N} = \rho g(h - X_2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}(\mathbf{X}) = -\rho g(h - X_2)\mathbf{I}$$

Then

$$\bar{\mathbf{t}}(\mathbf{x}, \theta) = \cos\theta \rho g(h - x_2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}(\mathbf{x}, \theta) = -\rho g(h - x_2)\begin{bmatrix} 1 & 0 & 0 \\ \tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.7.2 The Boundary Value Problem

The equations of motion 3.6.2, 3.6.9, are a set of three differential equations. In the solution of any problem, one would have to supplement these equations with others, for example a constitutive equation expressing a relationship between the stress and the kinematic variables (see Part IV). This constitutive relation will typically relate the stress to the strains, or rates of strain, for example $\boldsymbol{\sigma} = f(\mathbf{e}, \mathbf{d})$. Suppose then that the stresses are known in terms of the strains and hence the displacements $\mathbf{u}$. The equations of motion are then a set of three second order differential equations in the three unknowns u_i (assuming that the body force $\mathbf{b}$ is a prescribed function of the problem). They need to be subjected to certain boundary and initial conditions.

Assume that the boundary conditions are such that the displacements are specified over that part of the surface $s_{\mathbf{u}}$ and tractions are specified over that part $s_{\boldsymbol{\sigma}}$, with the total surface $s = s_{\mathbf{u}} + s_{\boldsymbol{\sigma}}$, with $s_{\mathbf{u}} \cap s_{\boldsymbol{\sigma}} = 0$ [1]. Thus

$$\begin{aligned} \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, &\qquad \text{on } s_\sigma \\ \mathbf{u} = \bar{\mathbf{u}}, &\qquad \text{on } s_u \end{aligned} \qquad \textbf{Boundary Conditions} \qquad (3.7.5)$$

where the overbar signifies quantities which are prescribed. Initial conditions are also required for the displacement and velocity, so that

$$\begin{aligned} \mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x}), &\qquad \text{at } t = 0 \\ \dot{\mathbf{u}}(\mathbf{x},\mathbf{t}) = \dot{\mathbf{u}}_0(\mathbf{x}), &\qquad \text{at } t = 0 \end{aligned} \qquad \textbf{Initial Conditions} \qquad (3.7.6)$$

and it is usually taken that $\mathbf{x} = \mathbf{X}$ at $t = 0$. Comparing 3.7.5 and 3.7.6, one also requires that $\mathbf{u}_0 = \bar{\mathbf{u}}$, $\dot{\mathbf{u}}_0 = \dot{\bar{\mathbf{u}}}$ over $s_{\mathbf{u}}$, so that the boundary and initial conditions are compatible.

These equations together, the differential equations of motion and the boundary and initial conditions, are called the **strong form** of the initial boundary value problem (BVP):

$$\boxed{\begin{array}{ll} \multicolumn{2}{c}{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\dot{\mathbf{v}} = \rho\ddot{\mathbf{u}}} \\ \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}, & \text{on } s_\sigma \\ \mathbf{u} = \bar{\mathbf{u}}, & \text{on } s_u \\ \mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x}), & \text{at } t = 0 \\ \dot{\mathbf{u}}(\mathbf{x},\mathbf{t}) = \dot{\mathbf{u}}_0(\mathbf{x}), & \text{at } t = 0 \end{array}} \qquad \textbf{Strong form of the Initial BVP} \qquad (3.7.7)$$

When the problem is quasi-static, so the accelerations can be neglected, the equations of motion reduce to the equations of equilibrium 3.6.3. In that case one does not need initial conditions and one has a boundary value problem involving 3.7.5 only.

It is only in certain special cases and in certain simple problems that an exact solution can be obtained to these equations. An alternative solution strategy is to convert these equations into what is known as the **weak form**. The weak form, which is in the form of integrals rather than differential equations, can then be solved approximately using a numerical technique, for example the Finite Element Method[2]. The weak form is discussed in §3.9.

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[1] It is possible to specify both traction and displacement over the same portion of the boundary, but not the same components. For example, if one specified $\mathbf{t} = t_1\mathbf{e}_1$ on a boundary, one could also specify $\mathbf{u} = u_2\mathbf{e}_2$, but not $\mathbf{u} = u_1\mathbf{e}_1$. In that case, one could imagine the boundary to consist of two separate boundaries, one with conditions with respect to $\mathbf{e}_1$ and one with respect to $\mathbf{e}_2$, and still write $s_{\mathbf{u}} \cap s_{\boldsymbol{\sigma}} = 0$.

[2] Further, it is often easier to prove results regarding the uniqueness and stability of solutions to the problem when it is cast in the weak form

In the material form, the boundary conditions are

$$\begin{aligned} &\mathbf{T} = \mathbf{P}\mathbf{N} = \overline{\mathbf{T}}, \quad && \text{on } S_\sigma \\ &\mathbf{U} = \overline{\mathbf{U}}, && \text{on } S_u \end{aligned} \qquad \textbf{Boundary Conditions} \qquad (3.7.8)$$

and the initial conditions are

$$\begin{aligned} &\mathbf{U}(\mathbf{X},\mathbf{t}) = \mathbf{U}_0(\mathbf{X}), \quad && \text{at } t = 0 \\ &\dot{\mathbf{U}}(\mathbf{X},\mathbf{t}) = \dot{\mathbf{U}}_0(\mathbf{X}), && \text{at } t = 0 \end{aligned} \qquad \textbf{Initial Conditions} \qquad (3.7.9)$$

and the initial vale problem is

$$\boxed{\begin{aligned} &\mathrm{Div}\mathbf{P} + \mathbf{B} = \rho_0 \dot{\mathbf{V}} = \rho \ddot{\mathbf{U}} \\ &\mathbf{T} = \mathbf{P}\mathbf{N} = \overline{\mathbf{T}}, \quad && \text{on } S_\sigma \\ &\mathbf{U} = \overline{\mathbf{U}}, && \text{on } S_u \\ &\mathbf{U}(\mathbf{X},\mathbf{t}) = \mathbf{U}_0(\mathbf{X}), && \text{at } t = 0 \\ &\dot{\mathbf{U}}(\mathbf{X},\mathbf{t}) = \dot{\mathbf{U}}_0(\mathbf{X}), && \text{at } t = 0 \end{aligned}} \qquad \textbf{Strong form of the Initial BVP} \qquad (3.7.10)$$