Cognome:........................................... Nome:...........................................
Data orale: ☐ 22 gennaio 2015.........☐ 19 febbraio 2015

**Prof. Beghin**
**II esonero**
**A**
**14-1-2015**

**Esercizio 1**

Siano $X$ e $Y$ v.a. indipendenti con distribuzione uniforme su $(0,1)$.

i) Si calcoli la seguente probabilità

$$P\left(\frac{X}{X+Y}<t\middle| X+Y<1\right)$$

per ogni $t \in \mathbb{R}$.

ii) Calcolare la funzione di ripartizione della v.a.

$$Z=\frac{X}{X+Y}.$$

-----------------------------------

**Esercizio 2**

Siano $X_1, ..., X_n$ v.a. indipendenti. La v.a. $X_i$ ha distribuzione normale di media 0 e varianza $\sigma_i^2 = \frac{1}{2^{i+1}}$, per ogni $i = 1, 2..., n$.

i) Calcolare la funzione caratteristica della v.a. $X_n$

ii) Calcolare la funzione caratteristica della v.a.

$$Z_n = X_1 + ... + X_n$$

iii) Studiare la convergenza in distribuzione della successione $\{Z_n\}_{n\geq 1}$ per $n \to \infty$.

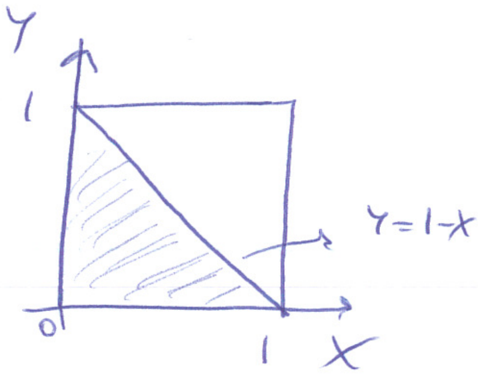
-----------------------------------

# SOLUZIONI (A)

## Es. 1

(i) $P\left( \frac{X}{X+Y} < t \,\middle|\, X+Y<1 \right) \qquad t \in \mathbb{R}$

$$= \frac{P\left( \frac{X}{X+Y} < t,\ X+Y<1 \right)}{P(X+Y<1)}$$



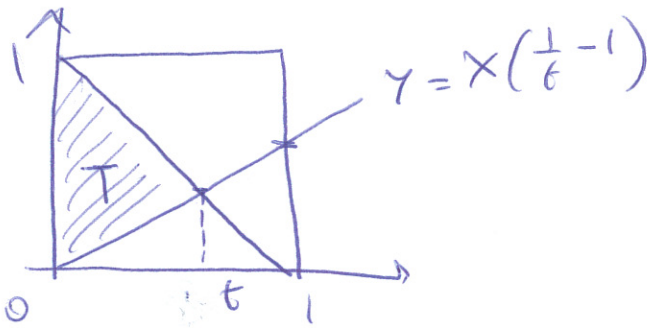

$P(X+Y<1) = P(Y<1-X) = \frac{1}{2}$

$P\left( \frac{X}{X+Y} < t,\ X+Y<1 \right) =$

per $t>0$ $\quad = P\left( X+Y > \frac{X}{t},\ X+Y<1 \right)$

$\quad = P\left( X\left(\frac{1}{t}-1\right) < Y < 1-X \right)$

per $0<t<1$




$= \text{Area}(T) = \frac{t}{2}$

$\Rightarrow P\left( \frac{X}{X+Y} < t \,\middle|\, X+Y<1 \right) = \frac{\frac{t}{2}}{\frac{1}{2}} = t$ $\qquad$ per $0<t<1$

$\Rightarrow \left( \frac{X}{X+Y} \,\middle|\, X+Y<1 \right) \sim \text{Unif}(0,1)$

per $t>1$ la prob. è 1 mentre è 0 per $t<0$

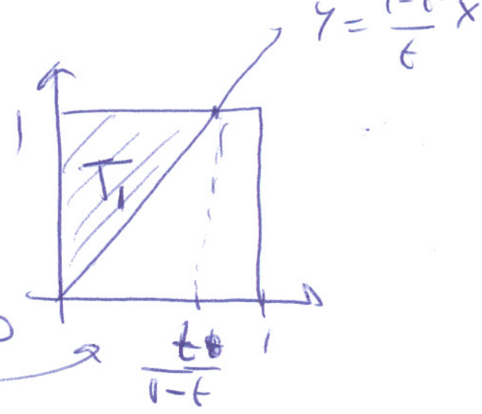
(ii) $P\left( \frac{X}{X+Y} < t \right) =$

$$= P(X < tX + tY) = P(X(1-t) < tY)$$

per $t \in (0,1)$

$$= P\left( X < \frac{t}{1-t} Y \right)$$

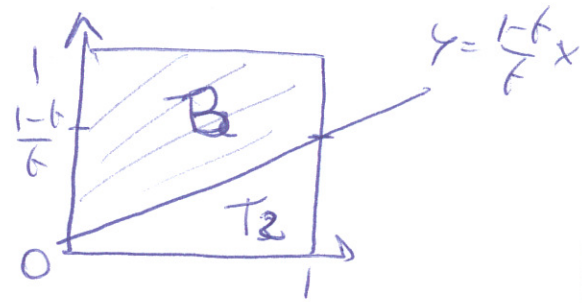
$$= P\left( Y > \frac{1-t}{t} X \right)$$




Distinguo i due casi $t \in (0, \frac{1}{2})$ e quindi $\frac{1-t}{t} > 1$

$$= \text{Area}(T_1) = \frac{t}{2(1-t)}$$

e $t \in (\frac{1}{2}, 1) \Rightarrow \frac{1-t}{t} < 1 \rightarrow$




$$= \text{Area}(B) = 1 - \text{Area}(T_2) = 1 - \frac{1-t}{2t} = \frac{3t-1}{2t}$$

$$F_T(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{2(1-t)} & 0 < t \leq \frac{1}{2} \\ \frac{3t-1}{2t} & \frac{1}{2} < t \leq 1 \\ 1 & t > 1 \end{cases}$$

## ES. 2

$X_n \sim N(0, \sigma_n^2) \qquad \sigma_n^2 = \frac{1}{2^{n+1}}$

(i) $$\mathbb{E}\, e^{itX_n} = \int_{-\infty}^{+\infty} e^{itx} \frac{e^{-\frac{x^2}{2\sigma_n^2}}}{\sqrt{2\pi\sigma_n^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma_n^2}(x^2 - 2itx\sigma_n^2)}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma_n^2}\left(x^2 - 2itx\sigma_n^2 + i^2t^2\sigma_n^4 - i^2t^2\sigma_n^4\right)} dx$$

$$= e^{-\frac{t^2\sigma_n^2}{2}}$$

(ii) $$H_{Z_n}(t) = \prod_{j=1}^{n} H_{X_j}(t)$$

$$= \prod_{j=1}^{n} e^{-\frac{t^2\sigma_j^2}{2}} = e^{-\frac{t^2}{2}\sum_{j=1}^{n}\frac{1}{2^{j+1}}}$$

$$= e^{-\frac{t^2}{\cancel{8}4}\frac{1-\frac{1}{2^n}}{\cancel{1-\frac{1}{2}}}} = e^{-\frac{t^2}{4}\left(1-\frac{1}{2^n}\right)}$$

$$Z_n \sim N\left(0, \frac{1}{2}\left(1-\frac{1}{2^n}\right)\right)$$

(iii) $$\lim_{n\to+\infty} H_{Z_n}(t) = \lim_{n\to+\infty} e^{-\frac{t^2}{4}\left(1-\frac{1}{2^n}\right)} = e^{-\frac{t^2}{4}}$$

$$\Rightarrow Z_n \xrightarrow{d} Z \sim N\left(0, \frac{1}{2}\right)$$

Cognome:.......................................... Nome:..........................................
Data orale: ☐ 22 gennaio 2015.........☐ 19 febbraio 2015

**Prof. Beghin**
**II esonero**
**B**
**14-1-2015**

**Esercizio 1**

Siano $X_1, ..., X_n$ v.a. indipendenti. Per ogni $i = 1, 2..., n$ la v.a. $X_i$ ha densità

$$f_{X_i}(x) = \begin{cases} \frac{\lambda^{\nu_i}}{\Gamma(\nu_i)} e^{-\lambda x} x^{\nu_i - 1}, & x > 0 \\ 0, & \text{altrove} \end{cases}$$

per $\lambda > 0$ e $\nu_i = 1/2^i$.

i) Calcolare la funzione caratteristica della v.a. $X_n$

ii) Calcolare la funzione caratteristica della v.a. $Z_n = X_1 + ... + X_n$.

iii) Studiare la convergenza in distribuzione della successione $\{Z_n\}_{n \geq 1}$ per $n \to \infty$.

------------------------------------

**Esercizio 2**

Siano $X$ e $Y$ v.a. indipendenti con distribuzione uniforme su $(0, 1)$.

i) Calcolare la funzione di ripartizione della v.a.

$$Z = \frac{X + Y}{X}.$$

ii) Si calcoli la seguente probabilità

$$P\left( \frac{X + Y}{X} < t \,\middle|\, X + Y > 1 \right)$$

al variare di $t \in \mathbb{R}$.

------------------------------------

# SOLUZIONI (B)

## ES. 1

(i) $$\varphi_{X_n}(t) = \int_0^{+\infty} e^{itx} x^{\nu_n - 1} \frac{\lambda^{\nu_n}}{\Gamma(\nu_n)} e^{-\lambda x} dx$$

$(\lambda - it)x = z$

$x = \frac{z}{\lambda - it}$

$$= \int_0^{+\infty} e^{-z} \frac{z^{\nu_n - 1}}{(\lambda - it)^{\nu_n}} \frac{\lambda^{\nu_n}}{\Gamma(\nu_n)} dz$$

$$= \frac{\cancel{\Gamma(\nu_n)}}{\cancel{\Gamma(\nu_n)}} \left(\frac{\lambda}{\lambda - it}\right)^{\nu_n} = \left(\frac{\lambda}{\lambda - it}\right)^{\frac{1}{2^n}}$$

(ii) $$\varphi_{Z_n}(t) = \prod_{j=1}^{n} \left(\frac{\lambda}{\lambda - it}\right)^{\frac{1}{2^j}}$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^{\sum_{j=1}^{n} \frac{1}{2^j}}$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^{\frac{1}{2}\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}} = \left(\frac{\lambda}{\lambda - it}\right)^{\frac{2}{2}\left(1 - \frac{1}{2^n}\right)}$$

(iii) $$\lim_{n \to +\infty} \varphi_{Z_n}(t) = \lim_{n \to +\infty} \left(\frac{\lambda}{\lambda - it}\right)^{\left(1 - \frac{1}{2^n}\right)}$$

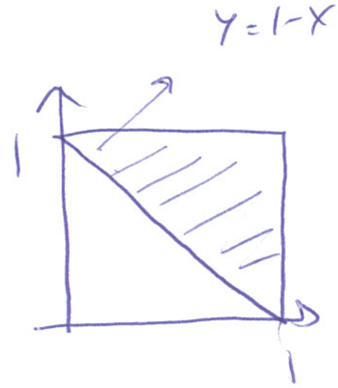
$$= \left(\frac{\lambda}{\lambda - it}\right)$$

$\Longrightarrow$ $Z_n \xrightarrow{d} Z \sim \text{Gamma}(\lambda, 1)$

e quindi $Z \sim \text{Exp}(\lambda)$

## Es. 2

(i) $$P\left(\frac{X+Y}{X}<t \,\middle|\, X+Y>1\right)$$




$$= \frac{P\left(\frac{X+Y}{X}<t,\ X+Y>1\right)}{P(X+Y>1)}$$

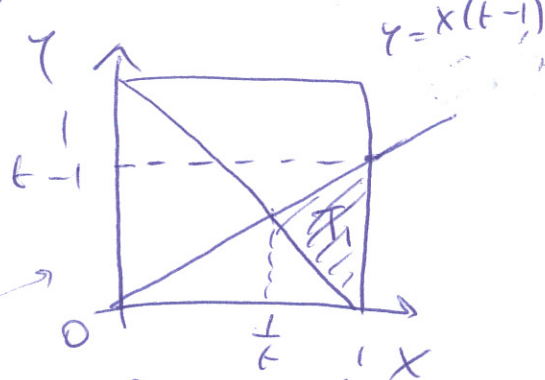
$$P(X+Y>1) = P(Y>1-X) = \frac{1}{2}$$

$$P\left(\frac{X+Y}{X}<t,\ X+Y>1\right)$$

$$= P(X+Y<Xt,\ X+Y>1)$$

$$= P(Y<X(t-1),\ \cancel{X+Y>1}\ Y>1-X)$$

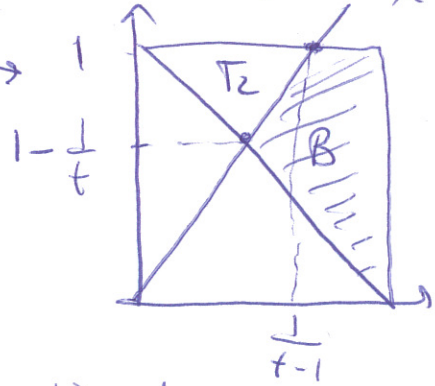
$$= P(\ 1-X<Y<X(t-1))$$




per $t>1$ devo distinguere i due casi

i) $1<t<2$

ii) $t>2$




per $t\in(1,2)$

$$= \text{Area}(T_1) = \frac{(t-1)\left(1-\frac{1}{t}\right)}{2}$$

$$= \frac{1}{2}\left(t-1-1+\frac{1}{t}\right) = \frac{1}{2}\left(t+\frac{1}{t}\right)-1$$

per $t>2$

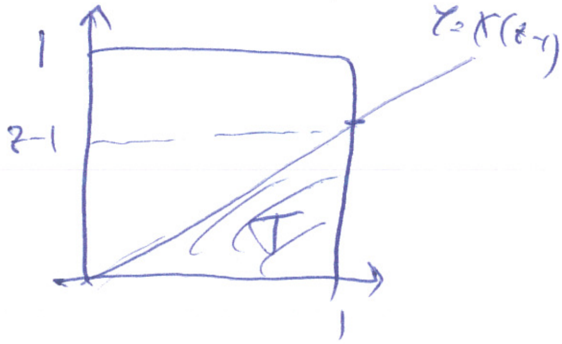
$$= \text{Area}(B) = \frac{1}{2} - \text{Area}(T_2) = \frac{1}{2} - \frac{\frac{1}{t}}{2}\left(\frac{1}{t-1}\right) = \frac{1}{2} - \frac{1}{2t(t-1)}$$

$$\Longrightarrow \quad P\left( \frac{X+Y}{X} < t \,\middle|\, X+Y>1 \right) = \begin{cases} 0 & t \le 1 \\ t + \frac{1}{t} - 2 = \frac{(t-1)^2}{t} & 1 < t \le 2 \\ 1 - \frac{1}{t(t-1)} & t > 2 \end{cases}$$

(ii) $F_Z(z) = P\left( \frac{X+Y}{X} < z \right)$

$= P(X+Y < zX)$

$= P(Y < X(z-1))$



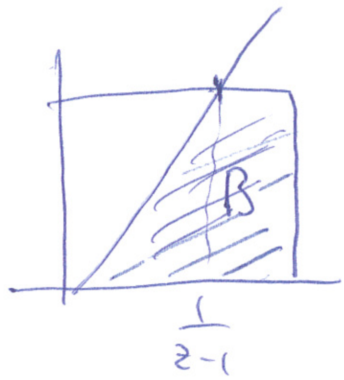

$z > 1$

$z-1>1$

$z>2$

$z \in (1,2)$

$= \text{Area}(T) = \frac{z-1}{2}$




$z > 2$

$1 - \text{Area}(B) = 1 - \frac{1}{z-1}\,\frac{1}{2}$

$$F_Z(z) = \begin{cases} 0 & z \le 1 \\ \frac{z-1}{2} & 1 < z \le 2 \\ 1 - \frac{1}{2(z-1)} & 2 < z \end{cases}$$