Cognome:.. Nome:..

Probabilità - II ESONERO
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Esercizio n.1

La v.a. doppia (X, Y) ha densità congiunta pari a

$$f_{X,Y}(x,y) = \begin{cases} \frac{ky}{(1+x)^2}, & 0 < y < 1, \quad x > 0, \quad k > 0 \\ 0 & \text{altrove} \end{cases}.$$

i) Calcolare la costante k.
ii) Le due v.a. X e Y sono indipendenti?
iii) Calcolare $\mathbb{E}(XY)$.
iv) Detereminare la distribuzione della v.a. definita come

$$Z = \frac{Y}{\sqrt{X}}.$$

Esercizio n.2

Siano $X_1, X_2, ...X_n$ v.a. indipendenti ed identicamente distribuite con la seguente distribuzione di probabilità (per ogni N):

$$\Pr\{X_k = r\} = p(1-p)^r, \qquad r = 0, 1, ..., \; 0 < p < 1.$$

i) Si calcoli la funzione caratteristica della seguente successione:

$$Z_n = \frac{\sum_{k=1}^n X_k - \frac{1-p}{p} n}{\sqrt{n}}.$$

ii) Si studi la convergenza della successione $\{Z_n\}_{n=1}^{\infty}$ per $n \to \infty$.

SUGGERIMENTO: si considerino le seguenti formule

$$\sum_{r=1}^{\infty} r a^r = \frac{a}{(1-a)^2}, \qquad |a| < 1,$$

$$\sum_{r=1}^{\infty} r^2 a^r = \frac{a^2 + a}{(1-a)^3}, \qquad |a| < 1.$$

SOLUZIONI

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ES. 1

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{kY}{(1+X)^2} & 0<y<1 \\ & x>0 \\ 0 & \text{altrove} \end{cases}$$

(i) Calcolare k:

$$k\int_0^1 dy\int_0^{+\infty} \frac{y}{(1+x)^2}\,dx = k\int_0^{+\infty} \left[\frac{y^2}{2}\right]_0^1 \frac{1}{(1+x)^2}\,dx$$

$$= \frac{k}{2}\left[-(1+x)^{-1}\right]_0^{+\infty}$$

$$= \frac{k}{2} = 1 \quad \Rightarrow \quad k=2$$

(ii) Calcolo $f_X(x)$

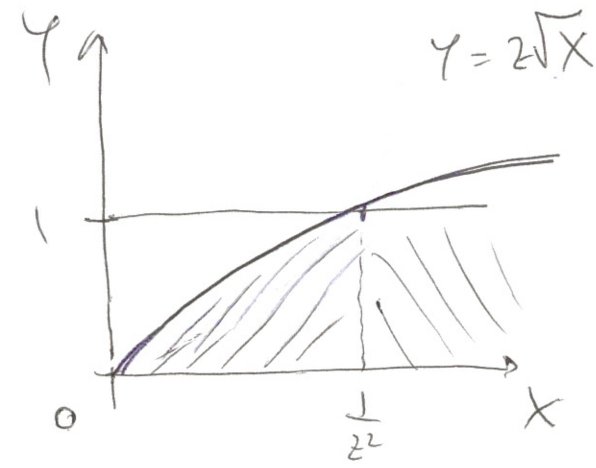
$$f_X(x) = \frac{2}{(1+x)^2}\int_0^1 y\,dy = \cancel{2}\frac{y^2}{\cancel{2}}\Big]_0^1 \frac{1}{(1+x)^2} = \frac{1}{(1+x)^2} \qquad x>0$$

Calcolo $f_Y(y)$

$$f_Y(y) = 2y\int_0^{+\infty}\frac{1}{(1+x)^2}\,dx = 2y\left[-(1+x)^{-1}\right]_0^{+\infty} = 2y \qquad y\in(0,1)$$

$\Rightarrow$ Le variabili sono indipendenti poiché $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$ $\forall x,y \in \mathbb{R}$

(iii) $Z = \dfrac{Y}{\sqrt{X}} \in (0,+\infty)$ q.c.



$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ ? & z > 0 \end{cases}$$

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$$P(Z<z) = P\left(\frac{Y}{\sqrt{X}} < z\right) = P\left(Y < z\sqrt{X}\right)$$

per $z>0$

$$= \int_{\frac{1}{z^2}}^{+\infty} dx \int_0^1 dy \, \frac{2y}{(1+x)^2} + \int_0^{\frac{1}{z^2}} dx \int_0^{z\sqrt{x}} dy \, \frac{2y}{(1+x)^2}$$

$$= \cancel{2}\,\frac{y^2}{\cancel{2}}\Big]_0^1 \int_{\frac{1}{z^2}}^{+\infty} \frac{1}{(1+x)^2}\, dx + \int_0^{1/z^2} dx \, \frac{\cancel{2}}{(1+x)^2}\, \frac{y^2}{\cancel{2}}\Big]_0^{z\sqrt{x}}$$

$$= -(1+x)^{-1}\Big]_{\frac{1}{z^2}}^{+\infty} + \int_0^{1/z^2} \frac{z^2 x}{(1+x)^2}\, dx$$

$$= \frac{1}{1+\frac{1}{z^2}} + z^2\left[-x(1+x)^{-1}\right]_0^{1/z^2} + z^2 \int_0^{1/z^2} (1+x)^{-1}\, dx$$

$$= \cancel{\frac{1}{1+\frac{1}{z^2}}} - \cancel{z^2\,\frac{1}{z^2}\,\frac{1}{1+\frac{1}{z^2}}} + z^2 \log\left(1+\frac{1}{z^2}\right)$$

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ z^2 \log\left(1+\frac{1}{z^2}\right) & z > 0 \end{cases}$$

verifica $F_Z(0^+) = 0$

$$\lim_{z\to+\infty} F_Z(z) = \lim_{z\to+\infty} \frac{\log\left(1+\frac{1}{z^2}\right)}{\frac{1}{z^2}} = 1$$

(iii) $\mathbb{E}(X\cdot Y) = \mathbb{E}(X)\cdot\mathbb{E}(Y)$

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$$\mathbb{E}(Y) = 2\int_0^1 y^2\,dy = 2\,\frac{y^3}{3}\Big)_0^1 = \frac{2}{3}$$

$$\mathbb{E}(X) = \int_0^{+\infty} \frac{x}{(1+x)^2}\,dx = \left[-x(1+x)^{-1}\right]_0^{+\infty} + \int_0^{+\infty}(1+x)^{-1}\,dx$$

$$= -1 + \infty = \infty$$

$$\Rightarrow \mathbb{E}(XY) = \infty$$

ES. 2

(i) $$H_{Z_n}(t) = H_{\frac{\sum_{k=1}^{n}\left(X_k - \frac{1-p}{p}\right)}{\sqrt{n}}}(t) = H_{\sum_{k=1}^{n}\left(X_k - \frac{1-p}{p}\right)}\left(\frac{t}{\sqrt{n}}\right)$$

per l'indip. e i.d.

$$= \left(H_{X_k - \frac{1-p}{p}}\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$= \left(e^{-i\frac{1-p}{p}\frac{t}{\sqrt{n}}}\, H_{X_k}\left(\frac{t}{\sqrt{n}}\right)\right)^n = e^{-i\frac{1-p}{p}\sqrt{n}t}\left(H_{X_k}\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$H_{X_k}(u) = \sum_{r=0}^{\infty} e^{iur} p(1-p)^r = p\,\frac{1}{1-e^{iu}(1-p)}$$

$\forall p\in(0,1)$ poichè $|e^{iu}|<1$ e $(1-p)<1$

$$\Rightarrow H_{Z_n}(t) = \left(\frac{p}{1-e^{\frac{it}{\sqrt{n}}}(1-p)}\right)^n e^{-i\frac{1-p}{p}\sqrt{n}t}$$

(ii) Valgono le ip. del TLC se $\mathbb{E} X_k < \infty$ e $V(X_k) < \infty$

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$$\mathbb{E} X_k = \sum_{r=0}^{\infty} r\, p (1-p)^r = p \sum_{r=1}^{\infty} r (1-p)^r = \frac{p \cdot (1-p)}{(1-(1-p))^2} = \frac{1-p}{p} < \infty$$

poiché $|1-p| < 1$

$$\mathbb{E} X_k^2 = \sum_{r=1}^{\infty} r^2 p (1-p)^r = p \frac{(1-p)^2 + 1-p}{(1-(1-p))^3}$$

$$= \frac{1}{p^2} (1 + p^2 - 2p + 1 - p)$$

$$= \frac{1}{p^2} (2 - 3p + p^2)$$

$$V(X_k) = \frac{1}{p^2}(2-3p+p^2) - \frac{(1-p)^2}{p^2} = \frac{2-3p+\cancel{p^2}-1-\cancel{p^2}+2p}{p^2}$$

$$= \frac{1-p}{p^2} < \infty$$

$\Rightarrow$ per TLC

$$\frac{\sum_{k=1}^{n} X_k - \frac{1-p}{p} n}{\sqrt{n \frac{1-p}{p^2}}} \xrightarrow{d} Z \sim N(0,1)$$

$\Rightarrow$ per il Teorema della conv. i.d. sotto funzione continua

$$Z_n \xrightarrow{d} \sqrt{\frac{1-p}{p^2}}\, Z \sim N\left(0, \frac{1-p}{p^2}\right)$$