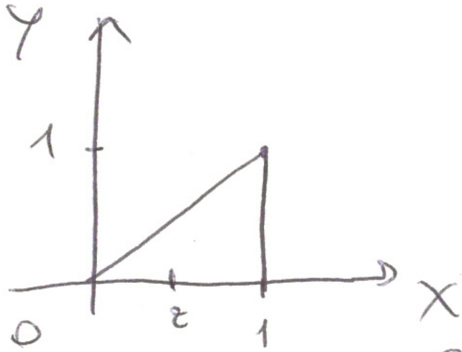
SOLUZIONI ESERC. 7BIS

ES. 1 $(X,Y) \sim \mathrm{Unif}(T)$ $\qquad Z \in (0,1)$ q.c.



$$f_{X,Y}(x,y) = 2\,\mathbb{1}_{\{0<x<1,\ 0<y<x\}}$$

$$P(\max(X,Y) < z) = P((X<z) \cap (Y<z))$$

$$= P(X<z)$$

perché $X<Y$ q.c.

$$= 2\int_0^z x\,dx$$

$$= 2\,\frac{z^2}{2} = z^2 \qquad \text{per } 0<z<1$$

Infatti

$$f_X(x) = 2\,\mathbb{1}_{\{0<x<1\}} \int_0^1 \mathbb{1}_{\{0<y<x\}}\,dy$$

$$= 2\,\mathbb{1}_{\{0<x<1\}} \int_0^x dy$$

$$= 2x\,\mathbb{1}_{\{0<x<1\}}$$

Quindi

$$f_Z(z) = \begin{cases} 2z & 0<z<1 \\ 0 & \text{altrove} \end{cases}$$

Es. 2 $\quad X, Y \sim Exp(\lambda)$ indip $\implies Z \in (0, +\infty)$ p. c.

$$P(Z < z) = P(\max(X,Y) < z)$$
$$= P(X < z, Y < z)$$
$$= \int_0^z \int_0^z \lambda^2 e^{-\lambda x - \lambda y} \, dx \, dy$$
$$= (1 - e^{-\lambda z})(1 - e^{-\lambda z}) = (1 - e^{-\lambda z})^2$$

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ (1 - e^{-\lambda z})^2 & z > 0 \end{cases}$$

per $z \to +\infty$ $\quad F_Z(z) \to 1$

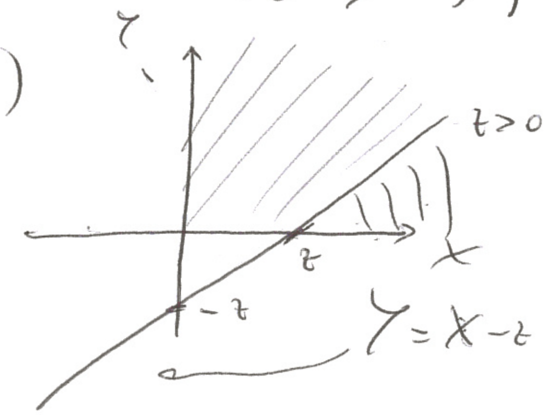
$$f_Z(z) = \begin{cases} 2\lambda (1 - e^{-\lambda z}) e^{-\lambda z} & z > 0 \\ 0 & \text{altrove} \end{cases}$$

E1.3

$X - Y = Z$ A) $X, Y \sim \exp(\lambda)$

$Z \in (-\infty, +\infty)$ p.c

$P(X - Y < z) = P(Y > X - z)$



$$= \iint_{\{(x,y):\, y > x - z\}} f_{X,Y}(x,y)\, dx\, dy$$

z>0

$$= 1 - \iint_{\{(x,y):\, y \le x - z\}} f_{X,Y}(x,y)\, dx\, dy$$

$$= 1 - \int_z^{+\infty} dx \int_0^{x-z} dy\, \lambda^2 e^{-\lambda x - \lambda y}$$

$$= 1 - \lambda \int_z^{+\infty} dx\, e^{-\lambda x} \left[-\frac{e^{-\lambda y}}{\cancel{\lambda}} \right]_0^{x-z}$$

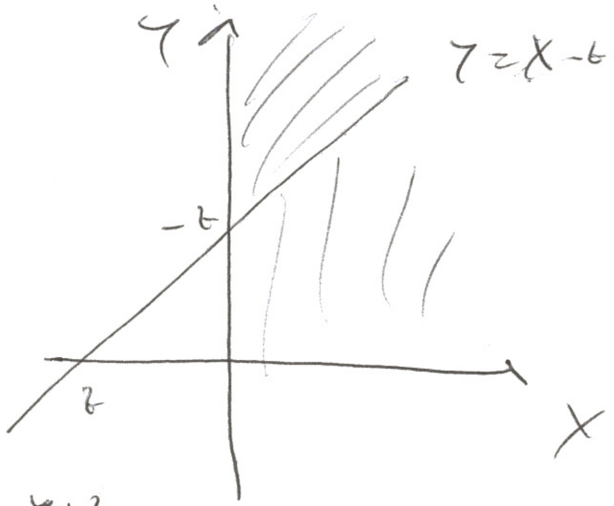
$$= 1 + \lambda \int_z^{+\infty} \left[e^{-\lambda x - \lambda x + \lambda z} - e^{-\lambda x} \right] dx$$

$$= 1 + \lambda \left[e^{\lambda z} \int_z^{+\infty} e^{-2\lambda x}\, dx - \left[-\frac{e^{-\lambda x}}{\lambda} \right]_z^{+\infty} \right]$$

$$= 1 + \cancel{\lambda} e^{\lambda z} \left[-\frac{e^{-2\lambda x}}{2\cancel{\lambda}} \right]_z^{+\infty} - e^{-\lambda z}$$

$$= 1 + \frac{e^{\lambda z - 2\lambda z}}{2} - e^{-\lambda z} = 1 - \frac{e^{-\lambda z}}{2}$$

$\underline{z<0}$



$$P(Y > X - z)$$

$$= \lambda^2 \int_{-z}^{+\infty} dy \int_0^{y+z} dx \; e^{-\lambda x} e^{-\lambda y}$$

$$= \lambda^2 \int_{-z}^{+\infty} dy \; e^{-\lambda y} \left(- \frac{e^{-\lambda x}}{\cancel{\lambda}} \right]_0^{z+y}$$

$$= -\lambda \int_{-z}^{+\infty} dy \left[e^{-\lambda y - \lambda y - \lambda z} - e^{-\lambda y} \right]$$

$$= \lambda e^{-\lambda z} \int_{-z}^{+\infty} dy \; e^{-2\lambda y} - \lambda \int_{-z}^{+\infty} e^{-\lambda y} dy$$

$$= \cancel{\lambda} e^{-\lambda z} \left[- \frac{e^{-2\lambda y}}{2\cancel{\lambda}} \right]_{-z}^{+\infty} - \cancel{\lambda} \left(- \frac{e^{-\lambda y}}{\cancel{\lambda}} \right)_{-z}^{+\infty}$$

$$= \frac{e^{-\lambda z + 2\lambda z}}{2} + e^{\lambda z}$$

$$= \frac{e^{\lambda z}}{2}$$

$$F_Z(z) = \begin{cases} \dfrac{e^{+\lambda z}}{2} & z \leq 0 \\[2ex] 1 - \dfrac{e^{-\lambda z}}{2} & z > 0 \end{cases}$$

ES.4

$X, Y \sim N(0,1)$ indip.

$Z = X - Y \qquad Z \in (-\infty, +\infty)$ q.c.

Per i momenti (che individuano univocam. la distribuzione delle normali) si ha

$EZ = EX - EY = 0$

$V(Z) = V(X) + V(Y) - 2\cancel{cov(X,Y)} = 2$

$$H_Z(t) = H_{X-Y}(t) = \mathbb{E}\, e^{it(X-Y)}$$

$$= \mathbb{E}\, e^{itX}\, \mathbb{E}\, e^{-itY}$$

$$= H_X(t) \cdot H_{-Y}(t) =$$

$$= H_X(t)\, H_Y(t) = e^{-\frac{t^2}{2}} e^{-\frac{t^2}{2}} = e^{-t^2}$$

oppure $\quad H_X(t)\, H_Y(-t) = e^{-\frac{t^2}{2}} e^{-\frac{(-t)^2}{2}} = e^{-t^2}$

$\Rightarrow \quad Z \sim N(0,2)$

Es. 5

$\Theta \sim \mathrm{Unif}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$X = \tan \Theta$$

$$\Rightarrow \quad f_\Theta(\theta) = \frac{1}{\pi} \mathbb{1}_{\{-\frac{\pi}{2} < \theta < \frac{\pi}{2}\}}$$

$$X \in (-\infty, +\infty) \text{ q.c.}$$

La tangente è invertibile quindi applico il teorema

$$x = g(\theta) = \tan \theta$$

$$\theta = g^{-1}(x) = \arctan x = h(x)$$

$$h'(x) = \frac{1}{1+x^2}$$

$$f_X(x) = f_\Theta(h(x))\, |h'(x)|$$

$$= \frac{1}{\pi} \mathbb{1}_{\{-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}\}} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{\pi(1+x^2)} \mathbb{1}_{\{-\infty < x < +\infty\}}$$

Quindi $X \sim \mathrm{Cauchy}(0, 1)$

La distribuzione di Cauchy è la seguente in generale

$$f_X(x) = \frac{\gamma}{\pi\left(\gamma^2 + (x-\mu)\right)^2} \qquad x \in \mathbb{R}$$

dove γ è il parametro di scala

μ " " " di posizione

Es. 6

$X \sim N(\mu, \sigma^2)$

$Y = |X| \implies Y \in (0, +\infty)$ p.c.

$$P(Y<y) = P(|X|<y, X>0) + P(|X|<y, X\leq 0)$$
$$= P(0<X<y) + P(-X<y, X\leq 0)$$
$$= P(0<X<y) + P(-y<X\leq 0)$$
$$= \int_0^y \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dx + \int_{-y}^0 \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dx$$

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} + \dfrac{e^{-\frac{(y+\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} & y>0 \\ 0 & \text{altrove} \end{cases}$$

<u>Es. 7</u> $(X,Y) \sim \mathrm{Unif}(0,1)$

$$Z = \frac{\max(X,Y)}{\min(X,Y)}$$

$$F_Z(z) = \begin{cases} 0 & z \le 1 \\ ? & z > 1 \end{cases}$$

$z > 1$ p.c.

$$P(Z < z) = P\left(\frac{\max(X,Y)}{\min(X,Y)} < z\right)$$

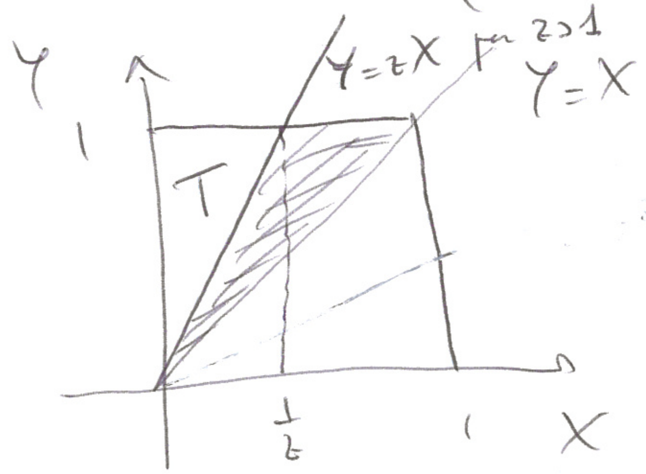
$$= P\left(\frac{\max(X,Y)}{\min(X,Y)} < z,\ X > Y\right) + P\left(\frac{\max(X,Y)}{\min(X,Y)} < z,\ X \le Y\right)$$

$$= P\left(\frac{X}{Y} < z,\ X > Y\right) + P\left(\frac{Y}{X} < z,\ X \le Y\right)$$

per la simmetria

$$= 2\, P\left(\frac{X}{Y} < z,\ X > Y\right)$$

$$= 2\, P(Y < X < zX) =$$



$= 2$ area Triangolo Trat.

$= 2\left(\frac{1}{2} - \text{area}(T)\right)$

$= 2\left(\frac{1}{2} - \frac{1}{2z}\right) = 1 - \frac{1}{z}$

$$F_Z(z) = \begin{cases} 0 & z \le 1 \\ 1 - \frac{1}{z} & z > 1 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{z^2} & z \ge 1 \\ 0 & \text{altrove} \end{cases}$$