COMPITO A

Es 1

Indichiamo con X = "n° palline bianche osservate su 3 estrazioni" senza reinserimento

i) $P((X=0) \cup (X=3)) = ?$

$X \sim$ Ipergeometrica

$\Rightarrow \boxed{P((X=0) \cup (X=3))} = P(X=0) + P(X=3)$

$$= \frac{\binom{5}{0}\binom{3}{3}}{\binom{8}{3}} + \frac{\binom{5}{3}\binom{3}{0}}{\binom{8}{3}} = \boxed{\frac{11}{56}}$$

ii) $\boxed{P((X=1) \cup (X=2))} = P(X=1) + P(X=2)$

$$= \frac{\binom{5}{1}\binom{3}{2}}{\binom{8}{3}} + \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} = \boxed{\frac{45}{56}} = 1 - \text{prob. del punto (i)}$$

iii) con reinserimento

$X \sim Bin(3, 5/8)$

$\Rightarrow \boxed{P(X=0) + P(X=3)} = \binom{3}{0}\left(\frac{5}{8}\right)^0\left(\frac{3}{8}\right)^3 + \binom{3}{3}\left(\frac{5}{8}\right)^3\left(\frac{3}{8}\right)^0$

$$= \boxed{\frac{19}{64}}$$

iv) T_1 e T_2 sono risp. l'istante di primo e secondo successo in una serie di prove indipendenti e di Bernoulli, ciascuna delle quali $p = \frac{5}{8}$.

$\Rightarrow T_1 \sim Geom\left(\frac{5}{8}\right)$ e $P(T_1 = k) = \frac{5}{8}\left(\frac{3}{8}\right)^{k-1} \quad k = 1, 2, \dots$

$T_2 \sim BinNeg\left(2, \frac{5}{8}\right)$ e $P(T_2 = k) = \binom{k-1}{1}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^{k-2}$ ②

$k = 2, 3, \ldots$

$\Rightarrow \boxed{P(T_1 \leq 3)} = P(T_1 = 1) + P(T_1 = 2) + P(T_1 = 3)$

$= \boxed{\frac{5}{8} + \frac{5}{8} \cdot \frac{3}{8} + \frac{5}{8}\left(\frac{3}{8}\right)^2}$

$\boxed{P(T_2 \in (1, 5))} = P(T_2 = 2) + P(T_2 = 3) + P(T_2 = 4)$

$= \boxed{\left(\frac{5}{8}\right)^2 + 2\left(\frac{5}{8}\right)^2 \frac{3}{8} + 3\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^2}$

ES. 2

A = "si utilizza la procedura A"

B = "si utilizza " " B"

E = "il programma è eseguito entro il limite"

i) $\boxed{P(E)} = P(E|A)P(A) + P(E|B)P(B)$

$= 0{,}75 \cdot 0{,}40 + 0{,}5 \cdot 0{,}6 = \boxed{0{,}6}$

ii) per la formula di Bayes

$\boxed{P(A|E)} = \frac{P(E|A)P(A)}{P(E)} = \frac{0{,}75 \cdot 0{,}4}{0{,}6} = \boxed{0{,}5}$

$T_2 \sim BinNeg\left(2, \frac{5}{8}\right)$ e $P(T_2 = k) = \binom{k-1}{1}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^{k-2}$

$k = 2, 3, \dots$

$$\Rightarrow \boxed{P(T_1 \le 3)} = P(T_1 = 1) + P(T_1 = 2) + P(T_1 = 3)$$

$$= \boxed{\frac{5}{8} + \frac{5}{8}\cdot\frac{3}{8} + \frac{5}{8}\left(\frac{3}{8}\right)^2}$$

$$\boxed{P(T_2 \in (1,5))} = P(T_2 = 2) + P(T_2 = 3) + P(T_2 = 4)$$

$$= \boxed{\left(\frac{5}{8}\right)^2 + 2\left(\frac{5}{8}\right)^2\frac{3}{8} + 3\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^2}$$

ES. 2

A = " si utilizza la procedura A "

B = " si utilizza " " B "

E = " il programma è eseguito entro il limite "

i) $\boxed{P(E)} = P(E|A)P(A) + P(E|B)P(B)$

$$= 0{,}75 \cdot 0{,}40 + 0{,}5 \cdot 0{,}6 = \boxed{0{,}6}$$

ii) per la formula di Bayes

$$\boxed{P(A|E)} = \frac{P(E|A)P(A)}{P(E)} = \frac{0{,}75 \cdot 0{,}4}{0{,}6} = \boxed{0{,}5}$$

COMPITO A

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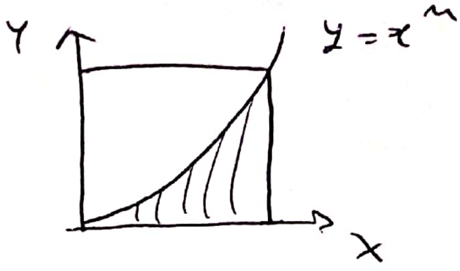

ES. 3

i) $f_{X,Y}(x,y) = 1_{(0,1)\times(0,1)}(x,y)$

$$\boxed{P(Y<X^n)} = \int_0^1 dx \int_0^{x^n} dy = \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big]_0^1 = \boxed{\frac{1}{n+1}}$$

ii) $X^n \in (0,1)$ q.c. $\quad F_{X^n}(z) = \begin{cases} 0 & z \le 0 \\ ? & 0 < z \le 1 \\ 1 & z > 1 \end{cases}$

$$F_{X^n}(z) = P(X^n < z) = P(X < z^{1/n}) = \sqrt[n]{z}$$

$$F_{X^n}(z) = \begin{cases} 0 & z \le 0 \\ \sqrt[n]{z} & 0 < z \le 1 \\ 1 & z > 1 \end{cases} \xrightarrow{n \to +\infty} \begin{cases} 0 & z \le 0 \\ 1 & z > 0 \end{cases}$$

$\Rightarrow$ $\boxed{X^n \xrightarrow{d} 0 \text{ e anche in probabilità}}$ (poiché nel caso di variabile limite degenere la convergenza in distr. implica quella i.p.)

iii) X_j^k i.i.d con f.r. $F_{X^k}(z) = \begin{cases} 0 & z \le 0 \\ z^{1/k} & 0 < z \le 1 \\ 1 & z > 1 \end{cases}$ $\quad \forall j$

$$\mathbb{E} X_j^k = \int_0^1 z \cdot f_{X^k}(z)\,dz$$

$$f_{X^k}(z) = \frac{d}{dz} F_{X^k}(z) = \frac{1}{k} z^{1/k - 1} 1_{0<z<1}$$

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$$\Rightarrow \mathbb{E}(X_j^k) = \int_0^1 \frac{z}{k} z^{1/k - 1} \, dz = \frac{1}{k} \left. \frac{z^{\frac{1}{k}+1}}{\frac{1}{k}+1} \right]_0^1 = \frac{1}{1+k}$$

$$\mathbb{E} X_j^{2k} = \mathbb{E}(X_j^k)^2 = \int_0^1 \frac{z^2}{k} z^{\frac{1}{k}-1} \, dz = \int_0^1 \frac{1}{k} \cdot z^{\frac{1}{k}+1} dz$$

$$= \frac{1}{k} \left. \frac{z^{\frac{1}{k}+2}}{\frac{1}{k}+2} \right|_0^1 = \frac{1}{2k+1}$$

$$\Rightarrow V(X_j^k) = \frac{1}{2k+1} - \left(\frac{1}{k+1}\right)^2 = \frac{k^2 \cancel{+1+2k-2k-1}}{(2k+1)(k+1)^2} = \frac{k^2}{(2k+1)(k+1)^2}$$

Per il TLC

$$\frac{\sum_{j=1}^{n} X_j^k - \frac{n}{k+1}}{\sqrt{n \left(\frac{k^2}{(2k+1)(k+1)^2}\right)}} \xrightarrow{d} N(0,1)$$

Quindi:

$$Z_n = \frac{\sum_{i=1}^{n} X_j^k - \frac{n}{k+1}}{\sqrt{n}} \cdot \frac{\sqrt{\frac{k^2}{(2k+1)(k+1)^2}}}{\sqrt{\frac{k^2}{(2k+1)(k+1)^2}}}$$

$$\xrightarrow{d} \sqrt{\frac{k^2}{(2k+1)(k+1)^2}} \cdot N(0,1)$$

ovvero $Z_n \xrightarrow{d} N\left(0, \frac{k^2}{(2k+1)(k+1)^2}\right)$

poiché per $Z \sim N(0,1)$

$aZ \sim N(0, a^2)$