AN EXAMPLE: HARMONIC OSCILLATOR

$$\dot{x} = -\lambda k x + \alpha \eta(t) \qquad x(t=0) = x_0$$

We can solve this equation in a standard fashion

Homogeneous eq: $\dot{x} = -\lambda k x \qquad x = A e^{-\lambda k t}$

Now, we take A t-dependent

$$x(t) = A(t) e^{-\lambda k t}$$

$$\dot{x}(t) = \dot{A} e^{-\lambda k t} - \lambda k A e^{-\lambda k t} = \dot{A} e^{-\lambda k t} - \lambda k x(t)$$

$\Downarrow$ Substitution in the equation

$$\dot{A} e^{-\lambda k t} - \lambda k x = -\lambda k x + \alpha \eta(t)$$

$$\dot{A} = \alpha e^{\lambda k t} \eta(t)$$

$$A(t) = \alpha \int_0^t ds\, e^{\lambda k s} \eta(s) + k$$

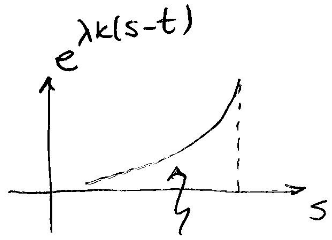
Solution

$$x(t) = \left[\alpha \int_0^t ds\, e^{\lambda k s} \eta(s) + k \right] e^{-\lambda k t}$$

$$x(0) = x_0 \quad \Longrightarrow \quad k = x_0$$

$$x(t) = x_0 e^{-\lambda k t} + \alpha \int_0^t \underbrace{e^{\lambda k (s-t)}} \eta(s)\, ds$$

initial conditions are irrelevant for $t \gg T = \frac{1}{\lambda k}$



memory on scales $T = 1/\lambda k$

Note that $x(t)$ is a Gaussian variable as it is a "sum" of Gaussian variables (η)

$$\langle x(t)\rangle = x_0 e^{-\lambda k t} + \alpha \int_0^t ds\, e^{\lambda k(s-t)} \underbrace{\langle \eta(s)\rangle}_{0}$$

$$= x_0 e^{-\lambda k t}$$

$$\langle x(t)^2\rangle = \left\langle \left[x_0 e^{-\lambda k t} + \alpha \int_0^t ds\, e^{\lambda k(s-t)} \eta(s) \right]^2 \right\rangle =$$

$$= \Big\langle (x_0 e^{-\lambda k t})^2 + 2\alpha x_0 e^{-\lambda k t} \int_0^t ds\, e^{\lambda k(s-t)} \eta(s)$$

$\boxed{\text{this term vanishes when averaged}}$

$$+ \alpha^2 \int_0^t ds_1\, e^{\lambda k(s_1-t)} \int_0^t ds_2\, e^{\lambda k(s_2-t)} \eta(s_1)\eta(s_2) \Big\rangle$$

$$= \langle x(t)\rangle^2 + \alpha^2 \int_0^t ds_1\, e^{\lambda k(s_1-t)} \int_0^t ds_2\, e^{\lambda k(s_2-t)} \delta(s_1-s_2)$$

$$\langle x(t)^2\rangle - \langle x(t)\rangle^2 = \alpha^2 \int_0^t ds\, e^{2\lambda k(s-t)}$$

$$= \frac{\alpha^2}{2\lambda k}\left(1-e^{-2\lambda k t}\right) = \sigma^2(t)$$

$$P(x(t)=y) = \frac{1}{\sqrt{2\pi}\,\sigma(t)} \exp\left(-\frac{1}{2\sigma^2(t)}\left(y - x_0 e^{-\lambda k t}\right)^2\right)$$

For t large

$$P(\sigma(t)=y) \propto \exp\left(-\frac{\lambda k}{\alpha^2} y^2\right)$$

DISCRETE VERSION

$$x(t+\Delta t) = x(t) - \lambda k \Delta t\, x(t) + \alpha\sqrt{\Delta t}\,\rho(t)$$
$$= (1-\lambda k\Delta t)\, x(t) + \alpha\sqrt{\Delta t}\,\rho(t)$$

Let us solve it iteratively

$$x(\Delta t) = (1-\lambda k\Delta t)\, x_0 + \alpha\sqrt{\Delta t}\,\rho(0)$$

$$x(2\Delta t) = (1-\lambda k\Delta t)\, x(\Delta t) + \alpha\sqrt{\Delta t}\,\rho(\Delta t)$$
$$= (1-\lambda k\Delta t)^2 x_0 + \alpha\sqrt{\Delta t}\left[(1-\lambda k\Delta t)\,\rho(0) + \rho(\Delta t)\right]$$

$$x(3\Delta t) = (1-\lambda k\Delta t)\, x(2\Delta t) + \alpha\sqrt{\Delta t}\,\rho(2\Delta t)$$
$$= (1-\lambda k\Delta t)^3 x_0 + \alpha\sqrt{\Delta t}\left[(1-\lambda k\Delta t)^2\rho(0) + \right.$$
$$\left. + (1-\lambda k\Delta t)\,\rho(\Delta t) + \rho(2\Delta t)\right]$$

Generalitation

$$x(N\Delta t) = (1-\lambda k\Delta t)^N x_0 +$$
$$+ \alpha\sqrt{\Delta t}\sum_{j=0}^{N-1}(1-\lambda k\Delta t)^j\,\rho\left((N-1-j)\Delta t\right)$$

It follows [remember $\langle\rho(t)\rangle = 0$]

$$\langle x(N\Delta t)\rangle = (1-\lambda k\Delta t)^N x_0$$

If $t = N\Delta t$

$$\langle x(t)\rangle = \left(1-\lambda k\frac{t}{N}\right)^N x_0$$

We recover the continuum result if we take $N \to \infty$ at t fixed
(i.e. $N \to \infty$, $\Delta t \to 0$ at fixed $N\Delta t = t$)

$$\langle x(t) \rangle \longrightarrow e^{-\lambda k t} x_0$$

We can also compute the variance

$$V = \langle x(N\Delta t)^2 \rangle - \langle x(N\Delta t) \rangle^2$$

$$= \alpha^2 \Delta t \sum_{j_1=0}^{N-1} (1-\lambda k \Delta t)^{j_1} \sum_{j_2=0}^{N-1} (1-\lambda k \Delta t)^{j_2} \langle p[(N-1-j_1)\Delta t] \times p[(N-1-j_2)\Delta t] \rangle$$

Now

$$\langle p[(N-1-j_1)\Delta t]\, p[(N-1-j_2)\Delta t] \rangle = \delta_{j_1 j_2}$$

$$V = \alpha^2 \Delta t \sum_{j=0}^{N-1} (1-\lambda k \Delta t)^{2j}$$

$$\boxed{\sum_{j=0}^{M} x^j = \frac{1-x^{M+1}}{1-x}}$$

$$= \alpha^2 \Delta t \, \frac{1-(1-\lambda k \Delta t)^{2N}}{1-(1-\lambda k \Delta t)^2}$$

$$= \frac{\alpha^2}{2\lambda k} \, \frac{1-(1-\lambda k \Delta t)^{2N}}{1-\lambda k \Delta t/2}$$

Instability if $1-\lambda k \Delta t/2 < 0$ $\boxed{\Delta t > \frac{2}{\lambda k}}$

$$\Big\downarrow \; N \to \infty \quad \Delta t \to 0 \quad t = N\Delta t$$

$$= \frac{\alpha^2}{2\lambda k} \left(1 - e^{-2\lambda k t}\right)$$