1 Computation of the conserved Hamiltonian: Sketch

We start from

$$iL_p = F\frac{\partial}{\partial p} \quad \rightarrow \quad H_p = U(q)$$
$$iL_q = \frac{p}{m}\frac{\partial}{\partial q} \quad \rightarrow \quad H_q = \frac{p^2}{2m}$$

Now we compute H_3 associated with $iL_3 = [iL_q, iL_p]$:

$$H_3 = -\frac{\partial H_q}{\partial q}\frac{\partial H_p}{\partial p} + \frac{\partial H_q}{\partial p}\frac{\partial H_p}{\partial q} = \frac{p}{m}\frac{\partial U(q)}{\partial q}.$$

Now we compute H_4 associated with $iL_4 = [iL_q, [iL_q, iL_p]] = [iL_q, iL_3]$:

$$H_4 = -\frac{\partial H_q}{\partial q}\frac{\partial H_3}{\partial p} + \frac{\partial H_q}{\partial p}\frac{\partial H_3}{\partial q} = \frac{p^2}{m^2}\frac{\partial^2 U(q)}{\partial q^2}$$

Now we compute H_5 associated with $iL_5 = [iL_p, [iL_q, iL_p]] = [iL_p, iL_3]$:

$$H_5 = -\frac{\partial H_p}{\partial q}\frac{\partial H_3}{\partial p} + \frac{\partial H_p}{\partial p}\frac{\partial H_3}{\partial q} = -\frac{1}{m}\left(\frac{\partial U(q)}{\partial q}\right)^2$$

Now, it can be shown that

$$i\hat{L} = iL_p + iL_q + \frac{\Delta t^2}{12}iL_4 + \frac{\Delta t^2}{24}iL_5$$

so that

$$\hat{H} = U(q) + \frac{p^2}{2m} + \frac{\Delta t^2}{12}H_4 + \frac{\Delta t^2}{24}H_5 = H(p,q) + \frac{\Delta t^2}{24m}\left[\frac{2p^2}{m}\frac{\partial^2 U(q)}{\partial q^2} - \left(\frac{\partial U(q)}{\partial q}\right)^2\right]$$

This formula holds with corrections of order Δt^3.
We can apply this result to the harmonic oscillator with $U(q) = \frac{1}{2}m\omega^2 q^2$. We find

$$\hat{H} = \left(1 + \frac{1}{6}\omega^2\Delta t^2\right)\frac{p^2}{2m} + \frac{1}{2}\left(1 - \frac{1}{12}\omega^2\Delta t^2\right)m\omega^2 q^2 + O(\Delta t^3).$$

To relate this result with the exact result obtained a few lessons ago, note that, if we take

$$\lambda = \frac{1}{1 + \frac{1}{6}\omega^2\Delta t^2},$$

we have

$$\lambda\hat{H} = \frac{p^2}{2m} + \frac{1}{2}\left(1 - \frac{1}{4}\omega^2\Delta t^2\right)m\omega^2 q^2 + O(\Delta t^3).$$