DEFINITION OF THE LIOUVILLIAN

Consider $A(p,q)$ which depends on p,q only (no explicit time dependence)

Let $p(t)$, $q(t)$ be the solution of the Hamilton equation and consider $A(p(t),q(t))$

$$\frac{d}{dt}A(p(t),q(t)) = \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial q}\dot{q}$$

$$= \left(-\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right) A(p(t),q(t))$$

The Liouvillian operator is defined as

$$iL = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

introduced to make L hermitian

$$\frac{d}{dt}A(p(t),q(t)) = iLA$$

If $H(p,q)$ does not depend explicitly on time t.

$$A(p(t),q(t)) = e^{iLt}A(p(0),q(0))$$

L hermitian $\Rightarrow e^{iLt}$ unitary

[Newton's dynamics is a unitary evolution in phase space

We define the Liouvillian op. as

$$iL = \frac{\partial H}{\partial p} \cdot \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \cdot \frac{\partial}{\partial p}$$

(For 1 particle, for N particles, add a sum over particles)

We wish to show that it is hermitean in the Hilbert space of complex functions of (p,q) with scalar product

$$\langle \psi_1 | \psi_2 \rangle = \int dp\,dq\, \psi_1^*(p,q)\, \psi_2(p,q)$$

[We use MQ notations]

Indeed, consider a 1D system of one particle

$$\langle \psi_1 | iL \psi_2 \rangle = \int dp\,dq\, \psi_1^*(p,q) \left(\frac{\partial H}{\partial p}\frac{\partial \psi_2}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \psi_2}{\partial p} \right)$$

We integrate by parts and use the fact that $\psi_{1,2}(\pm\infty, q) = 0$ $\psi_{1,2}(p, \pm\infty) = 0$ to guarantee integrability

$$= - \int dp\,dq\, \psi_2 \left[\frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p} \psi_1^* \right) - \frac{\partial}{\partial p}\left(\frac{\partial H}{\partial q} \psi_1^* \right) \right]$$

$$= - \int dp\,dq\, \psi_2 \left[\cancel{\frac{\partial^2 H}{\partial p \partial q}} \psi_1^* + \frac{\partial H}{\partial p}\frac{\partial \psi_1^*}{\partial q} - \cancel{\frac{\partial^2 H}{\partial p \partial q}} \psi_1^* - \frac{\partial H}{\partial q}\frac{\partial \psi_1^*}{\partial p} \right]$$

$$= - \left[\int dp\,dq\, \psi_2^* \left(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \right) \psi_1 \right]^*$$

$$= - \langle \psi_2 | iL \psi_1 \rangle^*$$

$$\Rightarrow \quad i \langle \psi_1 | L | \psi_2 \rangle = -(i)^* \langle \psi_2 | L | \psi_1 \rangle^* =$$

$$\langle \psi_1 | L | \psi_2 \rangle = \langle \psi_2 | L | \psi_1 \rangle^*$$ L is hermitian