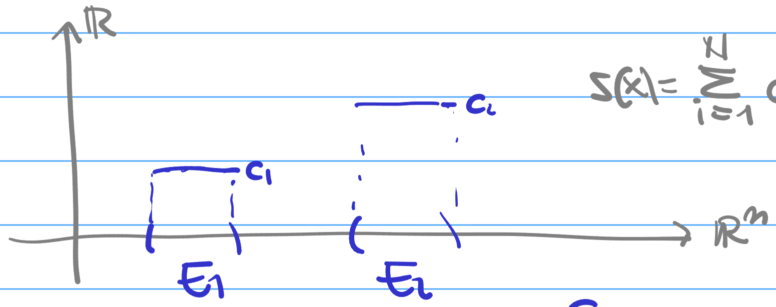
24/11/2025

DEF. $f: \mathbb{R}^n \to \mathbb{R}$ si dice sommabile secondo Lebesgue se $\exists \{f_k\}$ funzioni semplici tali che

1. $f_k(x) \to f(x)$ q.o. $x \in \mathbb{R}^n$
2. $\forall \varepsilon > 0 \ \exists k_0 = k_0(\varepsilon) \in \mathbb{N} : \int_{\mathbb{R}^n} |f_k(x) - f_{k+p}(x)| dx < \varepsilon \quad \forall k \geq k_0$



$$s(x) = \sum_{i=1}^{N} c_i \chi_{E_i}(x) \qquad c_i \in \mathbb{R}$$
E_i misurabile

$$\int_{\mathbb{R}^n} s(x) dx = \sum_{i=1}^{N} c_i |E_i|$$

$$\int_{\mathbb{R}^n} f(x) dx = \lim_{k \to +\infty} \int_{\mathbb{R}^n} f_k(x) dx$$

PROP. $\{f_k\}$ semplici e $\exists f: \mathbb{R}^n \to \mathbb{R}$ t.c. $\lim_{k \to +\infty} f_k(x) = f(x)$
allora
f è misurabile

DEF. $\Omega \subseteq \mathbb{R}^n$ misurabile (aperto) $|\Omega| > 0$ è possibile definire

$$L^1(\Omega) = \{ f: \Omega \to \mathbb{R} \mid \int_\Omega |f(x)| dx < +\infty \}$$

$$L^2(\Omega) = \{ f: \Omega \to \mathbb{R} \mid \int_\Omega |f(x)|^2 dx < +\infty \}$$

$$L^\infty(\Omega) = \{ f: \Omega \to \mathbb{R} \mid \exists M > 0 : |f(x)| \leq M \text{ q.o. in } \Omega \}$$

$f \in L^1(\Omega) \qquad \|f\|_1 = \int_\Omega |f(x)| dx$

$f \in L^\infty(\Omega) \qquad \|f\|_\infty = \inf \{ M > 0 : |f(x)| \leq M \text{ q.o. in } \Omega \}$

$f, g \in L^2(\Omega)$ $\quad (f|g)_{L^2} := \int_\Omega f(x) g(x) dx$, $\|f\|_2 = \left[\int_\Omega |f(x)|^2 dx\right]^{1/2}$

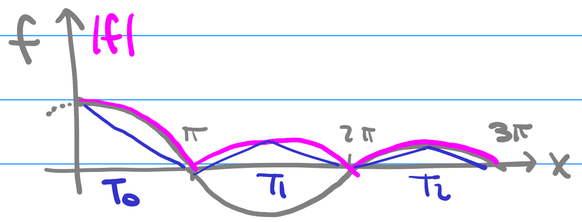
$\Rightarrow$ $L^1(\Omega)$, $L^\infty(\Omega)$ spazi di Banach, $L^2(\Omega)$ spazio di Hilbert

TEOR $f: \mathbb{R}^n \to \mathbb{R}$ misurabile allora vale che

f è sommabile sse $\exists\, \phi$ sommabile : $|f(x)| \leq \phi(x)$ q.o.

(f è sommabile sse $|f|$ è sommabile)

ESEMPIO
$$f(x) = \begin{cases} \frac{\operatorname{sen}(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



¿ f è sommabile?
¿ $f \in L^1(\mathbb{R})$? **NO**

$$\int_{\mathbb{R}} |f(x)| dx \geq \sum_{i=1}^{+\infty} |T_i| \qquad |T_k| = \pi \cdot \frac{1}{\pi(k+\frac{1}{2})} \sim \frac{1}{k}$$

considero l'integrale "secondo Riemann"

$\lim_{M \to +\infty} \int_0^M f(t) dx = \ell \in \mathbb{R}$ $\Rightarrow$ f è integrabile in senso improprio secondo Riemann

TEOREMA DELLA CONVERGENZA MONOTONA (B. LEVI)

$\{f_k\}$ sommabili tali che $f_k(x) \leq f_{k+1}(x)$ q.o.

$\Rightarrow$ $f(x) := \lim_{k \to +\infty} f_k(x)$ è misurabile e vale
$$\int_{\mathbb{R}^n} f_k(x) dx \longrightarrow \int_{\mathbb{R}^n} f(x) dx$$

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} f_k(x) dx = \int_{\mathbb{R}^n} \left[\lim_{k \to +\infty} f_k(x)\right] dx$$

TEOREMA DELLA CONVERGENZA DOMINATA (H. LEBESGUE)

$\{f_k\}$ sommabili tali che $f_k(x) \longrightarrow f(x)$ q.o. $x \in \mathbb{R}^n$

se $\exists\ \phi$ sommabile tale che $|f_k(x)| \leq \phi(x)$ q.o. $x \in \mathbb{R}^n$

allora
$$\int_{\mathbb{R}^n} f_k(x)\,dx \longrightarrow \int_{\mathbb{R}^n} f(x)\,dx$$
$$\int_{\mathbb{R}^n} |f_k(x) - f(x)|\,dx \longrightarrow 0$$

TEOREMA DI FUBINI

$$f : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R} \quad \text{sommabile, allora}$$
$$(x,y) \longmapsto f(x,y)$$

I. q.o. $y \in \mathbb{R}^k$ $\quad x \longmapsto f(x,y)$ è sommabile in $\mathbb{R}^n$,

II. $y \longmapsto \int_{\mathbb{R}^n} f(x,y)\,dx$ è sommabile in $\mathbb{R}^k$,

III. $\displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^k} f(x,y)\,dx\,dy = \int_{\mathbb{R}^k} \left[\int_{\mathbb{R}^n} f(x,y)\,dx \right] dy$.

TEOREMA DI TONELLI

$f : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ misurabile e non negativa se

I. q.o. $y \in \mathbb{R}^k$ $\quad x \longmapsto f(x,y)$ è sommabile in $\mathbb{R}^n$,

II. $y \longmapsto \int_{\mathbb{R}^n} f(x,y)\,dx$ è sommabile in $\mathbb{R}^k$,

allora f è sommabile in $\mathbb{R}^n \times \mathbb{R}^k$.

ESEMPI

$R = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \qquad R = \{ a \leq x_1 \leq b\ ;\ c \leq x_2 \leq d \}$

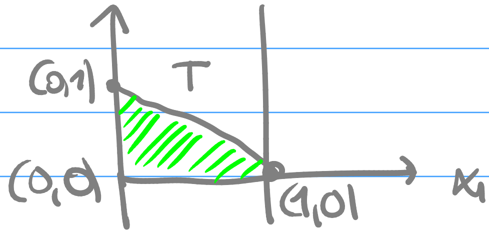
$$\int_R x_1\,dx = \int_{\underset{\mathbb{R}\times\mathbb{R}}{\mathbb{R}^2}} x_1 \chi_R(x)\,dx = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} x_1 \chi_R(x)\,dx_1 \right] dx_2$$

$$\left[\chi_R(x) = \chi_{[a,b]}(x_1) \cdot \chi_{[c,d]}(x_1) \right] = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} x_1 \chi_{[a,b]}(x_1)\, \chi_{[c,d]}(x_2)\,dx_1 \right] dx_2$$

$$= \int_{\mathbb{R}} \left[\int_a^b x_1 \, \chi_{[c,d]}(x_2) \, dx_1 \right] dx_2 = \int_{\mathbb{R}} \chi_{[c,d]}(x_2) \left[\int_a^b x_1 \, dx_1 \right] dx_2$$

$$= \int_{\mathbb{R}} \frac{1}{2}(b^2-a^2) \, \chi_{[c,d]}(x_2) \, dx_2 = \frac{1}{2}(b^2-a^2) \int_c^d dx_2 = \frac{1}{2}(b^2-a^2)(d-c)$$

$$T = \{ 0 \leq x_1 \leq 1 \, , \, 0 \leq x_2 \leq 1-x_1 \}$$



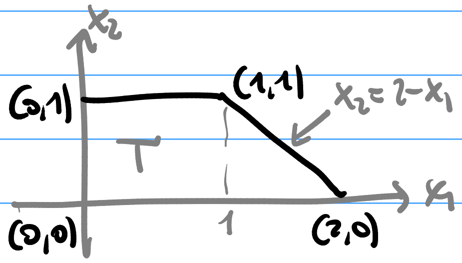
$$\int_T x_2 \, dx = \int_0^1 \left[\int_0^{1-x_1} x_2 \, dx_2 \right] dx_1 = \int_0^1 \left[\frac{1}{2} x_2^2 \right]_0^{1-x_1} dx_1$$

$$= \frac{1}{2} \int_0^1 (1-x_1)^2 dx_1 = -\frac{1}{6}(1-x_1)^3 \Big]_0^1 = +\frac{1}{6}$$

$$T = \{ 0 \leq x_2 \leq 1 \, , \, 0 \leq x_1 \leq 1-x_2 \}$$

$$\int_T x_2 \, dx = \int_0^1 \left[\int_0^{1-x_2} x_2 \, dx_1 \right] dx_2 = \int_0^1 \left[x_1 x_2 \right]_0^{1-x_2} dx_2$$

$$= \int_0^1 (1-x_2) x_2 \, dx_2 = \left[\frac{1}{2} x_2^2 - \frac{1}{3} x_2^3 \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



$$T = \{ 0 \leq x_1 \leq 2, \, 0 \leq x_2 \leq f(x_1) \}$$

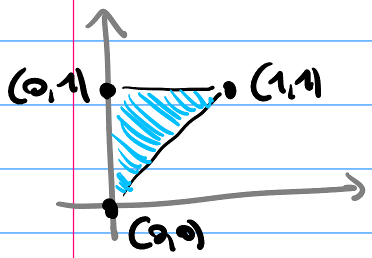
$$f(x_1) = \begin{cases} 1 & x_1 \in [0,1] \\ 2 - x_1 & x_1 \in [1,2] \end{cases}$$

$$T = \{ 0 \leq x_2 \leq 1, \, 0 \leq x_1 \leq 2 - x_2 \}$$

ESEMPIO $f(x) = e^{-x_2^2}$ $\quad T$ Triangolo di vertici $(0,0), (0,1), (1,1)$

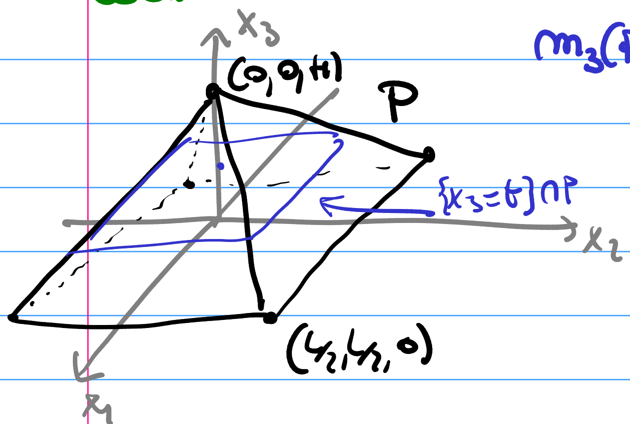
$$\int_T f(x)\,dx = \int_0^1 \left[\int_0^{x_2} e^{-x_2^2} dx_1 \right] dx_2 = \int_0^1 x_2 e^{-x_2^2} dx_2 = \left[-\frac{1}{2} e^{-x_2^2} \right]_0^1$$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right)$$



$$T = \{ 0 \le x_2 \le 1 \,;\, 0 \le x_1 \le x_2 \}$$

ESEMPIO



$$m_3(P) = |P| = \mathrm{vol}(P)$$

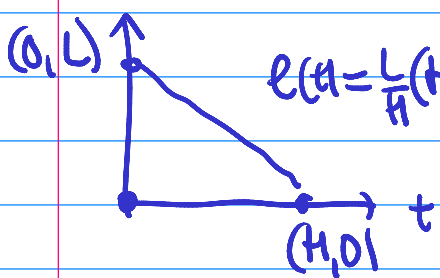
$$|P| = \int_{\mathbb{R}^3} \chi_P(x)\,dx = \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} \chi_P(x)\,dx_1 dx_2 \right] dx_3$$

$$= \int_0^H \left[\int_{P_t} 1 \cdot dx_1 dx_2 \right] dx_3$$

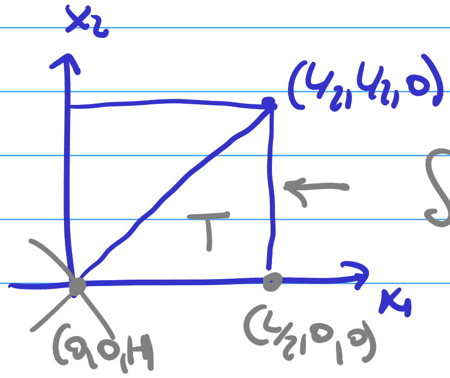
$P_t = P \cap \{x_3 = t\}$

$$= \int_0^H |P_t|\,dt = \int_0^H \frac{L^2}{H^2}(H-t)^2\,dt$$

$$|P_t| = m_2(P_t) = \ell^2 = \frac{L^2}{H^2}(H-t)^2 \quad = + \left[\frac{L^2}{H^2} \cdot \frac{(t-H)^3}{3} \right]_0^H = \frac{1}{3} L^2 H$$



$$\ell(t) = \frac{L}{H}(H-t) = L - \frac{L}{H}t$$



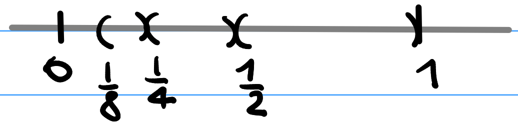
$$\int_T [c + a x_1 + b x_2]\,dx_1 dx_2$$

$$= \int_T \left[H - \frac{2H}{L} x_1 \right] dx_1 dx_2$$

ESERCIZIO 70. *Si spieghi perché i seguenti sottoinsiemi di* $\mathbb{R}$ *sono misurabili secondo Lebesgue*

$$A=\bigcup_{k\geq 0}\left(\frac{1}{2^{k+1}},\frac{1}{2^k}\right)\quad B=\bigcap_{k\geq 0}\left(-\frac{1}{2^{k+1}},\frac{2^k+1}{2^k}\right)\quad C=[0,1]\setminus\left[\bigcup_{k\geq 0}\left(\frac{1}{3^{2k+1}},\frac{1}{3^{2k}}\right)\right]\quad D=\bigcup_{k\geq 1}\left[0,\sum_{i=1}^{k}\frac{1}{i^2}\right]$$

e se ne calcoli la misura.



$I_k=\left(\frac{1}{2^{k+1}},\frac{1}{2^k}\right)$ è un intervallo aperto ed è misurabile

A è misurabile perché unione (numerabile) di insiemi misurabili (perché aperti)

$|I_k|=\frac{1}{2^k}-\frac{1}{2^{k+1}}=\frac{1}{2^{k+1}}$

$$\left(\frac{1}{2^{k+1}},\frac{1}{2^k}\right)\cap\left(\frac{1}{2^{k+2}},\frac{1}{2^{k+1}}\right)=\emptyset \Rightarrow I_k\cap I_j=\emptyset \ x\ j\neq k$$

$$|A|=\left|\bigcup_{k\geq 0}I_k\right|=\sum_{k=0}^{+\infty}|I_k|=\sum_{k=0}^{+\infty}\frac{1}{2^{k+1}}=\frac{1}{2}\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k=\frac{1}{2}\left(\frac{1}{1-1/2}\right)=1$$

ESERCIZIO 6. *Data l'applicazione*

$$x(u)=(u_1,u_2,au_1+bu_2)\qquad \text{con } u\in K=[0,\pi]^2$$

si provi che

i. (x,K) *è una superficie regolare,*

ii. $Im(x)$ *è un piano affine in* $\mathbb{R}^3$.

Infine si calcoli la lunghezza della curva sulla superficie prodotta dalla composizione della parametrizzazione della superficie con la curva contenuta in K di equazioni $\{\phi(s)=(s,s):s\in[0,\pi]\}$.

I. $\partial_1 x(u)=(1,0,a)\qquad \partial_2 x(u)=(0,1,b)$

$\dots\Rightarrow$ (x,K) è una superficie regolare

II. $Im(x)$ è il grafico della funzione $\{x_3=f(x_1,x_2)=ax_1+bx_2,\ (x_1,x_2)\in K\}$

$Im(x)=\Gamma_f\subseteq\{x_3-ax_1-bx_2=0\}$

$$(\partial_1 x\wedge\partial_2 x)(u)=\begin{vmatrix} e_1 & e_2 & e_3\\ 1 & 0 & a\\ 0 & 1 & b\end{vmatrix}=(-a,-b,1)$$

III. $\phi(s)=(s,s)$, $s\in[0,\pi]$ $\qquad Im(\phi)\subseteq K$

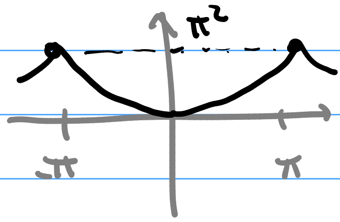
$x(\phi(s))=(s,s,(a+b)s)$, $\frac{d}{ds}(x(\phi(s)))=(1,1,(a+b))$

$L(\text{curva})=\sqrt{2+(a+b)^2}\,\pi$

INTERVALLO

$f(x)=x^2 \quad x\in[-\pi,\pi]$

f^*



$$P_N(x) = a_0 + \sum_{k=1}^{N} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$

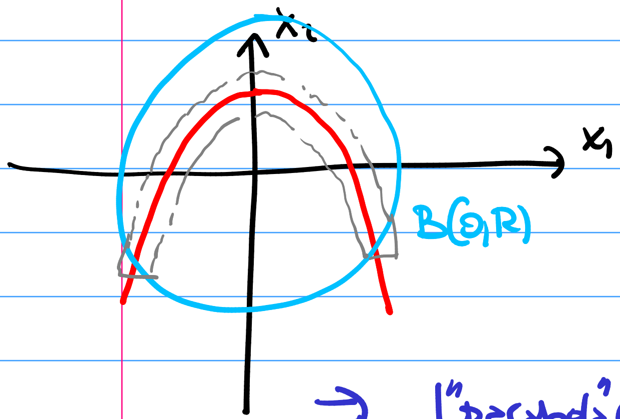
$$f^*(x) = f(x - 2k\pi) \quad \text{per} \quad x\in[(2k-1)\pi,(2k+1)\pi] \;, k\in\mathbb{Z}.$$

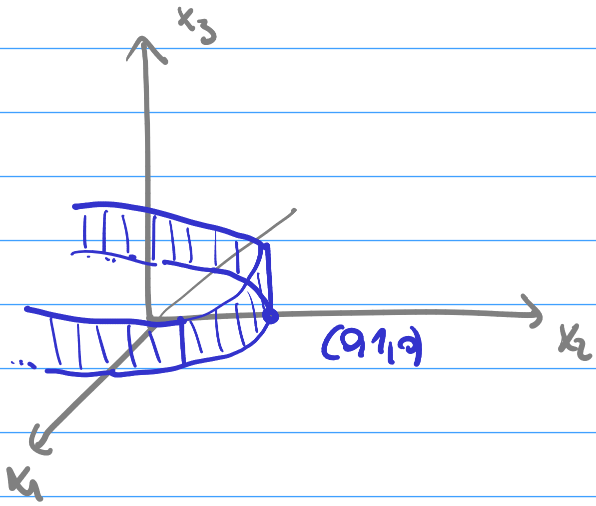
ESERCIZIO 3. *Si provi che i seguenti insiemi dello spazio* $\mathbb{R}^3$

$H=\{x_1=0\} \qquad S^2=\{x_1^2+x_2^2+x_3^2=1\} \qquad D=\{x_1^2+x_2\le 1, x_3=0\} \qquad C=\{x_1^2+x_2=1, 0\le x_3\le 1\}$

sono misurabili e se ne calcoli la misura.

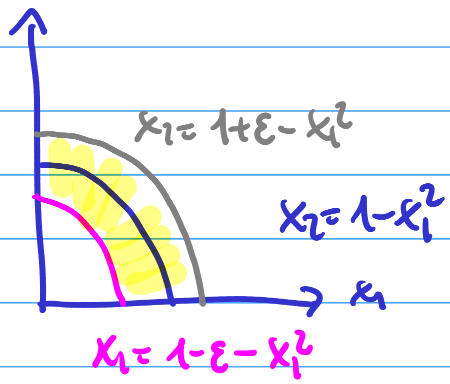
$$C = \begin{Bmatrix} x_2 = 1 - x_1^2 \\ 0 \le x_3 \le 1 \end{Bmatrix}$$





$\Rightarrow \; |\text{"parabola"} \cap B(0,R)| \le C(R)\,\varepsilon \;\Rightarrow\; |C \cap B(0,R)| = 0 \quad \forall R>0$

$\Rightarrow |C| = m_3(C) = 0$



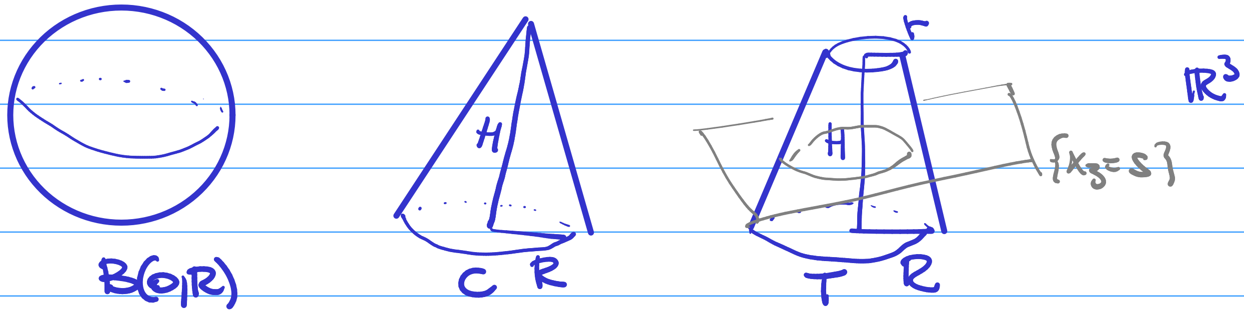
$$A = 2\left[\int_0^{\sqrt{1+\varepsilon}} \left((1+\varepsilon) - x_1^2\right) dx_1 - \int_0^{\sqrt{1-\varepsilon}} \left((1-\varepsilon) - x_1^2\right) dx_1\right]$$

$$= 2\left\{\left[(1+\varepsilon)x - \frac{1}{3}x_1^3\right]_0^{\sqrt{1+\varepsilon}} - \left[(1-\varepsilon)x - \frac{1}{3}x_1^3\right]_0^{\sqrt{1-\varepsilon}}\right\}$$

$$= 2\left\{\frac{2}{3}(1+\varepsilon)^{3/2} - \frac{2}{3}(1-\varepsilon)^{3/2}\right\} = A(\varepsilon)$$

N.B. $A(\varepsilon) \to 0$ per $\varepsilon \to 0^+$

ESERCIZIO



$B, C, T \cap \{x_3=s\} \longrightarrow$ cerchio $\longrightarrow m_2(\text{cerchio}) = \pi r^2$

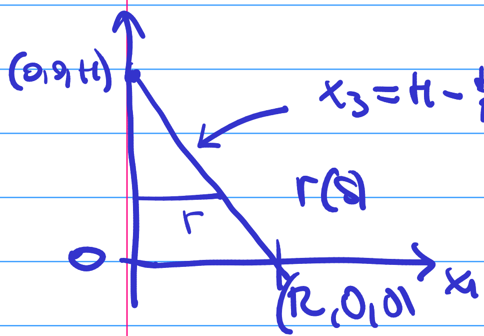
¿ $r = r(s)$?

$B(0,R) = B = \{x_1^2+x_2^2+x_3^2 < R^2\}$

$B \cap \{x_3=s\} = \{(x_1,x_2) : x_1^2+x_2^2 < R^2-s^2\} \longrightarrow r(s) = \sqrt{R^2-s^2}$

$$m_3(B) = \int_{-R}^{R} m_2(B\cap\{x_3=s\})\,ds = \int_{-R}^{R} \pi(R^2-s^2)\,ds$$

$$= \pi\left(2R^3 - \left[\tfrac{1}{3}s^3\right]_{-R}^{R}\right) = \pi\left(2R^3 - \tfrac{1}{3}R^3 - \tfrac{1}{3}R^3\right) = \tfrac{4}{3}\pi R^3$$

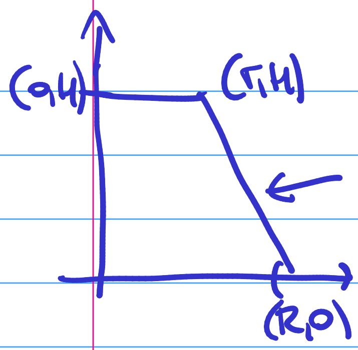


$\Rightarrow r(s) = \frac{R}{H}(H-s)$

$$m_3(C) = \int_0^H m_2(C\cap\{x_3=s\})\,ds = \int_0^H \pi\frac{R^2}{H^2}(H-s)^2\,ds$$

$$= \pi\frac{R^2}{H^2}\int_0^H (H^2-2Hs+s^2)\,ds = \frac{\pi R^2}{H^2}\left[\cancel{H^3} - \cancel{H\left[s^2\right]_0^H} + \left[\tfrac{1}{3}s^3\right]_0^H\right]$$

$$= \frac{\pi R^2}{H^2}\left(\tfrac{1}{3}H^3\right) = \tfrac{1}{3}\pi H R^2$$



$$x_3 = \frac{H}{r-R}(x_1 - R) \rightarrow \rho(s) = \frac{r-R}{H}s + R$$

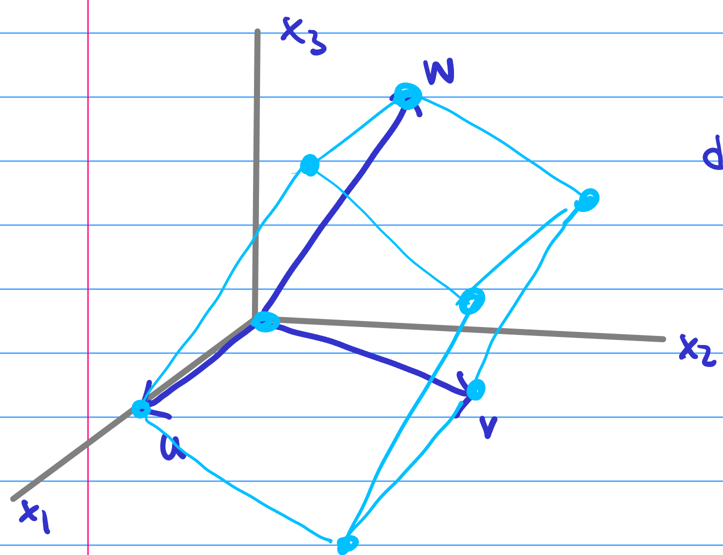
$$m_3(T) = \int_0^H m_2(T \cap \{x_3 = s\})\,ds = \int_0^H \pi\left[\frac{r-R}{H}s + R\right]^2 ds$$

$$= \pi \int_0^H \left(\left(\frac{r-R}{H}\right)^2 s^2 + 2R\frac{(r-R)}{H}s + R^2\right)ds$$

$$= \pi\left[R^2H + \frac{R(r-R)}{H}\left[s^2\right]_0^H + \frac{(r-R)^2}{H^2}\left[\frac{1}{3}s^3\right]_0^H\right]$$

$$= \pi\left[R^2H + RH(r-R) + \frac{1}{3}H(r-R)^2\right]$$

ESEMPIO IMPORTANTE



$(u \wedge v) \cdot w$ = volume del parallelepipedo

$$\det\begin{pmatrix} u \\ v \\ w \end{pmatrix} = m_3(P)$$

$$y = F(x) = \underbrace{\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}}_{= M}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$F(e_1) = u$

$F(e_2) = v$ $\qquad$ F lineare (e C^∞)

$F(e_3) = w$

se u, v, w sono linearmente indipendenti F è invertibile e C^∞

DEF. $\phi : A \to B$, $A, B \subseteq \mathbb{R}^n$ aperti, iniettiva, suriettiva e ϕ, ϕ^{-1} di classe C^1

$\Rightarrow$ ϕ si dice diffeomorfismo tra A e B

$$\det(M) = m_3(P) = \int_P 1\, dy = \int_C \det(M)\, dx = \int_C \det(JF)\, dx = \int_{F^{-1}(P)} |\det(JF)|\, dx$$

$$C = [0,1]^3 \Rightarrow F(C) = P$$

$$y = F(x) = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} u_1x_1 + v_1x_2 + w_1x_3 \\ u_2x_1 + v_2x_2 + w_2x_3 \\ u_3x_1 + v_3x_2 + w_3x_3 \end{pmatrix}$$

$$JF(x) = \left(\partial_j F_i(x)\right)_{i,j=1,2,3} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} = M$$

TEOREMA $f: B \to \mathbb{R}$ sommabile, $B \subseteq \mathbb{R}^n$ aperto

se $\exists\, \phi : A \to B$ diffeomorfismo ($A \subseteq \mathbb{R}^n$ aperto)

allora

$$\int_E f(y)\, dy = \int_{\phi^{-1}(E)} f(\phi(x))\, |\det(J\phi)(x)|\, dx, \quad \forall E \subseteq B \text{ misurabile}$$

ESEMPIO $B(0,R) \subseteq \mathbb{R}^3$

$$m_3(B(0,R)) = \int_{B(0,R)} dx = \int_{[0,R]\times[0,\pi]\times[0,2\pi]} 1 \cdot \rho^2 \operatorname{sen}(\phi)\, d\rho\, d\phi\, d\theta = \int_0^R \rho^2 d\rho \cdot \int_0^\pi \operatorname{sen}(\phi)\, d\phi \cdot \int_0^{2\pi} d\theta$$

$$\begin{cases} x_1 = \rho \operatorname{sen}(\phi) \cos(\theta) \\ x_2 = \rho \operatorname{sen}(\phi) \operatorname{sen}(\theta) \\ x_3 = \rho \cos(\phi) \end{cases} \qquad = \frac{R^3}{3} \cdot 2 \cdot 2\pi$$

$$= \frac{4}{3}\pi R^3$$

$\rho \geq 0$, $\phi \in [0,\pi]$, $\theta \in [0,2\pi]$

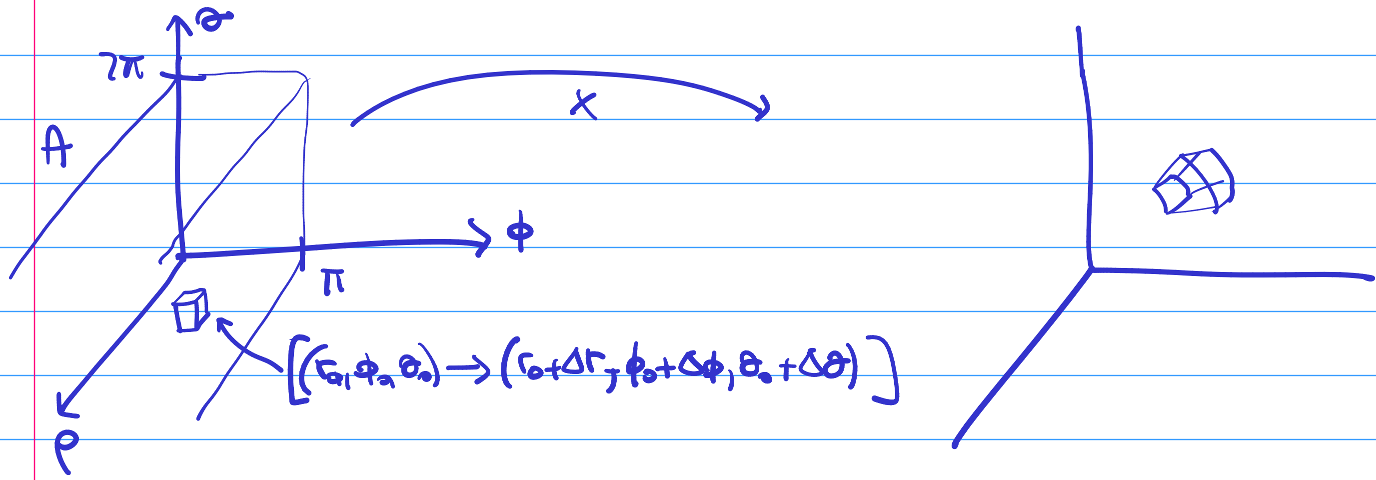
$$A = [0,+\infty]\times[0,\pi]\times[0,2\pi] \xrightarrow[\text{diff.}]{} \mathbb{R}^3 = B$$

$$J = \begin{pmatrix} \operatorname{sen}(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\operatorname{sen}(\phi)\operatorname{sen}(\theta) \\ \operatorname{sen}(\phi)\operatorname{sen}(\theta) & \rho\cos(\phi)\operatorname{sen}(\theta) & \rho\operatorname{sen}(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\operatorname{sen}(\phi) & 0 \end{pmatrix}$$

$$= \cos(\phi) \begin{vmatrix} \rho\cos(\phi)\cos(\theta) & -\rho\operatorname{sen}(\phi)\operatorname{sen}(\theta) \\ \rho\cos(\phi)\operatorname{sen}(\theta) & \rho\operatorname{sen}(\phi)\cos(\theta) \end{vmatrix} + \rho\operatorname{sen}(\phi) \begin{vmatrix} \operatorname{sen}(\phi)\cos(\theta) & -\rho\operatorname{sen}(\phi)\operatorname{sen}(\theta) \\ \operatorname{sen}(\phi)\operatorname{sen}(\theta) & \rho\operatorname{sen}(\phi)\cos(\theta) \end{vmatrix}$$

$$= \cos(\phi)\cdot\rho^2\left(\cos^2(\theta)\cos(\phi)\mathrm{sen}(\phi) + \mathrm{sen}^2(\theta)\cos(\phi)\mathrm{sen}(\phi)\right) + \rho\,\mathrm{sen}(\phi)\cdot\rho\left(\mathrm{sen}^2(\phi)\cos^2(\theta) + \mathrm{sen}^2(\phi)\,\mathrm{sen}^2(\theta)\right)$$

$$= \rho^2\,\mathrm{sen}(\phi)\cos^2(\phi) + \rho^2\,\mathrm{sen}^3(\phi) = \rho^2\,\mathrm{sen}(\phi)$$



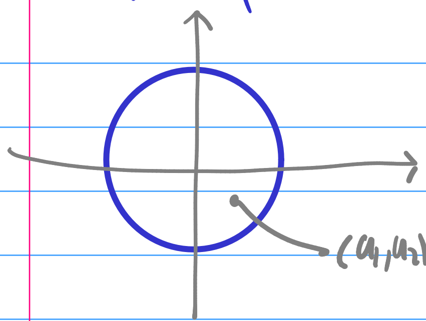
ESEMPIO

$$\pi = \int_{\{x_1^2+x_2^2<1\}} dx = \int_{\{u_1^2+u_2^2<1\}} 4(u_1^2+u_2^2)\,du_1\,du_2 \overset{*}{=} \int_0^1\left(\int_0^{2\pi} 4\rho^2\cdot\rho\, d\theta\right)d\rho = 2\pi$$

$$\begin{cases} x_1(u_1,u_2) = u_1^2 - u_2^2 \\ x_2(u_1,u_2) = 2u_1u_2 \end{cases} \qquad J = \begin{pmatrix} \partial_1 x_1 & \partial_2 x_1 \\ \partial_1 x_2 & \partial_2 x_2 \end{pmatrix} = \begin{pmatrix} 2u_1 & -2u_2 \\ 2u_2 & 2u_1 \end{pmatrix}$$

$$x_1^2 + x_2^2 = (u_1^2 - u_2^2)^2 + 4u_1^2u_2^2 = (u_1^2+u_2^2)^2 < 1$$

$$* \begin{cases} u_1 = \rho\cos(\theta) \\ u_2 = \rho\,\mathrm{sen}(\theta) \end{cases} \qquad J' = \begin{pmatrix} \cos(\theta) & -\rho\,\mathrm{sen}(\theta) \\ \mathrm{sen}(\theta) & +\rho\cos(\theta) \end{pmatrix} \Rightarrow \det(J') = \rho$$



$(u_1,u_2) = u_1 + iu_2 = z$

$$z^2 = (u_1+iu_2)^2 = u_1^2 + 2iu_1u_2 + i^2u_2^2$$
$$= (u_1^2 - u_2^2) + i(2u_1u_2)$$
$$= x_1 + ix_2$$

$$z^2 = (re^{i\theta})^2 = r^2e^{i2\theta}$$

$x_1 = a u_1$
$x_2 = b u_2$
$x_3 = c u_3$

$$x = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} u \qquad a,b,c>0$$

$$|E| = \int_E dx = \int_{B(0,1)} abc \, du = abc \, m_3(B(0,1)) = \frac{4}{3}\pi abc$$

$$E = \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1 \right\} \quad \text{ellisoide}$$

$$x^{-1}(E) = \left\{ \frac{a^2u_1^2}{a^2} + \frac{b^2u_2^2}{b^2} + \frac{c^2u_3^2}{c^2} = u_1^2 + u_2^2 + u_3^2 \leq 1 \right\} = B(0,1)$$