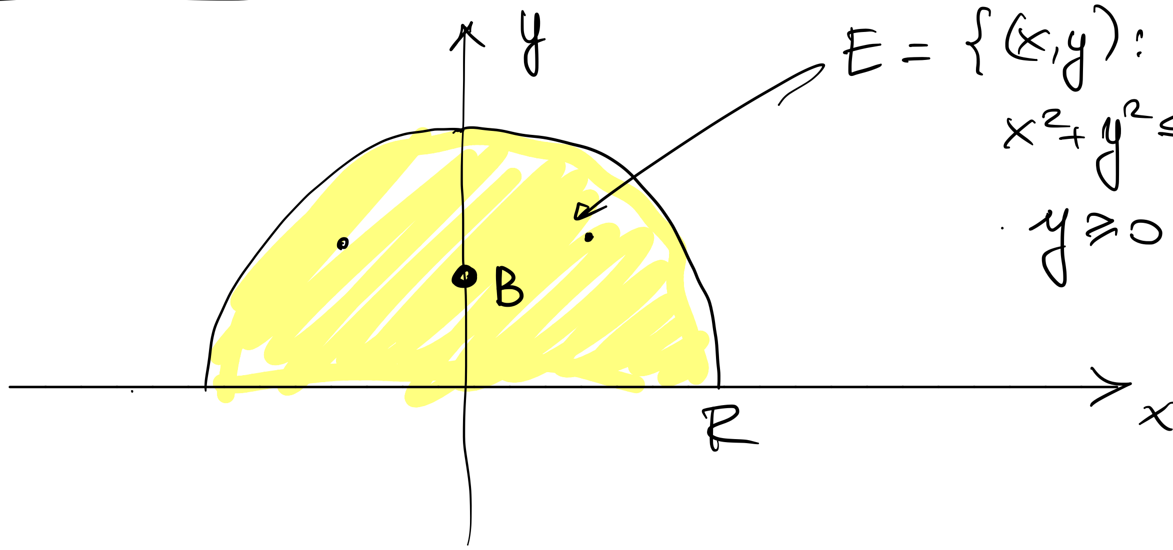
Esempio 2

Baricentro di un semicerchio



$E = \{(x,y): x^2+y^2 \leq R^2, \; y \geq 0\}$

Il baricentro è il punto (x_B, y_B) t.c.

$$x_B = \frac{1}{\text{area } E} \iint_E x \, dx\,dy = 0$$

$$y_B = \frac{1}{\text{area } E} \iint_E y \, dx\,dy = ?$$

$$\text{area } E = \frac{\pi R^2}{2}$$

passo a coordinate polari

$$\iint_E y \, dx\,dy = \iint_{\tilde{E}} \rho \operatorname{sen}\theta \cdot \rho \; d\rho\, d\theta =$$

dove $\tilde{E} = \{(\rho, \theta): 0 \leq \rho \leq R, \; 0 \leq \theta \leq \pi\}$.

$$= \int_0^\pi d\theta \int_0^R d\rho \; \rho^2 \operatorname{sen}\theta$$

$$\iint_E y \, dx\,dy = \int_0^{\pi} d\theta \int_0^R d\rho \quad \rho^2 \operatorname{sen}\theta =$$

$$= \underbrace{\int_0^{\pi} d\theta \operatorname{sen}\theta}_{=\, 2} \cdot \underbrace{\int_0^R \rho^2 \, d\rho}_{=\, \frac{R^3}{3}} = \frac{2R^3}{3}$$

$$y_B = \frac{2 \cdot 2 R^{\cancel{3}}}{\pi \cancel{R^2} \, 3} = \frac{4}{3\pi} R$$

$$x_B = \frac{1}{\text{area } E} \iint_E x \, dx\,dy$$

$$\iint_E x \, dx\,dy = \underbrace{\int_0^{\pi} d\theta \cos\theta}_{=\, 0} \int_0^R \rho^2 \, d\rho = 0$$

DEF. Un dominio normale E della forma

$$E=\{(x,y):\ a\le x\le b\ ,\ \alpha(x)\le y\le\beta(x)\}$$

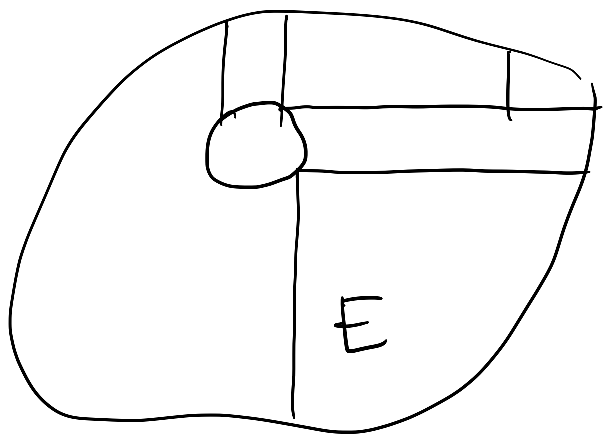
si dice regolare se $\alpha(x),\beta(x)$ sono di classe C^1

e se $\alpha(x)<\beta(x)\quad\forall x\in(a,b)$

e analogamente a variabili scambiate.

Un dominio regolare è unione finita di domini normali regolari senza pti interni in comune

a due a due



E

TEOREMA Siano T, D due domini regolari

t.c. $T\subset[0,+\infty)\times[0,2\pi]$

$D\subset\mathbb{R}^2$. t.c. la trasformazione

$$\phi(\rho,\theta):\ T\longrightarrow D$$

$$(\rho,\theta)\longmapsto(x,y)=(\rho\cos\theta,\rho\,\mathrm{sen}\,\theta)$$

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verifichi $\phi(T) = D$

Allora, $\forall$ funzione $f(x,y)$ continua in D

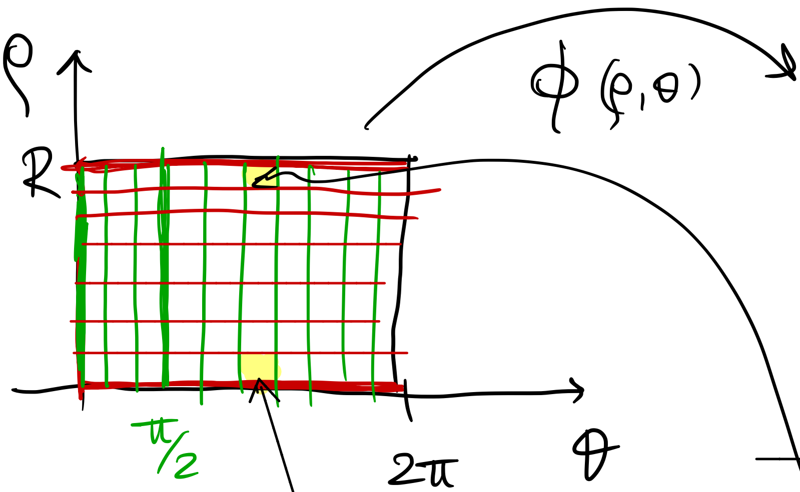
si ha

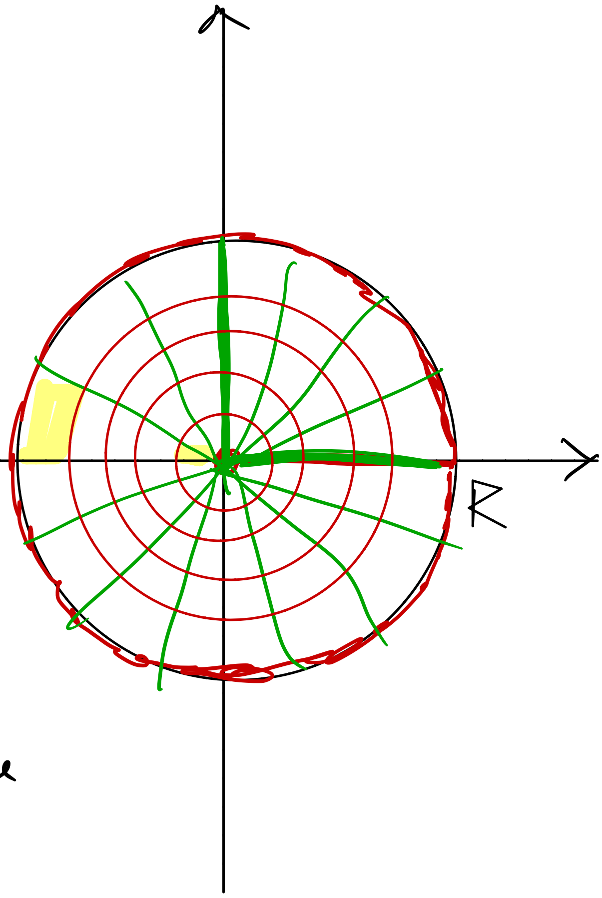
$$\iint_D f(x,y)\,dx\,dy = \iint_T f(\rho\cos\theta, \rho\operatorname{sen}\theta)\,\rho\,d\rho\,d\theta$$

Esempio

$$D = \{(x,y): \ x^2+y^2 \le R^2\}$$
$$T = \{(\rho,\theta): \ 0 \le \theta \le 2\pi, \ 0 \le \rho \le R\}$$

Allora $$\iint_D f(x,y)\,dx\,dy = \iint_T f(\rho\cos\theta, \rho\operatorname{sen}\theta)\,\rho\,d\rho\,d\theta$$



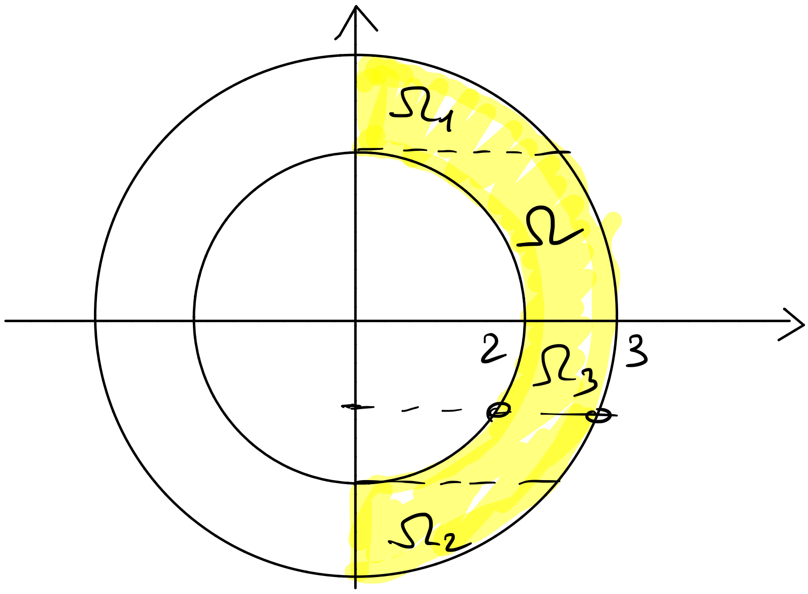


questo quadratino
deve "pesare"
meno di questo.
questo spiega il fattore
correttivo ρ.

Calcolare

$$\iint_\Omega \frac{xy^2}{x^2+y^2}\,dx\,dy$$

dove Ω è dato dall'intersezione della corona circolare di raggi 2 e 3 e centro l'origine con il semipiano dei pti di ascissa positiva.



Se dovessi farlo in coord. cartesiane

$$\iint_\Omega \frac{xy^2}{x^2+y^2}\,dx\,dy = \int_2^3 dy \int_0^{\sqrt{9-y^2}} dx \frac{xy^2}{x^2+y^2} + \int_{-3}^{-2} dy \int_0^{\sqrt{9-y^2}} dx \frac{xy^2}{x^2+y^2} + \int_{-2}^{2} dy \int_{\sqrt{4-y^2}}^{\sqrt{9-y^2}} dx \frac{xy^2}{x^2+y^2}$$

Non è questa la maniera più semplice.

Meglio passare a coord. polari

$$\iint_\Omega \frac{xy^2}{x^2+y^2}\,dx\,dy = \int_{-\pi/2}^{\pi/2} d\theta \int_2^3 d\rho\, \frac{\rho\cos\theta\, \cancel{\rho^2} \operatorname{sen}^2\theta}{\cancel{\rho^2}}\,\rho =$$

$$= \underbrace{\int_{-\pi/2}^{\pi/2} d\theta \cos\theta \operatorname{sen}^2\theta}_{=\, 2\int_0^{\pi/2} d\theta \cos\theta \operatorname{sen}^2\theta} \quad \underbrace{\int_2^3 d\rho\, \rho^2}_{=\, \frac{\rho^3}{3}\Big|_2^3 = \frac{1}{3}\cdot(27-8) = \frac{19}{3}} = \frac{38}{9}$$

$$2\int_0^{\pi/2} d\theta \cos\theta \operatorname{sen}^2\theta = \frac{2}{3}\operatorname{sen}^3\theta\Big|_0^{\pi/2} = \frac{2}{3}$$

<u>Esercizio</u> Calcolare l'area (e il baricentro) della regione piana

$$D = \{ (x,y) \in \mathbb{R}^2 : \quad 4 \le x^2+y^2 \le 4x \}$$

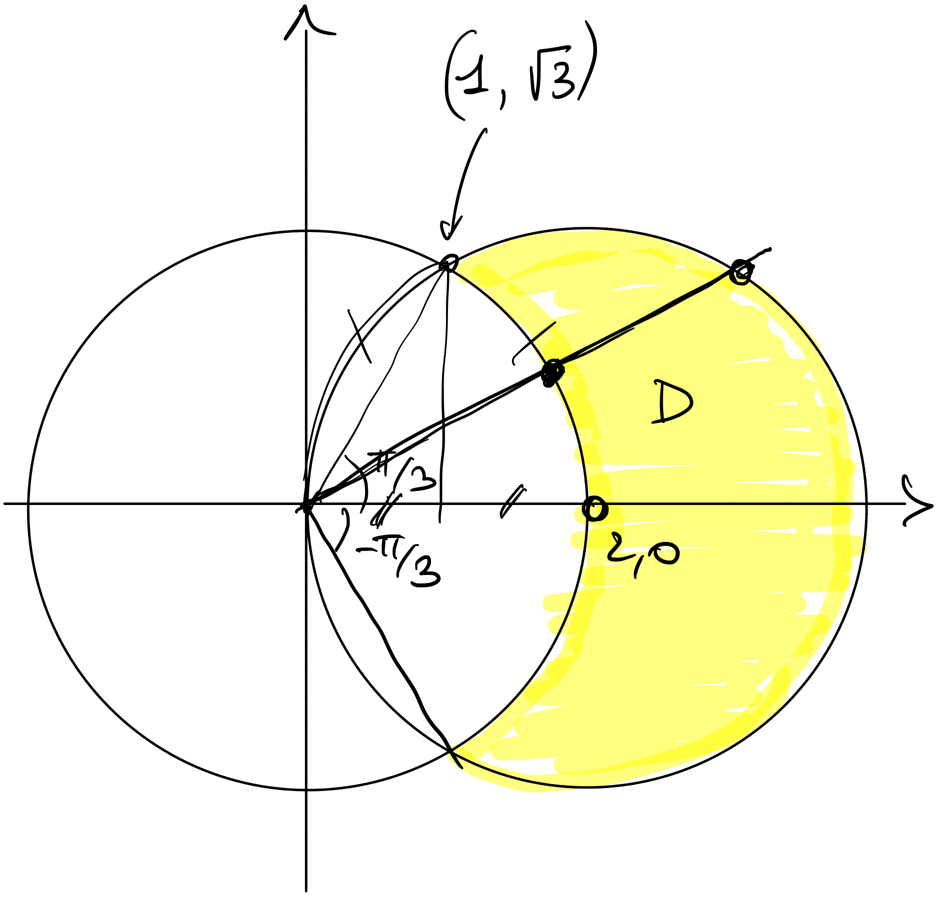
$4 \le x^2+y^2$ è la parte esterna alla circonf. $x^2+y^2=4$

$$x^2+y^2 \le 4x \iff (x-2)^2+y^2 \le 4$$

$$x^2+y^2=4x \iff (x^2-4x+4)+y^2=4$$

$$\iff (x-2)^2+y^2=4$$

circonf di centro $(2,0)$ e raggio 2



$$\text{Area } D = \iint_D 1 \, dx\,dy$$

La disuguaglianza $x^2+y^2 \geq 4$ corrisponde, in coord. polari a $\rho \geq 2$

$x^2+y^2 \leq 4x$ corrisponde a

$$\rho^{\cancel{2}} \leq 4\cancel{\rho}\cos\theta \iff \rho \leq 4\cos\theta$$

L'insieme D, in coord. polari, diventa:

$$\tilde{D} = \{(\rho,\theta): -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3},\ 2 \leq \rho \leq 4\cos\theta\}$$

$$\text{area } D = \iint_D 1 \, dx\,dy = \iint_{\tilde{D}} 1 \cdot \rho \, d\rho\, d\theta =$$

$$= \int_{-\pi/3}^{\pi/3} d\theta \int_2^{4\cos\theta} d\rho \, \rho =$$

$$= \int_{-\pi/3}^{\pi/3} d\theta \quad \frac{\rho^2}{2}\Bigg|_{\rho=2}^{\rho=4\cos\theta} =$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} d\theta \, (16\cos^2\theta - 4) =$$

$$= 2 \int_{-\pi/3}^{\pi/3} (4\cos^2\theta - 1) = 2 \cdot 2 \int_0^{\pi/3} (4\cos^2\theta - 1)\, d\theta$$

$$= 4\int_0^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

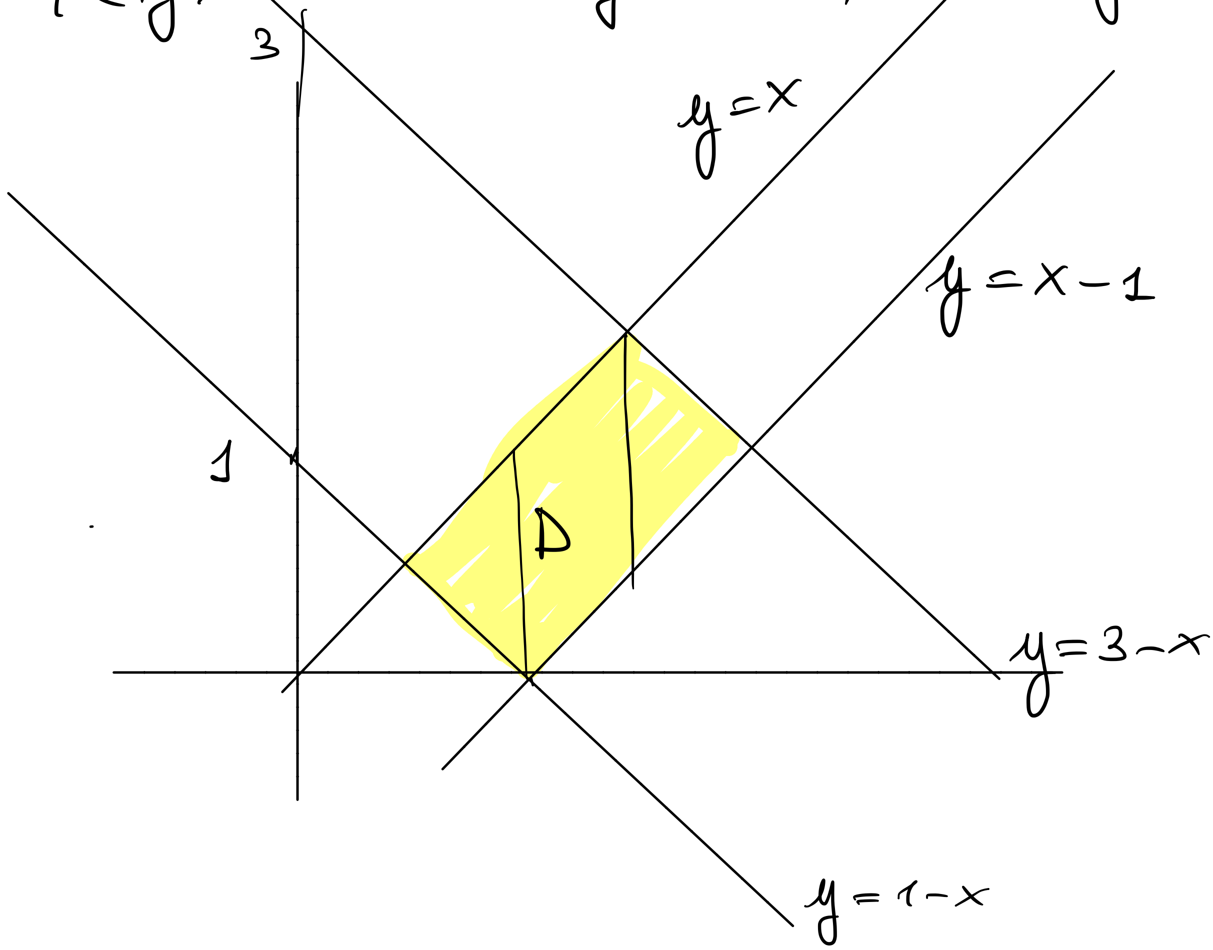
$$= 4\int_0^{\pi/3} \left(2 + 2\cos(2\theta) - 1\right) d\theta =$$

$$= 4\left(\frac{\pi}{3} + \operatorname{sen}(2\theta)\right)\Big|_0^{\pi/3} = \frac{4\pi}{3} + 4\cdot\frac{\sqrt{3}}{2} =$$

$$= \frac{4\pi}{3} + 2\sqrt{3}$$

$$\iint_D (x+y)\ln(x-y)\,dx\,dy$$

$$D=\{(x,y):\ 1-x\le y\le 3-x,\ x-1\le y\le x\}$$



$$D=\{(x,y):\ 1\le x+y\le 3,\ -1\le y-x\le 0\}.$$

E' naturale porre

$$\begin{cases} u = x+y \\ v = y-x \end{cases} \iff \begin{cases} x = \dfrac{u-v}{2} \\ y = \dfrac{u+v}{2} \end{cases}$$

L'insieme D diventa

$$\tilde D=\{(u,v):\ 1\le u\le 3,\ -1\le v\le 0\}.$$

L'integrale diventa $\displaystyle\iint_{\tilde D} u\ln(-v)\ ??\ du\,dv$

cosa va qui?

$$= \iint_{\tilde{D}} u \ln(-v) \ ?? \ du\,dv$$

cosa va qui?

ci va $\left| \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \right| = \frac{1}{2}$

$$x(u,v) = \frac{u-v}{2} \implies x_u = \frac{1}{2} \qquad x_v = -\frac{1}{2}$$

$$y(u,v) = \frac{u+v}{2} \implies y_u = \frac{1}{2} \qquad y_v = 1/2$$

$$\iint_{\tilde{D}} u \ln(-v) \frac{1}{2} \, du\,dv =$$

$$= \frac{1}{2} \int_1^3 du \int_{-1}^0 dv \ u \left(\ln(-v)\right) =$$

$$= \frac{1}{2} \underbrace{\int_1^3 u \, du}_{\frac{u^2}{2}\Big|_1^3} \underbrace{\int_{-1}^0 \ln(-v)\,dv}_{\text{per parti}}$$

nel passaggio a coord. polari:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases} \qquad x_\rho = \cos\theta \qquad x_\theta = -\rho\,\mathrm{sen}\,\theta$$

$$y_\rho = \mathrm{sen}\,\theta \qquad y_\theta = \rho\cos\theta$$

$$\iint_D f(x,y)\,dx\,dy =$$

$$= \iint_{\tilde{D}} f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta) \left| \det \begin{bmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{bmatrix} \right| d\rho\, d\theta$$

$$\left| \det \begin{bmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{bmatrix} \right| = \left| \det \begin{pmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \rho\cos\theta \end{pmatrix} \right| = |\rho| = \rho$$

Cosa sarà oggetto di verifica al 1° esonero?

- funzioni di 2 variabili: proprietà differenziali

- Curve e integrali curvilinei.
  Campi conservativi.

- Integrali doppi (eccetto cambi di variabile)

potranno essere chiesti all'orale e negli esami scritti