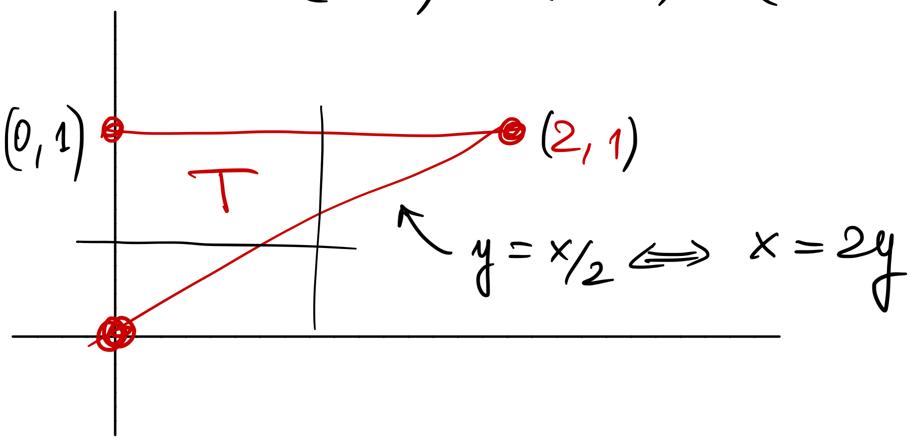
ESERCIZIO Calcolare

$$\iint_T e^{y^2}\,dx\,dy$$ dove T è il triangolo di vertici $(0,0)$, $(0,1)$, $(2,1)$



Farlo nei due ordini possibili; uno dei due è <u>molto</u> più facile.

1) Come dominio x-normale

$$\iint_T e^{y^2}\,dx\,dy = \int_0^2 dx \left(\int_{\frac{x}{2}}^1 dy\, e^{y^2} \right)$$ non si riesce a calcolare

2) Come dominio y-normale

$$\iint_T e^{y^2}\,dx\,dy = \int_0^1 dy \left(\int_0^{2y} dx\, e^{y^2} \right) =$$

$$= \int_0^1 2y\, e^{y^2}\, dy = \int_0^1 e^t\, dt = e - 1$$

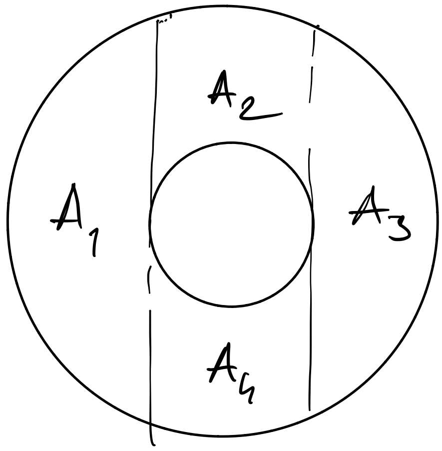
$y^2 = t$, $2y\,dy = dt$

Principali proprietà degli integrali doppi.

A dominio normale, oppure unione finita di domini normali a due a due senza punti interni in comune.

f, g continue in A.

se $A = A_1 \cup A_2 \cup A_3 \cup A_4$



$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f\,dx\,dy + \iint_{A_2} \cdots + \iint_{A_3} \cdots + \iint_{A_4} \cdots$$

1) $$\iint_A \left(f(x,y) + g(x,y)\right) dx\,dy = \iint_A f(x,y)\,dx\,dy + \iint_A g(x,y)\,dx\,dy$$

2) se $c \in \mathbb{R}$, allora

$$\iint_A c\, f(x,y)\,dx\,dy = c \iint_A f(x,y)\,dx\,dy.$$

linearità dell' $\iint$.

3) se $f(x,y) \leq g(x,y) \quad \forall (x,y) \in A$

$$\Rightarrow \iint_A f(x,y)\,dx\,dy \leq \iint_A g(x,y)\,dx\,dy$$

monotonia dell' $\iint$.

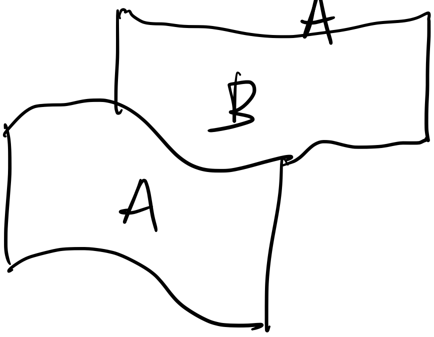
4) $$\iint_A 1 \, dx\,dy = \text{area}(A)$$

5) $$\left| \iint_A f(x,y)\, dx\,dy \right| \leq \iint_A |f(x,y)|\, dx\,dy$$

dis. triangolare

6) se A e B sono due domini come sopra, privi di pti interni in comune, allora

$$\iint_{A \cup B} f(x,y)\, dx\,dy = \iint_A f(x,y)\, dx\,dy + \iint_B f(x,y)\, dx\,dy$$



additività rispetto all' insieme di integrazione

7) $$\text{area}(A) \min_A f(x,y) \leq \iint_A f(x,y)\, dx\,dy \leq \overset{\max}{\cancel{\sup_A}}\, f(x,y)\ \text{area}(A)$$

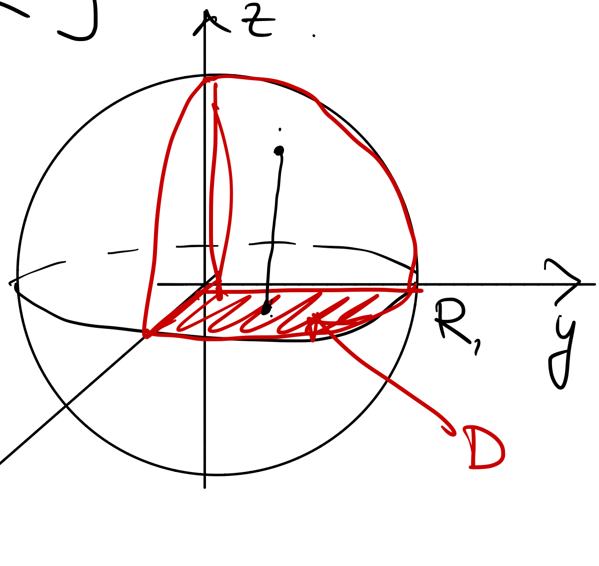
segue da (2),(3),(4)

$$\min_A f(x,y) \leq \underbrace{\frac{1}{\text{area}\,A} \iint_A f(x,y)\, dx\,dy}_{\text{media / valor medio di } f \text{ su } A} \leq \max_A f(x,y)$$

Volume di una palla di raggio R.

$$B_R = \{(x,y,z) : x^2+y^2+z^2 \le R^2\}$$

$$\mathrm{vol}(B_R) = ?$$



Si può calcolare in vari modi:

Calcolo il volume della parte di B_R che sta nel primo ottante

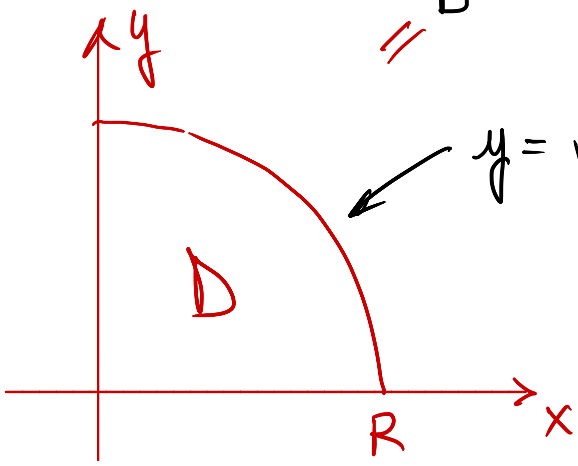
$$B_R \cap \{x \ge 0, y \ge 0, z \ge 0\}$$

1° modo:

$$\mathrm{vol}\, B_R = 8 \iint_D \sqrt{R^2-x^2-y^2}\, dx\, dy =$$

in seguito lo faremo in coord. polari

$$=$$



$$= 8 \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \sqrt{R^2-x^2-y^2}$$

$$= 8 \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \sqrt{R^2-x^2-y^2} = (*)$$

$$\int \sqrt{R^2-x^2-y^2}\, dy = \int \sqrt{a^2-y^2}\, dy =$$

per parti

$1 = g'(y) \Rightarrow g(y) = y$

$\sqrt{a^2-y^2} = f(y)$

$f'(y) = -\frac{y}{\sqrt{a^2-y^2}}$

$$R^2-x^2 = a^2$$

$$= y\sqrt{a^2-y^2} + \int \frac{(y^2-a^2)+a^2}{\sqrt{a^2-y^2}} dy =$$

$$= \quad " \quad - \int \sqrt{a^2-y^2}\, dy + a^2 \int \frac{dy}{\sqrt{a^2-y^2}}$$

$$\Rightarrow \int \sqrt{a^2-y^2}\, dy = \frac{1}{2} y \sqrt{a^2-y^2} + \frac{1}{2} a^2 \int \frac{dy}{\sqrt{a^2-y^2}}$$

$$= \quad " \quad + \frac{a^{\cancel{2}}}{2\cancel{a}} \int \frac{dy}{\sqrt{1-\left(\frac{y}{a}\right)^2}} =$$

$$= \frac{1}{2} y \sqrt{a^2-y^2} + \frac{a^2}{2} \arcsen\left(\frac{y}{a}\right)$$

$$(*) = 8 \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \sqrt{R^2-x^2-y^2} =$$

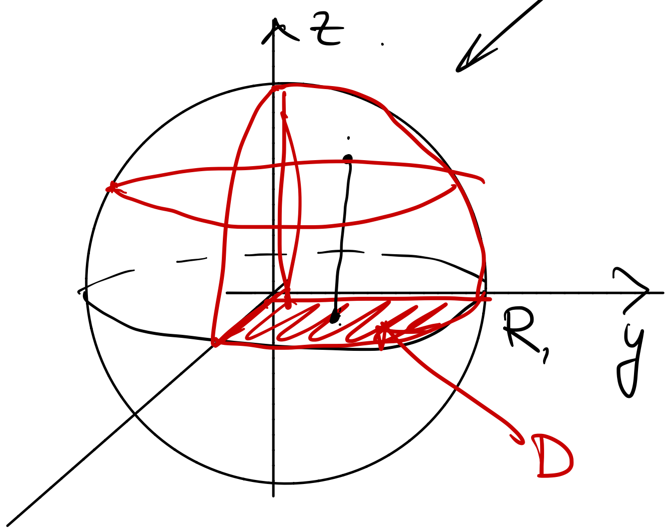
$$= 8\int_0^R dx \left(\cancel{\frac{1}{2} y \sqrt{R^2-x^2-y^2}} + \frac{R^2-x^2}{2} \operatorname{arcsen}\left(\frac{y}{\sqrt{R^2-x^2}}\right)\right)\Bigg|_{y=0}^{y=\sqrt{R^2-x^2}}$$

$$= 4\int_0^R dx \left(R^2-x^2\right)\left(\frac{\pi}{2}\right) = 2\pi \int_0^R dx \left(R^2-x^2\right) =$$

$$= 2\pi \left(R^3 - \frac{R^3}{3}\right) = 2\pi \frac{2}{3} R^3 = \frac{4}{3}\pi R^3 .$$

2° modo (più facile): affettando la palla:

$x^2+y^2+z^2 \leq R^2$



Se fisso $z=z_0$ e taglio la palla con il piano $z=z_0$, ottengo un cerchio di centro $(0,0)$ e raggio $\sqrt{R^2-z_0^2}$

$(x^2+y^2 \leq R^2-z_0^2)$

La sua area sarà

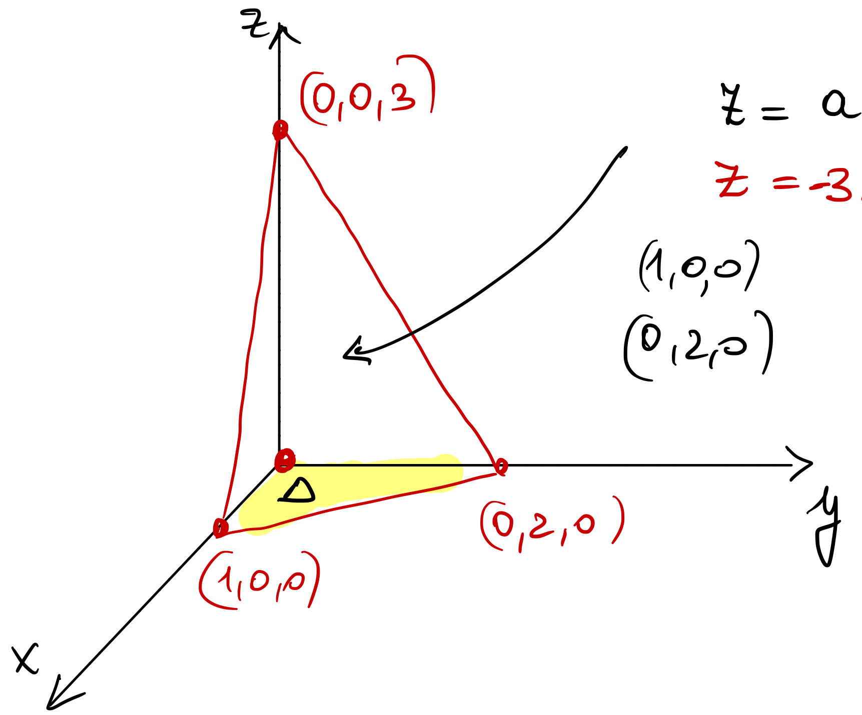
$\pi(R^2-z_0^2)$

$$\mathrm{Vol}\, B_R = 2\int_0^R dz \text{ area(cerchio intersezione a quota } z)$$

$$= 2\int_0^R dz\ \pi(R^2-z^2) = 2\pi\int_0^R (R^2-z^2)\,dz =$$

$$= \frac{4}{3}\pi R^3$$

Calcolare il volume del tetraedro T di vertici $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$.



$z = ax + by + 3$

$z = -3x - \frac{3}{2}y + 3$

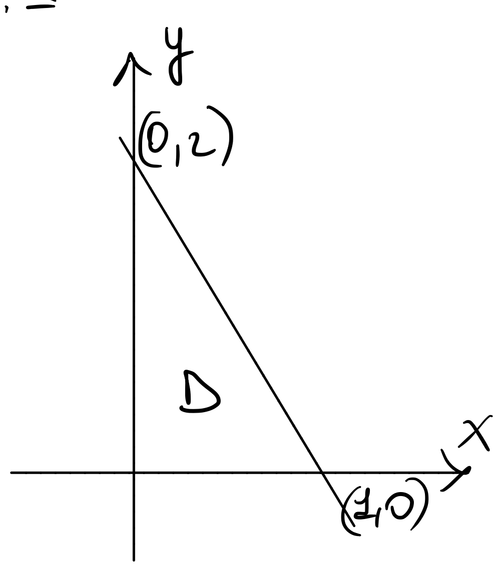
$(1,0,0)$ $\quad 0 = a + 3 \quad a = -3$

$(0,2,0)$ $\quad 0 = 2b + 3 \quad b = -\frac{3}{2}$

$$\mathrm{vol}(T) = \iint_D \left(-3x - \frac{3}{2}y + 3\right) dx\,dy =$$

$$= 3 \iint_D \left(-x - \frac{y}{2} + 1\right) dx\,dy =$$

$$= 3 \int_0^1 dx \int_0^{2-2x} dy \left(-x - \frac{y}{2} + 1\right) =$$



$$= 3 \int_0^1 dx \left[(1-x)\,2(1-x) - \frac{y^2}{4}\Big|_{y=0}^{y=2(1-x)} \right] =$$

$$= 3 \int_0^1 dx \left[2(1-x)^2 - (1-x)^2 \right] = 3 \int_0^1 (1-x)^2\, dx =$$

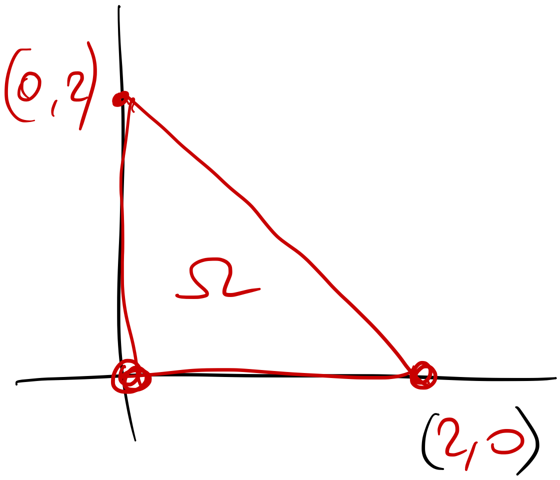
$$= 3 \int_0^1 (x-1)^2\, dx = \cancel{3}\, \frac{(x-1)^3}{\cancel{3}} \Big|_0^1 = 1$$

in accordo con la formula

$$\text{Vol } T = \frac{\text{area base} \cdot h}{3}$$

$$\iint_{\Omega} (x-2y)\,dx\,dy = (*)$$

Ω è il triangolo di vertici $(0,0)$ $(0,2)$, $(2,0)$.



$$(*) = \int_0^2 dx \left(\int_0^{2-x} dy\, (x-2y) \right) =$$

$$= \int_0^2 dx \left[x(2-x) - (2-x)^2 \right]$$

$$= \int_0^2 dx \left[2x - x^2 - 4 + 4x - x^2 \right] =$$

$$= \int_0^2 dx \left(-2x^2 + 6x - 4 \right) =$$

$$= -\frac{2}{3} \cdot 8 + \frac{6}{2} \cdot 4 - 4 \cdot 2 = -\frac{16}{3} + 12 - 8 =$$

$$= -\frac{4}{3}.$$

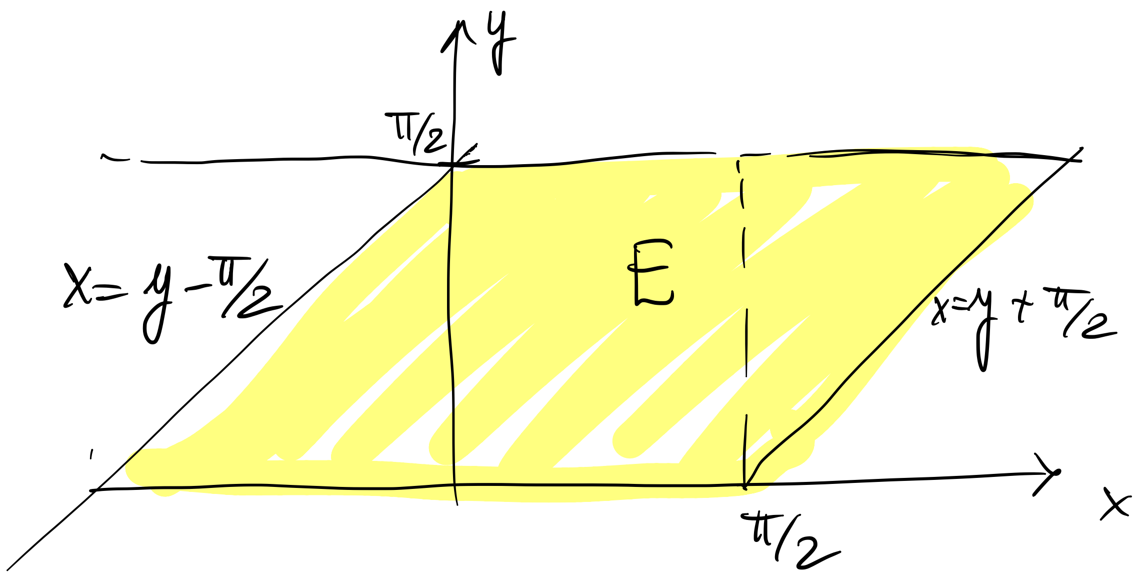
$$\iint_E y \cos^2(y-x)\, dx\, dy$$

$$E = \left\{ (x,y) \in \mathbb{R}^2 : \quad 0 \le y \le \frac{\pi}{2}, \quad |y-x| \le \frac{\pi}{2} \right\}$$

$$\Updownarrow$$

$$-\frac{\pi}{2} \le x - y \le \frac{\pi}{2}$$

$$y - \frac{\pi}{2} \le x \le y + \frac{\pi}{2}$$



$$\iint_E y \cos^2(y-x)\, dx\, dy = \int_0^{\pi/2} dy \int_{y-\frac{\pi}{2}}^{y+\pi/2} dx\; y \cos^2(y-x) =$$

$$= \frac{1}{2}\int_0^{\pi/2} dy \; y \int_{y-\pi/2}^{y+\pi/2} \left(1 + \cos\left(2(y-x)\right)\right) dx$$

$$\cos^2 t = \frac{1+\cos(2t)}{2}$$

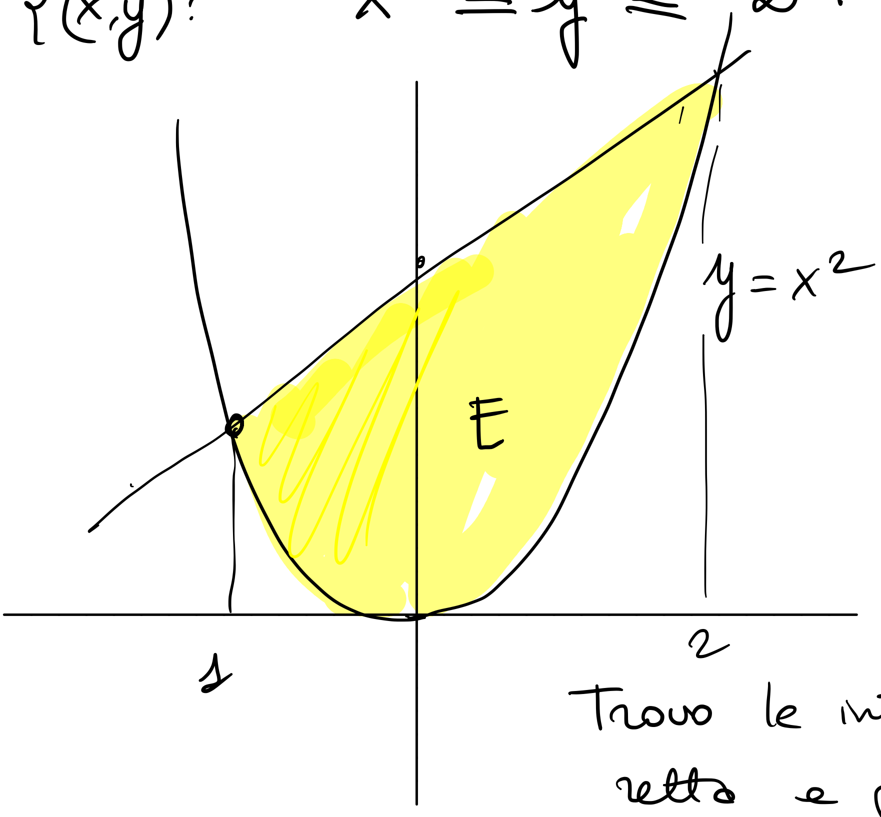
$$= \frac{1}{2}\int_0^{\pi/2} dy \; y \left(\int_{y-\pi/2}^{y+\pi/2} \left(1 + \cos\left(2(y-x)\right)\right) dx \right) =$$

$$= \frac{1}{2}\int_0^{\pi/2} dy \; y \left(\pi - \frac{1}{2}\cancel{\left(\operatorname{sen}\left(2(y-x)\right)\right)\Big|_{x=y-\pi/2}^{x=y+\pi/2}} \right) =$$

$$= \frac{\pi}{2} \int_0^{\pi/2} dy \; y = \frac{\pi}{2} \, \frac{1}{2} \left(\frac{\pi}{2}\right)^2 = \frac{\pi^3}{16}$$

$$\iint_E xy \, dx\,dy$$

$$E = \{(x,y): \quad x^2 \le y \le 2+x\}$$



Trovo le intersezioni tra retta e parabola.

$$x^2 = 2+x \qquad x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} -1 \\ 2. \end{cases}$$

$$\Rightarrow E = \{(x,y): \quad -1 \le x \le 2, \quad x^2 \le y \le 2+x\}$$

$$\iint_E xy \, dx\,dy = \int_{-1}^{2} dx \int_{x^2}^{2+x} dy \; x \, y =$$

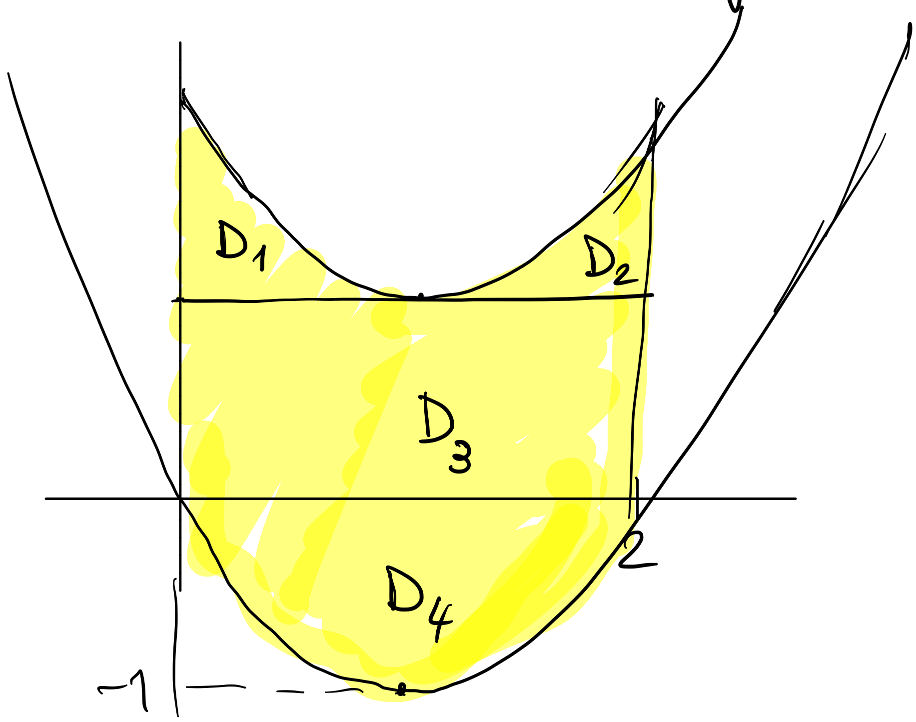
$$= \frac{1}{2}\int_{-1}^{2} dx \; x\left[(2+x)^2 - x^4\right] = \frac{1}{2}\int_{-1}^{2} dx \; x\left(4+x+x^2-x^4\right) =$$

Esercizio: Disegnare l'insieme D t.c.

$$\iint_D f(x,y)\,dx\,dy = \int_0^2 \left(\int_{x^2-2x}^{x^2-2x+2} f(x,y)\,dy \right) dx$$

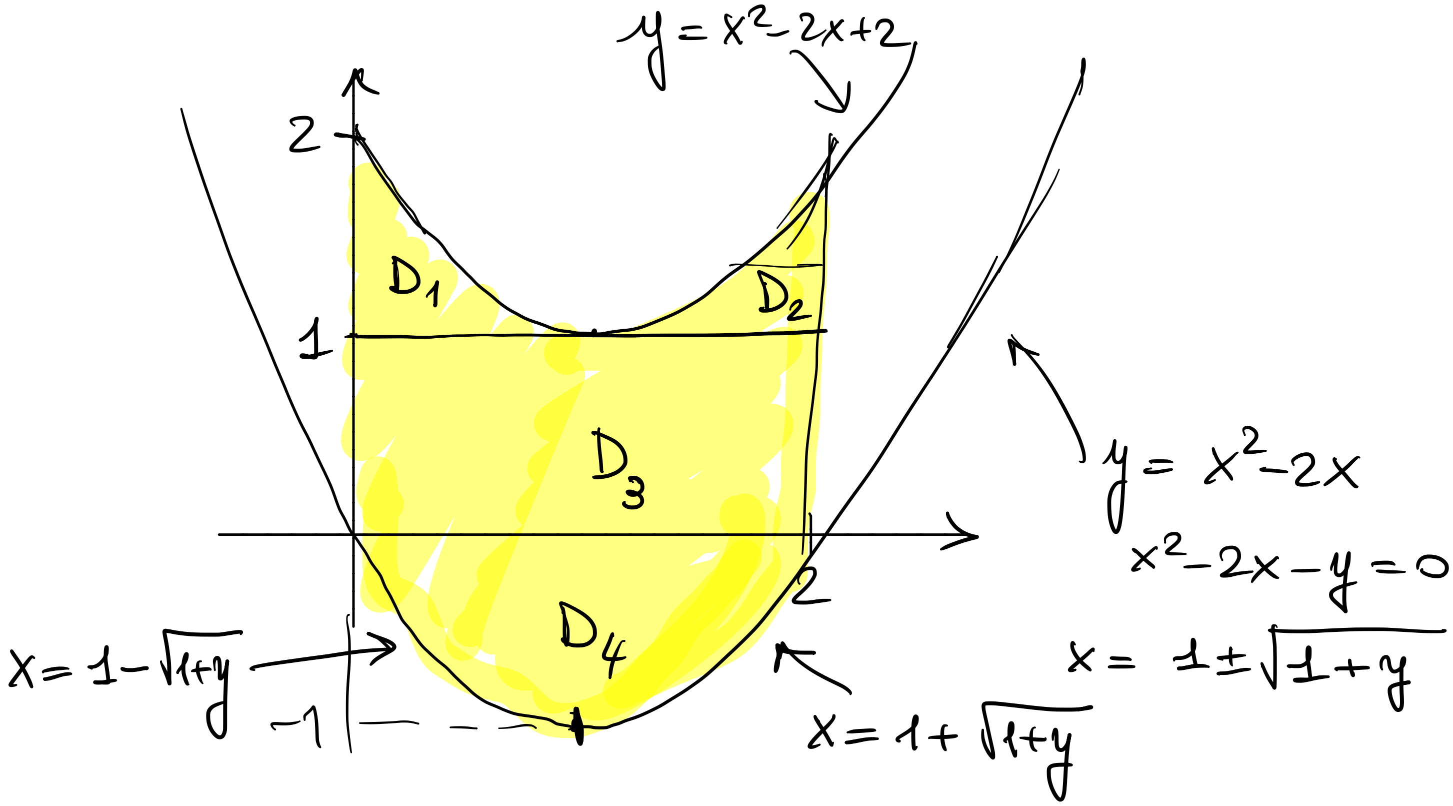
$\forall f$ funzione continua, e scrivere la formula per invertire l'ordine di integrazione delle variabili.

$$D = \{(x,y):\ 0 \le x \le 2,\ x^2-2x \le y \le x^2-2x+2\}$$



$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f\,dx\,dy + \iint_{D_2} \cdots + \iint_{D_3} \cdots +$$

$$+ \iint_{D_4} \cdots$$



$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f\,dx\,dy + \iint_{D_2} \cdots + \iint_{D_3} \cdots + \iint_{D_4} \cdots$$

$$\iint_{D_3} f(x,y)\,dx\,dy = \int_0^1 dy \left(\int_0^2 dx \; f(x,y) \right)$$

$$\iint_{D_4} f(x,y)\,dx\,dy = \int_{-1}^0 dy \left(\int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dx \; f(x,y) \right)$$

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_1^2 dy \left(\int_0^{1-\sqrt{y-1}} dx \; f(x,y) \right)$$

$$\iint_{D_2} f(x,y)\,dx\,dy = \int_1^2 dy \left(\int_{1+\sqrt{y-1}}^2 dx \; f(x,y) \right)$$

$$y = x^2 - 2x + 2$$

$$x^2 - 2x + 2 - y = 0$$

$$x = 1 \pm \sqrt{1-(2-y)} = 1 \pm \sqrt{y-1}$$