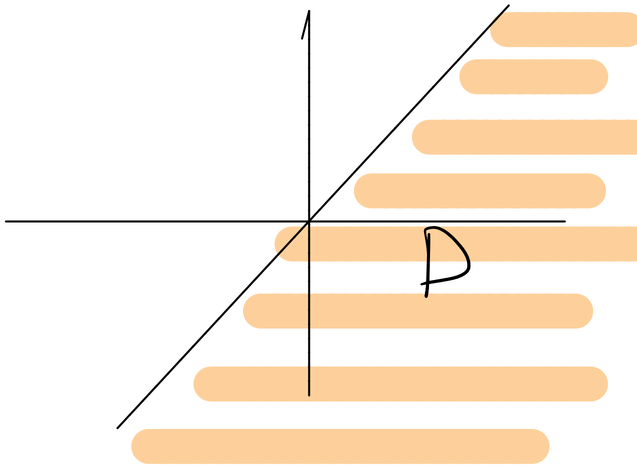
Limiti di funzioni di più variabili

$f(x,y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

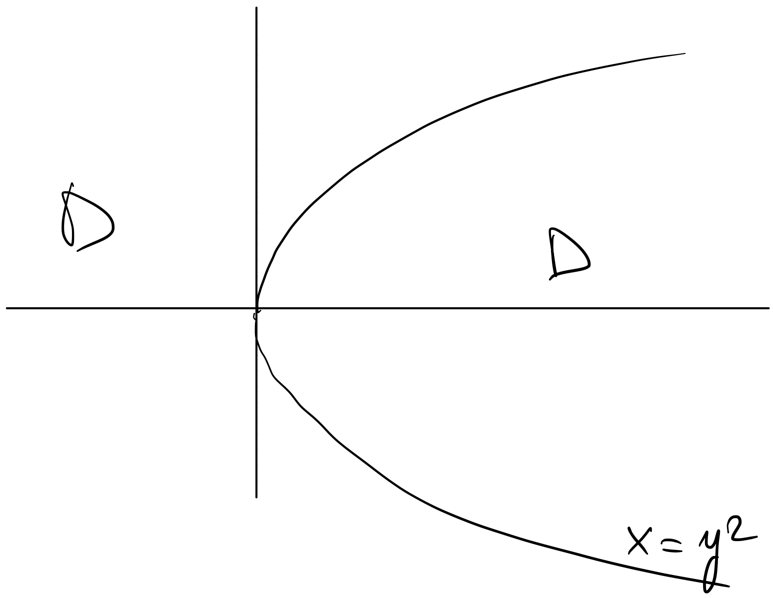
Esempi:

$f(x,y) = x^2 + y^2$ $\qquad D = \mathbb{R}^2$

$f(x,y) = \sqrt{x-y}$ $\qquad D = \{(x,y) \in \mathbb{R}^2 : x \geq y\}$



$f(x,y) = \dfrac{x^2 y + 3x}{x - y^2}$ $\qquad D = \{(x,y) \text{ t.c. } x \neq y^2\}$



Limiti in 1 variabile $\quad f(x): D \subset \mathbb{R} \to \mathbb{R}$

$\lim_{x \to 1} f(x) = 5$ significa:

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ t.c. $|f(x) - 5| < \varepsilon \quad \forall x \in D$ t.c.

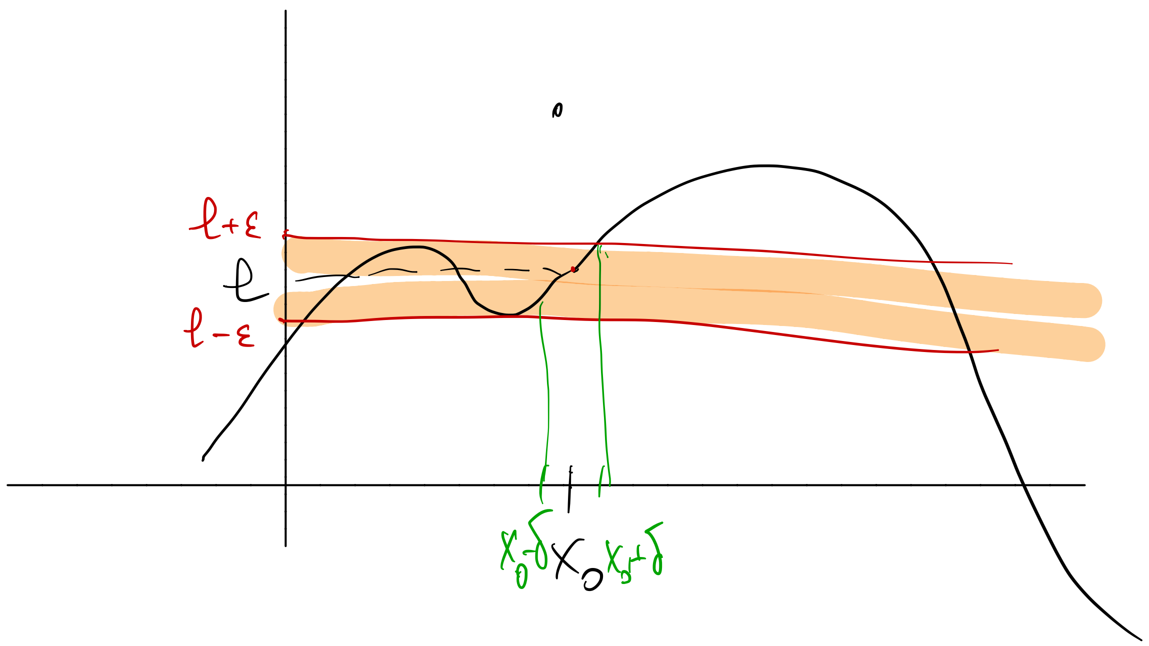
$0 < |x - 1| < \delta$

$\lim_{x \to x_0} f(x) = \ell$ significa: $\quad (x_0, \ell \in \mathbb{R})$

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ t.c. $|f(x) - \ell| < \varepsilon \quad \forall x \in D$ t.c.

$0 < |x - x_0| < \delta$

Graficamente:



$$\lim_{x \to x_0} f(x) = \ell \quad \text{significa:}$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{t.c.} \quad \forall x \in D \ \text{verificante} \ 0 < |x - x_0| < \delta$$

$$\text{si ha} \quad |f(x) - \ell| < \varepsilon$$

Cosa vuol dire

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = \ell \in \mathbb{R} \qquad ?$$

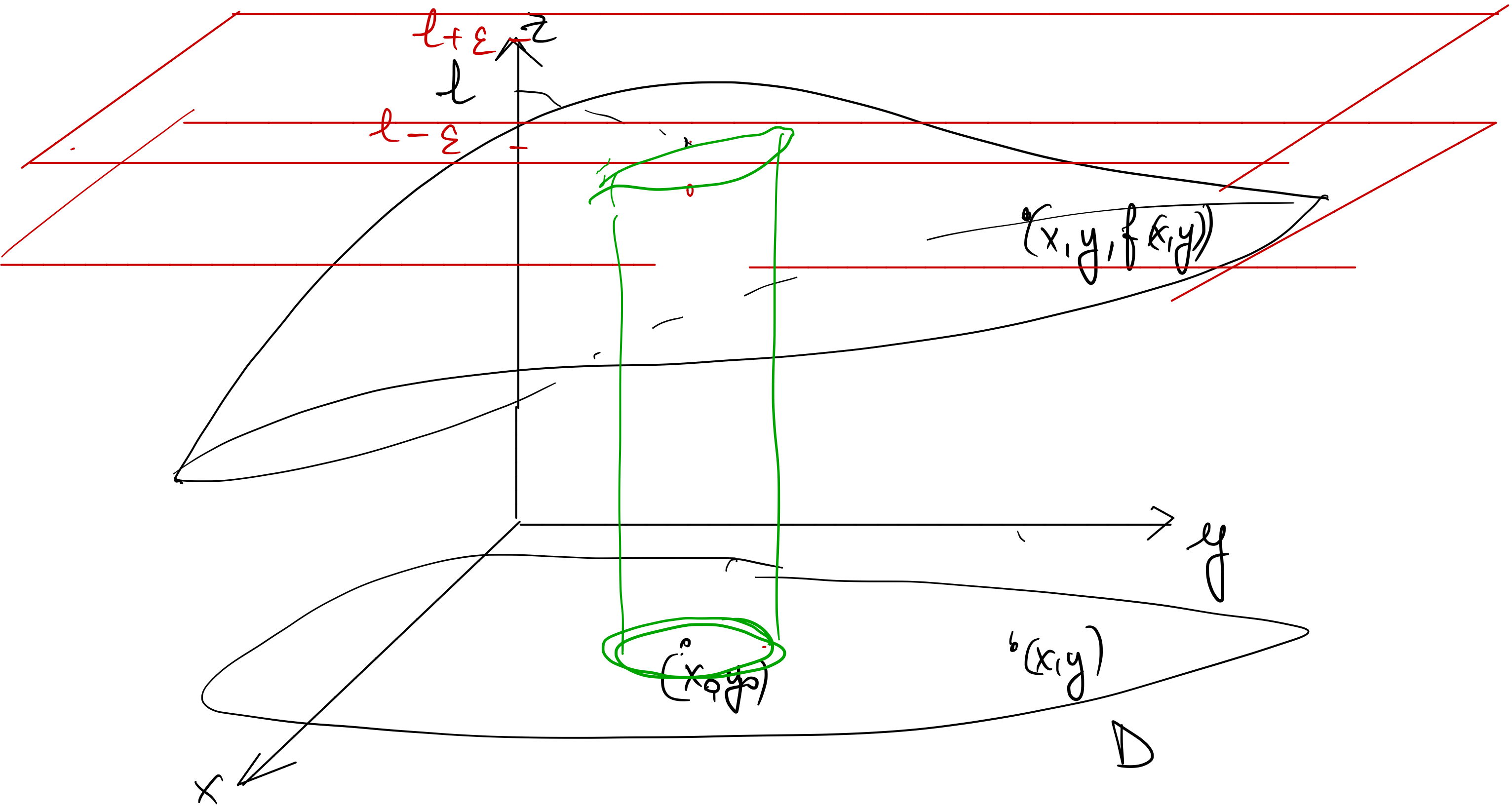
$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ t.c. $\forall (x,y) \in D$ verificante

$$0 < d\big((x,y),(x_0,y_0)\big) < \delta \quad \text{si ha} \quad |f(x,y) - \ell| < \varepsilon$$

$$d\big((x,y),(x_0,y_0)\big) = |(x - x_0, y - y_0)| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \text{t.c.} \quad \forall (x,y)\in D \text{ verificante}$$

$$0<|(x,y)-(x_0,y_0)|<\delta \quad \text{si ha} \quad |f(x,y)-\ell|<\varepsilon$$



Esempio: $\lim_{(x,y)\to(0,0)} x^2y = 0$ $\quad D=\mathbb{R}^2$

Fisso $\varepsilon>0$. Cerco $\delta>0$ t.c. se (x,y) verifica

$0<\sqrt{x^2+y^2}<\delta$ si ha $|x^2y-0|<\varepsilon$.

$$|x^2y| = x^2|y| \le (x^2+y^2)|y| \le (x^2+y^2)\sqrt{x^2+y^2} =$$

OSS $x^2 \le x^2+y^2<\delta^2$

$|y| = \sqrt{y^2} \le \sqrt{x^2+y^2}<\delta$

$$= \left(\sqrt{x^2+y^2}\right)^3 < \delta^3 \le \varepsilon$$

se scelgo $\delta \le \sqrt[3]{\varepsilon}$

Basta scegliere $\delta = \sqrt[3]{\varepsilon}$

OSS. Se prendiamo $\tilde{f}(x,y) = \begin{cases} x^2y & \text{se } (x,y)\neq(0,0) \\ 5 & \text{se } (x,y)=(0,0) \end{cases}$

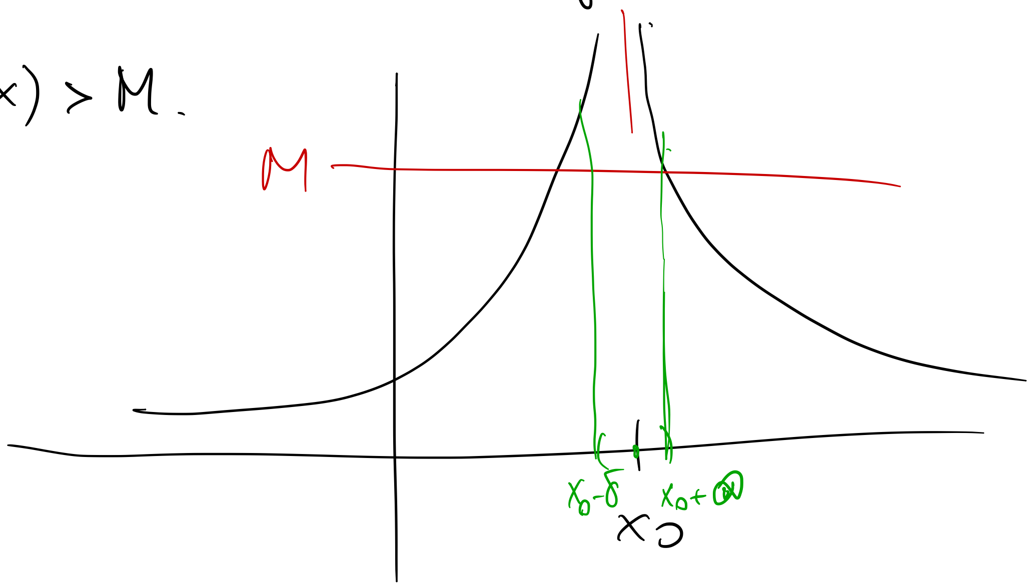
il limite $\lim_{(x,y)\to(0,0)} \tilde{f}(x,y)$ resta zero.

In 1 variabile

$$\lim_{x \to x_0} f(x) = +\infty \quad \text{significa:}$$

$\forall M > 0 \ \exists \delta > 0$ t.c. $\forall x \in D$ verificante $0 < |x - x_0| < \delta$

si ha $f(x) > M$.

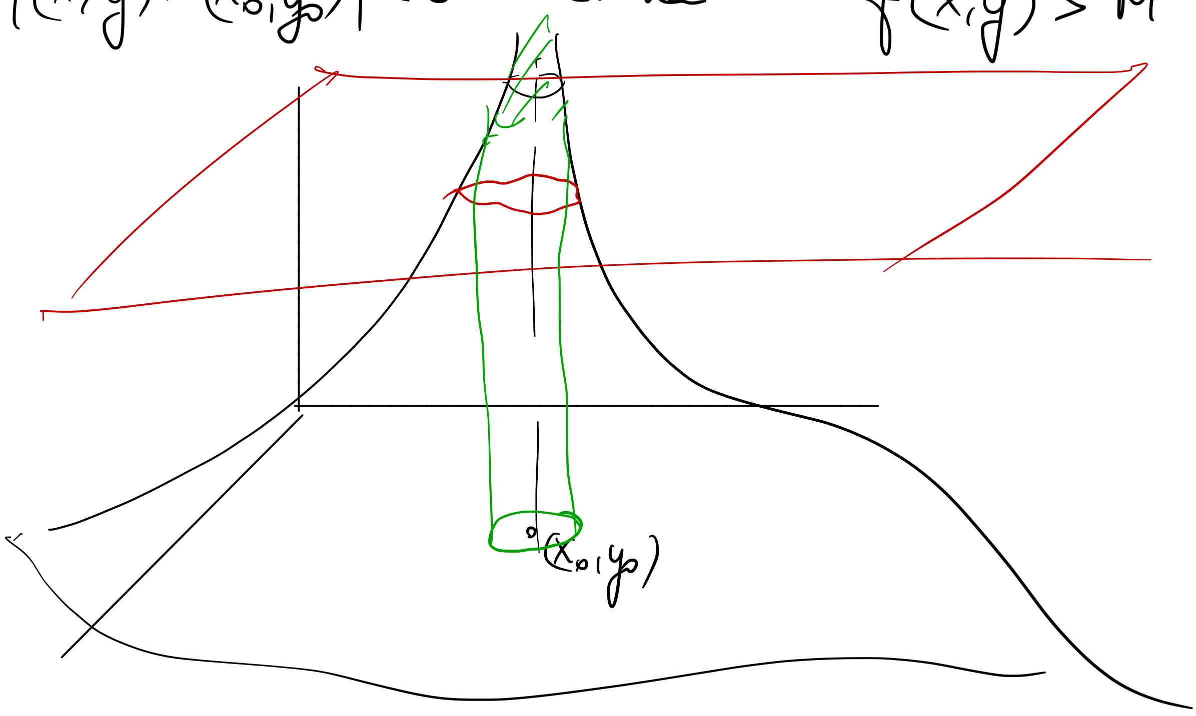


In due variabili diventa:

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = +\infty \quad \text{significa:}$$

$\forall M > 0 \ \exists \delta > 0$ t.c. $\forall (x,y) \in D$ verificante

$0 < |(x,y) - (x_0,y_0)| < \delta$ si ha $f(x,y) > M$



$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty .$$

Verifica:

Fisso $M>0$. Cerco $\delta>0$, t.c.

se (x,y) verifica $0<\underbrace{\sqrt{x^2+y^2}<\delta}_{x^2+y^2<\delta^2}$ si ha

$$\frac{1}{x^2+y^2} > M.$$

$$\frac{1}{x^2+y^2} > \frac{1}{\delta^2} \geq M$$

basta prendere $\delta = \frac{1}{\sqrt{M}}$