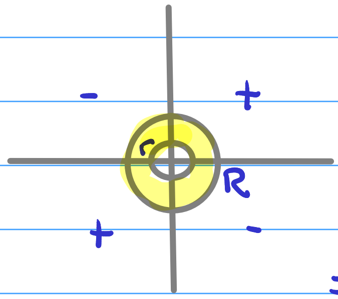
01/12/2025 $\quad C_{R,r} = \{ r^2 \leq x_1^2+x_2^2 \leq R^2 \} \subseteq \mathbb{R}^2 \qquad 0 \leq r < R$

$$\int_{C_{R,r}} |x_1 x_2|\,dx = 4 \int_{C_{R,r}\cap\{x_1,x_2\geq 0\}} x_1 x_2\,dx \overset{(C.P.)}{=} 4 \int_r^R \left[\int_0^{\pi/2} \rho^2 \sin(\vartheta)\cos(\vartheta)\cdot \underbrace{\rho}_{|\det(J)|}\, d\vartheta\right] d\rho$$



$$C^+_{R,r} = \left\{ r \leq \rho \leq R,\ 0 \leq \vartheta \leq \frac{\pi}{2} \right\} = [r,R]\times\left[0,\frac{\pi}{2}\right]$$

$$= 2\left[\int_r^R \rho^3\,d\rho\right]\cdot\left[\int_0^{\pi/2} \sin(2\vartheta)\,d\vartheta\right]$$

$$= 2\left[\frac{1}{4}\rho^4\right]_r^R \cdot \left[-\frac{1}{2}\cos(2\vartheta)\right]_0^{\pi/2} = \frac{1}{4}\left(R^4-r^4\right)\cdot 2 = \frac{1}{2}\left(R^4-r^4\right)$$

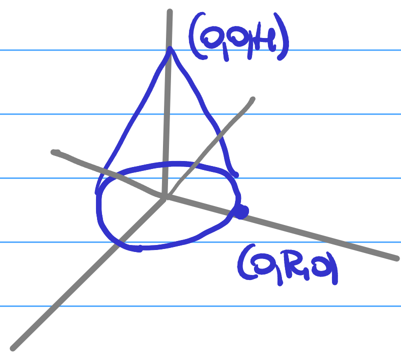
ESERCIZIO

C cono di raggio R e altezza H

densità di massa $d(x)=\delta_0>0$

¿ Baricentro e momento di inerzia di C ?

$$M = \int_C \delta_0\,dx = \delta_0 |C| = \frac{\pi}{3}\delta_0 H R^2$$

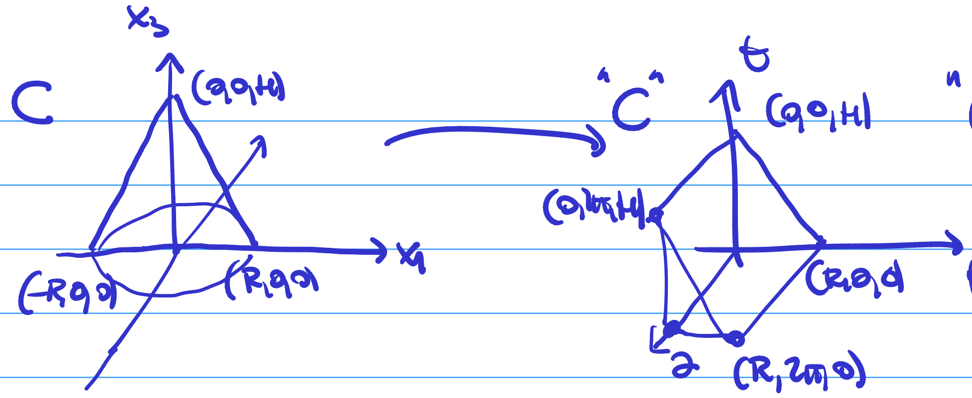


$$G_i = \frac{1}{M}\int_C \delta_0 x_i\,dx \qquad (G_1 = G_2 = 0)$$

$$G_3 = \frac{1}{M}\delta_0 \int_C x_3\,dx = \frac{\delta_0}{M}\int_{C''} t\,\rho\,d\vartheta\,d\rho\,dt$$

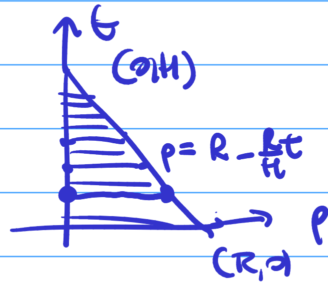
$$\begin{cases} x_1 = \rho\cos(\vartheta) \\ x_2 = \rho\sin(\vartheta) \\ x_3 = t \end{cases} \qquad J = \begin{pmatrix} \cos(\vartheta) & -\rho\sin(\vartheta) & 0 \\ \sin(\vartheta) & \rho\cos(\vartheta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(J) = \rho$$



"C" $= T \times [0,2\pi]$

$\uparrow \{\rho,t\}$ $\quad \uparrow \vartheta$



$$T \times [0,2\pi] = \{0 \le t \le H,\ 0 \le \rho \le R(1-\tfrac{1}{H}t)\} \times [0,2\pi]$$

$$= \frac{\delta_0}{m}\int_0^H \left[\int_0^{R(1-\frac{1}{H}t)} \left[\int_0^{2\pi} \rho t \, d\vartheta\right] d\rho\right] dt = \frac{2\pi\delta_0}{m}\int_0^H t\left[\tfrac{1}{2}\rho^2\right]_0^{R(1-\frac{1}{H}t)} dt$$

$$= \frac{\pi\delta_0}{m}\int_0^H R^2\left(1-\tfrac{1}{H}t\right)^2 t \, dt = \frac{\pi\delta_0 R^2}{m}\int_0^H \left(t - \tfrac{2}{H}t^2 + \tfrac{1}{H^2}t^3\right)dt$$

$$= \frac{\pi\delta_0 R^2}{m}\left[\tfrac{1}{2}t^2 - \tfrac{2}{3H}t^3 + \tfrac{1}{4H^2}t^4\right]_0^H = \frac{\pi\delta_0}{m}R^2\left[\tfrac{1}{2}H^2 - \tfrac{2}{3}H^2 + \tfrac{1}{4}H^2\right]$$

$$= \frac{\pi\delta_0}{12m}R^2H^2 = G_3 \Rightarrow G = \left(0,0,\frac{\pi\delta_0}{12m}R^2H^2\right)$$

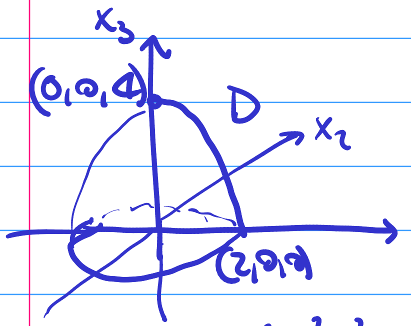
$$I_3 = \int_C \delta_0(x_1^2+x_2^2)\,dx \underset{(C.C.)}{=} \int_{\text{"C"}} \delta_0 \rho^2 \cdot \rho \, d\rho\, d\vartheta\, dt = \int_0^H \int_0^{R(1-\frac{1}{H}t)} \delta_0 2\pi \rho^3 \, d\rho\, dt$$

$$= \int_0^R \delta_0 \int_0^{H(1-\frac{1}{R}\rho)} 2\pi\rho^3 \, dt\, d\rho = \int_0^R \delta_0 2\pi\rho^3 \cdot H\left(1-\tfrac{1}{R}\rho\right)d\rho = 2\pi H\delta_0\left[\tfrac{1}{4}\rho^4 - \tfrac{1}{5R}\rho^5\right]_0^R$$

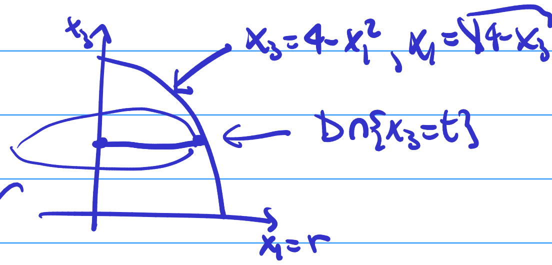
$$\rho = R\left(1-\tfrac{1}{H}t\right) \longrightarrow t = H\left(1-\tfrac{\rho}{R}\right)$$

$$= \delta_0 2\pi H \cdot \tfrac{1}{20}R^4 = \tfrac{1}{10}\pi H R^4 \delta_0$$

ESERCIZIO. $m_3(D)$, dove $D=\{0\le x_3\le 4-x_1^2-x_2^2\}\subseteq\mathbb{R}^3$



$$\begin{cases} x_1=\rho\cos(\vartheta) \\ x_2=\rho\,\mathrm{sen}(\vartheta) \\ x_3=t \end{cases} \longrightarrow x_1^2+x_2^2=\rho^2$$

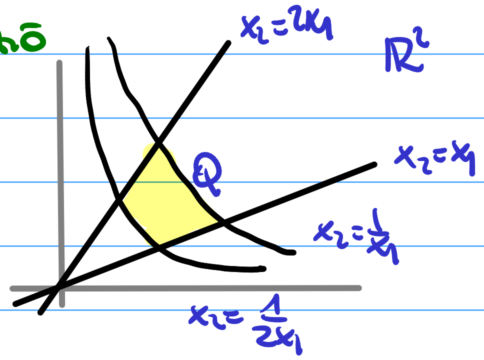


$$|D|=\iint_{\{x_1^2+x_2^2\le 4\}}\left[\int_0^{4-x_1^2-x_2^2}dx_3\right]dx_1dx_2$$

$$|D|=\int_0^4\left[\iint_{D\cap\{x_3=t\}}dx_1dx_2\right]dx_3=\int_0^4\pi(4-x_3)\,dx_3=\pi\left[4x_3-\frac{1}{2}x_3^2\right]_0^4=8\pi$$

$$|D|\underset{(c.c.)}{=}\int_0^{2\pi}\left[\iint_T\rho\,d\rho\,dt\right]d\vartheta$$

ESERCIZIO



$\mathbb{R}^2$

$m_2(Q)=|Q|$

$$Q=\left\{1\le\frac{x_2}{x_1}\le 2;\ \frac{1}{2}\le x_1x_2\le 1\right\}$$

$$\begin{cases} u_1=\dfrac{x_2}{x_1} \\ u_2=x_1x_2 \end{cases}$$

$$x_2=u_1x_1\longrightarrow u_2=x_1\cdot u_1x_1=u_1x_1^2\longrightarrow\begin{cases} x_1=\sqrt{\dfrac{u_2}{u_1}}=u_2^{1/2}u_1^{-1/2} \\ x_2=u_1\sqrt{\dfrac{u_2}{u_1}}=u_2^{1/2}u_1^{1/2} \end{cases}$$

$$J=\begin{pmatrix} -\frac{1}{2}u_2^{1/2}u_1^{-3/2} & \frac{1}{2}u_2^{-1/2}u_1^{-1/2} \\ \frac{1}{2}u_1^{-1/2}u_2^{1/2} & \frac{1}{2}u_1^{1/2}u_2^{-1/2} \end{pmatrix}$$

$$|\det(J)|=\frac{1}{4}\left(u_1^{-1}+u_1^{-1}\right|=\frac{1}{2u_1}$$

$$\Rightarrow |D| = \int_1^2 \left[\int_{1/2}^{1} \frac{1}{2u_1}\, du_2\right] du_1 = \frac{1}{4}\int_1^2 \frac{1}{u_1}\, du_1 = \frac{1}{4} \ln(u_1)\Big]_1^2$$

$$= \frac{1}{4}\ln(2)$$

ESERCIZIO IMPORTANTE

$$B(0,R) \subseteq \mathbb{R}^n \qquad ¿\, m_n(B(0,R))?$$

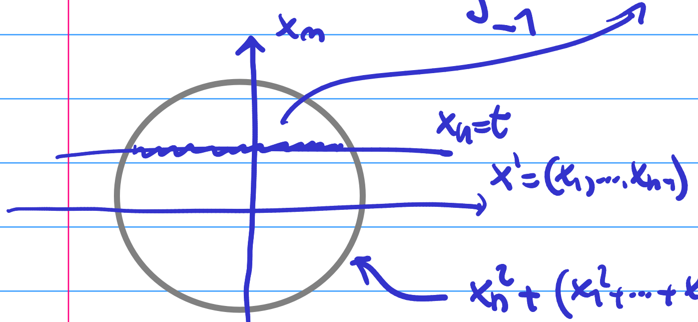
OSSERVAZIONE $\quad x = Ry \quad x, y \in \mathbb{R}^n \quad (R>0)$

$$B(0,R) \qquad B(0,1) \qquad x = R I_n y = My$$

$$|\det(J)| = |\det(M)| = |\det(R I_n)| = R^n$$

$$|B(0,R)| = \int_{B(0,R)} dx = \int_{B(0,1)} |\det(R I_n)|\, dy = R^n \int_{B(0,1)} dy = R^n |B(0,1)|$$

$$m_n(B(0,1)) = \int_{-1}^{1} m_{n-1}(B(0,1) \cap \{x_n = t\})\, dt = \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} m_{n-1}(B(0,1))\, dt$$



$$x_n^2 + (x_1^2 + \dots + x_{n-1}^2) = 1$$

$B(0,1) \subseteq \mathbb{R}^n$

$$(x_1^2 + \dots + x_{n-1}^2) = 1 - t^2 \Rightarrow r = \sqrt{1-t^2}$$

$$\Rightarrow \underbrace{m_n(B(0,1))}_{\omega_n} = \underbrace{m_{n-1}(B(0,1))}_{\omega_{n-1}} \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} dt$$

$$\int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} dt \underset{t = \operatorname{sen}(w)}{=} 2\int_0^{\pi/2} \cos^{n-1}(w)(+\cos(w))\, dw = +2\int_0^{\frac{\pi}{2}} \cos^n(w)\, dw$$

$$\int_0^{\frac{\pi}{2}} \underbrace{\cos^{n-1}(w)}_{g(w)} \underbrace{\cos(w)}_{(\mathrm{sen}(w))'} dw = \cancel{\left[\cos^{n-1}(w)\,\mathrm{sen}(w)\right]_0^{\frac{\pi}{2}}} + \int_0^{\frac{\pi}{2}} \mathrm{sen}^2(w)\cdot(n-1)\cos^{n-2}(w)\,dw$$

$$= (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2(w))\cos^{n-2}(w)\,dw$$

$$\Rightarrow \quad n\int_0^{\pi/2} \cos^n(w)\,dw = (n-1)\int_0^{\pi/2}\cos^{n-2}(w)\,dw$$

$$\Rightarrow \quad \int_0^{\pi/2} \cos^n(w)\,dw = \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}(w)\,dw$$

$$n=2k \Rightarrow \int_0^{\pi/2}\cos^{2k}(w)\,dw = \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}$$

$$n=2k+1 \Rightarrow \int_0^{\pi/2}\cos^{2k+1}(w)\,dw = \frac{(2k)!!}{(2k+1)!!}$$

$$\omega_{2k} = m_{2k}(B(0,1)) = \frac{(2k-1)!!}{(2k)!!}\,\pi\,\omega_{2k-1}$$

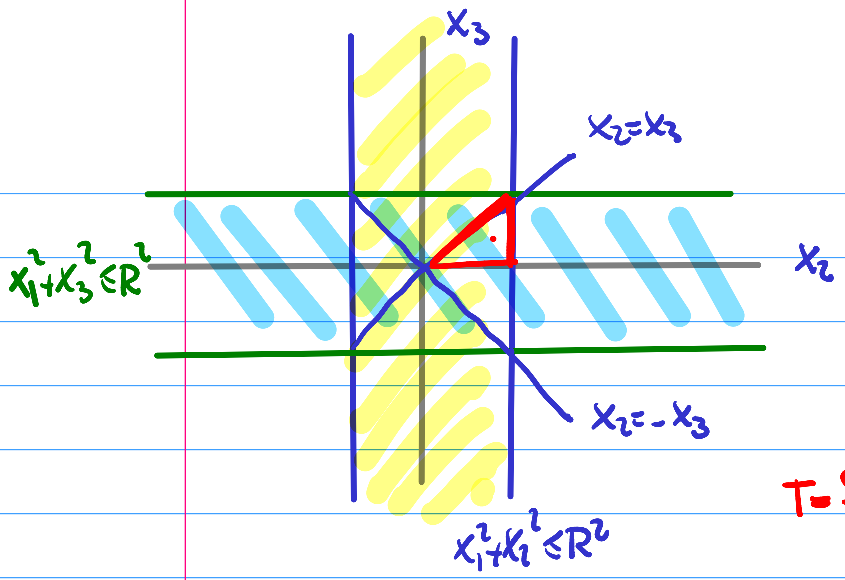
$$\omega_{2k+1} = m_{2k+1}(B(0,1)) = \frac{(2k)!!}{(2k+1)!!}\,2\,\omega_{2k}$$

$$\frac{\omega_{2k+1}}{\omega_{2k-1}} = \frac{(2k)!!}{(2k+1)!!}\cdot 2\cdot\frac{(2k-1)!!}{(2k)!!}\,\pi = \frac{2\pi}{2k+1}$$

$$\Rightarrow \quad \omega_n = m_n(B(0,1)) \xrightarrow[n\to+\infty]{} 0.$$

ESERCIZIO

¿ $m_3(D)$? dove $D = \{x_1^2+x_2^2 \le R^2\} \cap \{x_1^2+x_3^2 \le R^2\} \subset \mathbb{R}^3$



$$\begin{cases} x_1^2+x_2^2=R^2 \\ x_1^2+x_3^2=R^2 \end{cases} \longrightarrow x_2^2=x_3^2 \;\Downarrow\; x_2=\pm x_3$$

$$T=\{0\le x_2\le R\,;\,0\le x_3\le x_2\}$$

$$x_1=\sqrt{R^2-x_3^2}$$

$$\Rightarrow m_3(D)=16\int_T\sqrt{R^2-x_3^2}\,dx_2\,dx_3=16\int_0^R\left(\int_0^{x_3}\sqrt{R^2-x_3^2}\,dx_2\,dx_3\right)$$

Esercizio

$$\int_{\mathbb{R}^2}e^{-(x_1^2+x_2^2)}dx=\left[\int_{\mathbb{R}}e^{-x_1^2}dx_1\right]\left[\int_{\mathbb{R}}e^{-x_2^2}dx_2\right]=\left[\int_{\mathbb{R}}e^{-t^2}dt\right]^2$$

$$\Big\downarrow (C.P.)$$

$$\int_0^{+\infty}\int_0^{2\pi}e^{-\rho^2}\rho\,d\rho\,d\theta=2\pi\left(-\frac{1}{2}\right)\int_0^{+\infty}-2\rho e^{-\rho^2}d\rho=-\pi\left[e^{-\rho^2}\right]_0^{+\infty}$$

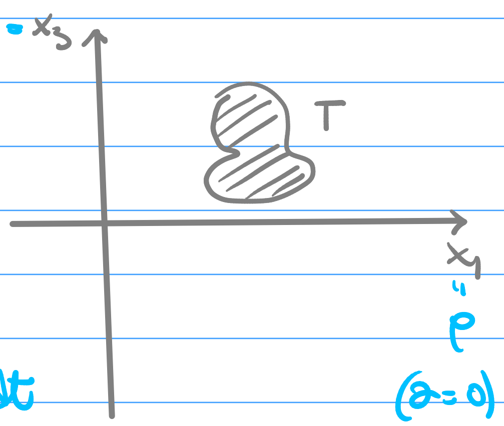
"

$$\pi=\lim_{R\to+\infty}\pi\left[-e^{-R^2}+1\right]=-\pi\lim_{R\to+\infty}\left[e^{-\rho^2}\right]_0^R$$

$$\Rightarrow\int_{-\infty}^{+\infty}e^{-t^2}dt=\sqrt{\pi}$$

" $B(0,R)\xrightarrow[R\to+\infty]{}\mathbb{R}^2$ "

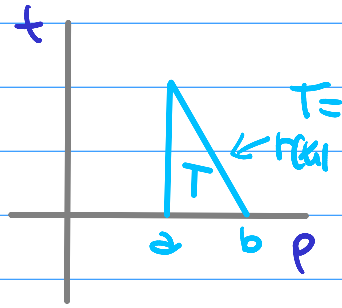
ESERCIZIO IMPORTANTE



$T \subseteq \{x_2 = 0\}$ limitata e misurabile

S solido generato dalla rotazione di T attorno all'asse x_3

$$m_3(S) = \int_0^{2\pi} \left[\iint_{"T"} \rho \, d\rho \, dt \right] d\theta = 2\pi \iint_T \rho \, d\rho \, dt$$



$T = \{a \le x_1 \le b, \ 0 \le x_3 \le r(x_1)\}$

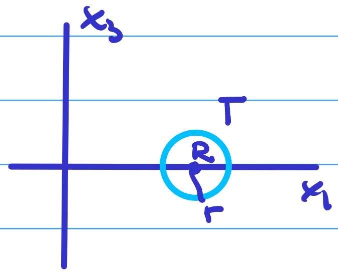
$T = \{a \le \rho \le b, \ 0 \le t \le r(\rho)\}$

$$\iint_T \rho \, d\rho \, dt = \iint_T x_1 \, dx_1 \, dx_3 = \iint_T dx_1 \, dx_3 \, G_1(T) \leftarrow$$ coordinata 1 del baricentro di T (x_1, x_3)

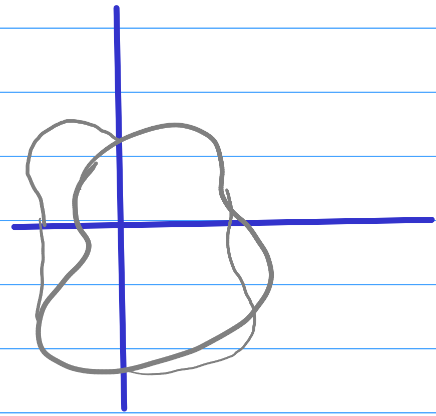
$$= m_2(T) \, G_1(T)$$

$$\Rightarrow \boxed{m_3(S) = 2\pi G_1(T) \cdot m_2(T)}$$

Teorema di Pappo & Guldino



T^2 $\quad m_3(T^2) = 2\pi \cdot \pi r^2 \cdot R = 2\pi^2 R r^2$



$$|D| = \int_{\{x_1^2+x_2^2\le 4\}} \left[\int_0^{4-x_1^2-x_2^2} dx_3\right] dx_1\,dx_2 = \int_{B(0,2)} (4-x_1^2-x_2^2)\,dx_1\,dx_2$$

$$\{-2\le x_1\le 2 \; ; \; -\sqrt{4-x_1^2}\le x_2\le \sqrt{4-x_1^2}\} = B(0,2)$$

$$= 4\int_0^2\int_0^{\sqrt{4-x_1^2}} (4-x_1^2-x_2^2)\,dx_2\,dx_1$$

$$= 4\int_0^2 \left[4\sqrt{4-x_1^2} - x_1^2\sqrt{4-x_1^2} - \frac{1}{3}(4-x_1^2)\sqrt{4-x_1^2}\right]dx_1$$

$$= \frac{8}{3}\int_0^2 \left[4\sqrt{4-x_1^2} - x_1^2\sqrt{4-x_1^2}\right]dx_1 \underset{\substack{x_1=2\sin(s)\\ 0\le s\le \frac{\pi}{2}}}{=} \frac{16}{3}\int_0^{\frac{\pi}{2}} \left[8\cos^2(s) - 8\cos^2(s)\sin^2(s)\right]ds$$

$$= \frac{16}{3}\int_0^{\pi/2} \left[8\cos^2(s) - 2\sin^2(2s)\right]ds$$

$$= \frac{16}{3}\left(8\cdot\frac{\pi}{4} - 2\cdot\frac{\pi}{4}\right) = 8\pi$$

$\sin^2(w)+\cos^2(w) = 1$

$\int_0^{2\pi}\left[\sin^2(w)+\cos^2(w)\right]dw = 2\pi$

$\Rightarrow \int_0^{2\pi}\sin^2(w)\,dw = \pi$

$\int_0^{\pi/2}\sin^2(w)\,dw = \frac{\pi}{4}$

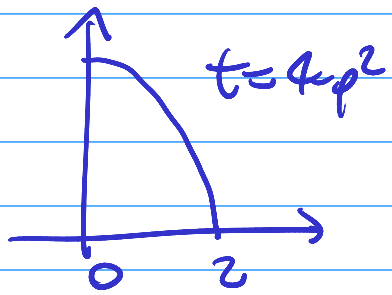
$$|D| = \int_0^{2\pi}\left[\iint_T \rho\,d\rho\,dt\right]d\theta$$

$$= 2\pi\int_T \rho\,d\rho\,dt$$

$$= 2\pi\int_0^2 \rho\left(\int_0^{4-\rho^2} dt\right)d\rho$$

$$= 2\pi\int_0^2 \rho(4-\rho^2)\,d\rho$$

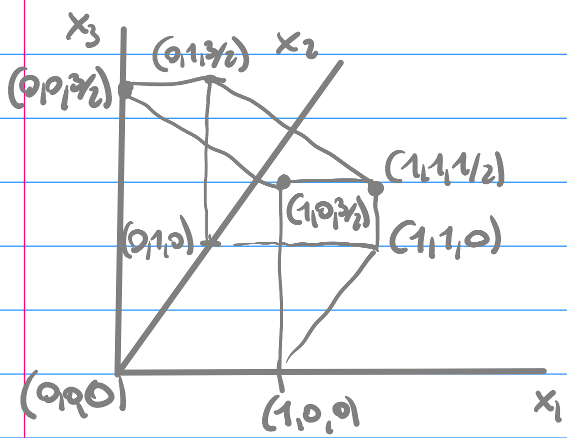
$$= 2\pi\left[2\rho^2 - \frac{1}{4}\rho^4\right]_0^2 = 8\pi$$



ESERCIZIO 8. *Si calcoli l'integrale*

$$\iiint_E x_3 dx_1 dx_2 dx_3$$

dove $E \subseteq \mathbb{R}^3$ *è la porzione del solido* $C = \{|x_1|, |x_2| \le 1\}$ *contenuta nell'ottante* $\{x_1, x_2, x_3 \ge 0\}$*, delimitata dai piani* $\{x_3 = 0\}$ *e* $\{x_1 + x_2 + 2x_3 = 3\}$



$$C = \{0 \le x_1 \le 1;\ 0 \le x_2 \le 1;\ 0 \le x_3 \le \tfrac{1}{2}(3 - x_1 - x_2)\}$$

$$\int_E x_3 dx = \int_0^1 \left[\int_0^1 \left[\int_0^{\frac{1}{2}(3-x_1-x_2)} x_3 dx_3\right] dx_1\right] dx_2$$

$$= \int_0^1 \left[\int_0^1 \frac{1}{8}(3-x_1-x_2)^2 dx_1\right] dx_2$$

$$= \frac{1}{8}\int_0^1 \left[-\frac{1}{3}(3-x_1-x_2)^3\right]_0^1 dx_2 = -\frac{1}{24}\int_0^1 \left[(2-x_2)^3 - (3-x_2)^3\right] dx_2$$

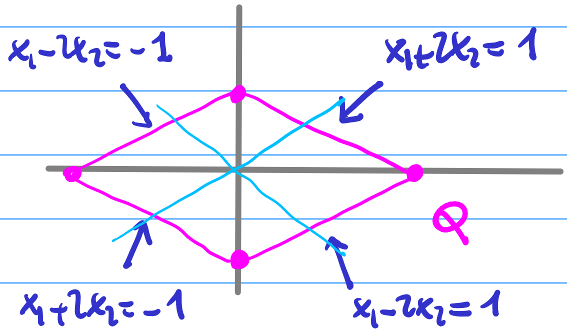
$$= \frac{1}{96}\left[[2-x_2]^4 - [3-x_2]^4\right]_0^1 = \frac{1}{96}(1 - 16 - 16 + 81) = \frac{25}{48}$$

$$= \int_0^1 \left[\int_0^1 \frac{1}{8}(3-x_1-x_2)^2 dx_1\right] dx_2 = \frac{1}{8}\int_0^1\int_0^1 (9 + x_1^2 + x_2^2 - 6x_1 - 6x_2 + 2x_1x_2) dx_1 dx_2$$

$$= \frac{1}{8}\int_0^1 \left[9x_1 + \frac{1}{3}x_1^3 + x_2^2x_1 - 3x_1^2 - 6x_2x_1 + x_1^2x_2\right]_0^1 dx_2 = \dots$$

ESERCIZIO 6. *Sia Q il quadrilatero di vertici* (1, 0), (0, 1/2), (−1, 0) *e* (0, −1/2). *Calcolare*

$$\int_Q |x_1^2 - 4x_2^2| e^{(x_1+2x_2)^2} dx_1 dx_2$$



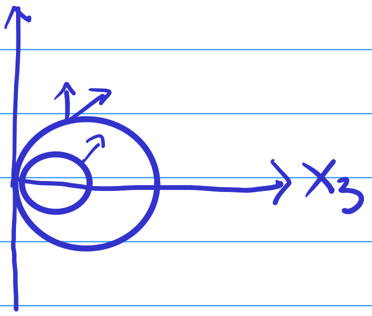
$$\begin{cases} u_1 = x_1 - 2x_2 \\ u_2 = x_1 + 2x_2 \end{cases}$$

$$u = \begin{pmatrix} 1 & -2 \\ 1 & +2 \end{pmatrix} x$$

$$\int_Q |x_1^2 - 4x_2^2| e^{(x_1+2x_2)^2} dx_1 dx_2 = \int_{[-1,1]^2} |u_1 u_2| e^{u_2^2} \underbrace{\frac{1}{4}}_{\det(M^{-1})} du_1 du_2$$

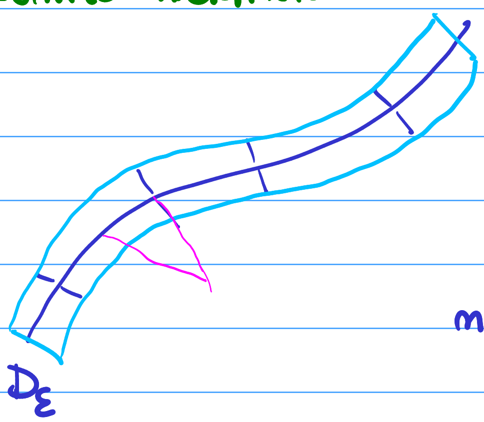
$$= \frac{2}{4} \int_{[-1,1]\times[0,1]} |u_1| u_2 e^{u_2^2} du_1 du_2 = \int_{[0,1]^2} u_1 u_2 e^{u_2^2} du_1 du_2$$

$$= \left[\int_{[0,1]} u_1 du_1\right] \cdot \left[\int_{[0,1]} u_2 e^{u_2^2} du_2\right]$$



$$x - a e_3 = \lambda x$$

ESEMPIO IMPORTANTE



$$\Sigma = \phi([0,1]^2)$$

$$\phi : [0,1]^2 \longrightarrow \mathbb{R}^3$$
$$u \longmapsto \phi(u)$$

ϕ superficie regolare di classe C^2

$$n(u) = \frac{\partial_1 \phi(u) \wedge \partial_2 \phi(u)}{\|\partial_1 \phi(u) \wedge \partial_2 \phi(u)\|_2}$$

$$x(u_1, u_2, u_3) = \phi(u) + u_3 \varepsilon n(u) \qquad (u_1, u_2, u_3) \in [0,1]\times[0,1]\times[-1,1]$$

$$\frac{1}{2\varepsilon}\left[m_3(D_\varepsilon)=\int_{D_\varepsilon} dx=\int_{[0,1]\times[0,1]\times[-1,1]} |\det(Jx)|\,du\right]\xrightarrow[\varepsilon\to 0^+]{} A(\Sigma) \text{ area di } \Sigma$$

$$\det(Jx)=\det\begin{bmatrix}\partial_1\phi_1+\varepsilon u_3\partial_1 m_1 & \partial_2\phi_1+\varepsilon u_3\partial_2 m_1 & \varepsilon m_1\\ \partial_1\phi_2+\varepsilon u_3\partial_1 m_2 & \partial_2\phi_2+\varepsilon u_3\partial_2 m_2 & \varepsilon m_2\\ \partial_1\phi_3+\varepsilon u_3\partial_1 m_3 & \partial_2\phi_3+\varepsilon u_3\partial_2 m_3 & \varepsilon m_3\end{bmatrix}$$

$$=\det\begin{bmatrix}\partial_1\phi+\varepsilon u_3\partial_1 m & \partial_2\phi+\varepsilon u_3\partial_2 m & \varepsilon m\end{bmatrix}$$

$$=\det\begin{bmatrix}\partial_1\phi & \partial_2\phi+\varepsilon u_3\partial_2 m & \varepsilon m\end{bmatrix}+\det\begin{bmatrix}\varepsilon u_3\partial_1 m & \partial_2\phi+\varepsilon u_3\partial_2 m & \varepsilon m\end{bmatrix}$$

$$=\det\begin{bmatrix}\partial_1\phi & \partial_2\phi & \varepsilon m\end{bmatrix}+\det\begin{bmatrix}\partial_1\phi & \varepsilon u_3\partial_2 m & \varepsilon m\end{bmatrix}$$

$C\varepsilon^2$

$$+\det\begin{bmatrix}\varepsilon u_3\partial_1 m & \partial_2\phi & \varepsilon m\end{bmatrix}+\det\begin{bmatrix}\varepsilon u_3\partial_1 m & \varepsilon u_3\partial_2 m & \varepsilon m\end{bmatrix}$$

$C\varepsilon^3$

$$\Rightarrow \frac{1}{2\varepsilon}\int_{D_\varepsilon} dx=\frac{1}{2\varepsilon}m_3(D_\varepsilon)\longrightarrow \lim_{\varepsilon\to 0^+}\int_{[0,1]^2\times[-1,1]}\frac{1}{2}\left|\det\begin{pmatrix}\partial_1\phi & \partial_2\phi & m\end{pmatrix}\right|du_1du_2du_3$$

$$\|$$

$$\int_{[0,1]^2}\left|\det\begin{pmatrix}\partial_1\phi & \partial_2\phi & m\end{pmatrix}\right|du_1du_2$$

$$\det\begin{pmatrix}\partial_1\phi & \partial_2\phi & m\end{pmatrix}=\det\begin{pmatrix}\partial_1\phi_1 & \partial_2\phi_1 & m_1\\ \partial_1\phi_2 & \partial_2\phi_2 & m_2\\ \partial_1\phi_3 & \partial_2\phi_3 & m_3\end{pmatrix}$$

$$=m_1(\partial_1\phi_2\,\partial_2\phi_3-\partial_2\phi_2\,\partial_1\phi_3)-m_2(\partial_1\phi_1\,\partial_2\phi_3-\partial_2\phi_1\,\partial_1\phi_3)+m_3(\partial_1\phi_1\,\partial_2\phi_2-\partial_2\phi_1\,\partial_1\phi_2)$$

$$n(u_1,u_2)=\frac{\left(\partial_1\phi_2\partial_2\phi_3-\partial_2\phi_2\partial_1\phi_3,\ \partial_2\phi_1\partial_1\phi_3-\partial_1\phi_1\partial_2\phi_3,\ \partial_1\phi_1\partial_2\phi_2-\partial_2\phi_1\partial_1\phi_2\right)}{\left[(\partial_1\phi_2\partial_2\phi_3-\partial_2\phi_2\partial_1\phi_3)^2+(\partial_2\phi_1\partial_1\phi_3-\partial_1\phi_1\partial_2\phi_3)^2+(\partial_1\phi_1\partial_2\phi_2-\partial_2\phi_1\partial_1\phi_2)^2\right]^{1/2}}$$

$$\to \det(\partial_1\phi\ \partial_2\phi\ n)=\|\partial_1\phi\wedge\partial_2\phi\|_2$$

$$\Rightarrow \boxed{A(\Sigma)=\int_K\|\partial_1\phi\wedge\partial_2\phi\|_2\,du_1du_2=:\int_\Sigma d\sigma}$$

ESEMPIO

$$S^2=\{\underbrace{(\operatorname{sen}(\phi)\cos(\theta),\operatorname{sen}(\phi)\operatorname{sen}(\theta),\cos(\phi))}_{x(\phi,\theta)};(\phi,\theta)\in K=[0,\pi]\times[0,2\pi]\}$$

$$\partial_1x=(\cos(\phi)\cos(\theta),\cos(\phi)\operatorname{sen}(\theta),-\operatorname{sen}(\phi))$$

$$\partial_2x=(-\operatorname{sen}(\phi)\operatorname{sen}(\theta),\operatorname{sen}(\phi)\cos(\theta),0)$$

$$(\partial_1x\wedge\partial_2x)=(\operatorname{sen}^2(\phi)\cos(\theta),\operatorname{sen}^2(\phi)\operatorname{sen}(\theta),\operatorname{sen}(\phi)\cos(\phi))$$

$$\|\partial_1x\wedge\partial_2x\|_2=\left[\operatorname{sen}^4(\phi)\cos^2(\theta)+\operatorname{sen}^4(\phi)\operatorname{sen}^2(\theta)+\operatorname{sen}^2(\phi)\cos^2(\phi)\right]^{1/2}$$

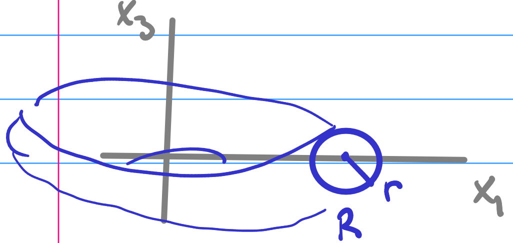
$$=\operatorname{sen}(\phi)$$

$$A(S^2)=\int_K\operatorname{sen}(\phi)\,d\phi d\theta=\int_0^\pi\left(\int_0^{2\pi}\operatorname{sen}(\phi)\,d\theta\right)d\phi$$

$$=2\pi\int_0^\pi\operatorname{sen}(\phi)\,d\phi=4\pi.$$

ESEMPIO

$$x(u,w)=\begin{pmatrix}(R+r\cos(u))\cos(w)\\(R+r\cos(u))\operatorname{sen}(w)\\r\operatorname{sen}(u)\end{pmatrix}\qquad\begin{array}{l}(u,w)\in[0,2\pi]^2\\R>r>0\end{array}$$



$$\partial_1x=(-r\operatorname{sen}(u)\cos(w),-r\operatorname{sen}(u)\operatorname{sen}(w),r\cos(u))$$

$$\partial_2 x = \left(-(R+r\cos(u))\operatorname{sen}(w),\ (R+r\cos(u))\cos(w),\ 0\right)$$

$$\partial_1 x \wedge \partial_2 x = \big((R+r\cos(u))\, r\cos(u)\cos(w),\ -(R+r\cos(u))\, r\cos(u)\operatorname{sen}(w),$$
$$-(R+r\cos(u))\, r\operatorname{sen}(u)\cos^2(w) - (R+r\cos(u))\, r\operatorname{sen}(u)\operatorname{sen}^2(w)\big)$$

$$= (R+r\cos(u))\left(r\cos(u)\cos(w),\ -r\cos(u)\operatorname{sen}(w),\ -r\operatorname{sen}(u)\right)$$

$$\|\partial_1 x\wedge\partial_2 x\|_2 = (R+r\cos(u))\left[r^2\cos^2(u)\cos^2(w) + r^2\cos^2(u)\operatorname{sen}^2(w) + r^2\operatorname{sen}^2(u)\right]^{1/2}$$

$$= r(R+r\cos(u))$$

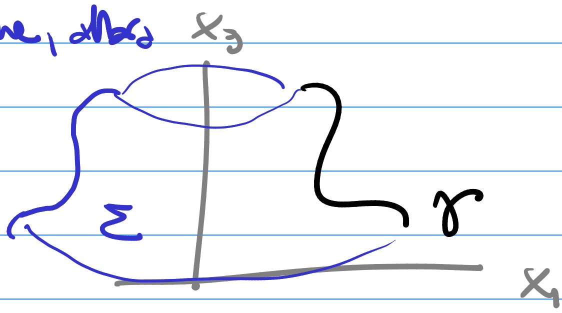
$$\Rightarrow A(T^2) = \int_{T^2} d\sigma = \int_{[0,2\pi]^2} r(R+r\cos(u))\,du\,dw$$

$$= 2\pi\int_0^{2\pi}\left[Rr + \cancel{r^2\cos(u)}\right]du = 4\pi^2 Rr$$

TEOREMA DI GULDINO

Σ superficie di rotazione, allora

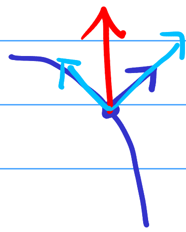
$$A(\Sigma) = 2\pi \underset{\substack{\uparrow\\ G_1(\gamma)}}{G_1}\, L(\gamma)$$



$$\Rightarrow A(T^2) = 2\pi\, G\, L(\gamma) = 2\pi\cdot R\cdot 2\pi r = 4\pi^2 Rr$$

DEF. $A\subseteq\mathbb{R}^3$ aperto, (ϕ, K) superficie regolare $\phi(K)\subseteq A$
$F\in C^1(A,\mathbb{R}^3)$ campo vettoriale
diremo che il flusso di F attraverso $\Sigma=\phi(K)$ è

$$\Phi_\Sigma(F) =: \int_\Sigma \underbrace{F(x)\cdot n(x)}_{\text{proiezione di } F \text{ su } n}\, d\sigma$$



ESEMPIO $\quad F(x) = \frac{k}{\|x\|_2^3} x \qquad \Sigma = S^2 = \partial B(0,R)$

$$\frac{k\|x\|_2^2}{\|x\|_2^3} \cdot \frac{1}{R} = \frac{k}{R^2}$$

$$\Phi_\Sigma(F) = \Phi_{S^2}(F) = \int_{S^2} F \cdot n \, d\sigma = \int_{S^2} \frac{kx}{\|x\|_2^3} \cdot \frac{x}{R} \, d\sigma$$

$$= \int_{S^2} \frac{k}{R^2} d\sigma = \frac{k}{R^2} \int_{S^2} d\sigma = \frac{k}{R^2} \cdot 4\pi R^2 = 4\pi k$$