Risoluzione dell'eq^{ne} di Laplace nel semispazio con condizioni al bordo di Dirichlet

$$(P)\quad \begin{cases} -\Delta u = 0 & \text{su } \mathbb{R}^n_+ = \{\underline{x} \in \mathbb{R}^n : x_n > 0\} \\ u(x) = \underbrace{g(x)}_{\text{assegnata}} & \text{su } \partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n = 0\} = \mathbb{R}^{n-1} \end{cases}$$

$$(*)\quad u(x) = \frac{2x_n}{n\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy = \int_{\partial\mathbb{R}^n_+} K(x,y)\, g(y)\, dy$$

dove

$$K(x,y) = \frac{2x_n}{n\omega_n} \frac{1}{|x-y|^n} \quad \text{nucleo di Poisson}$$

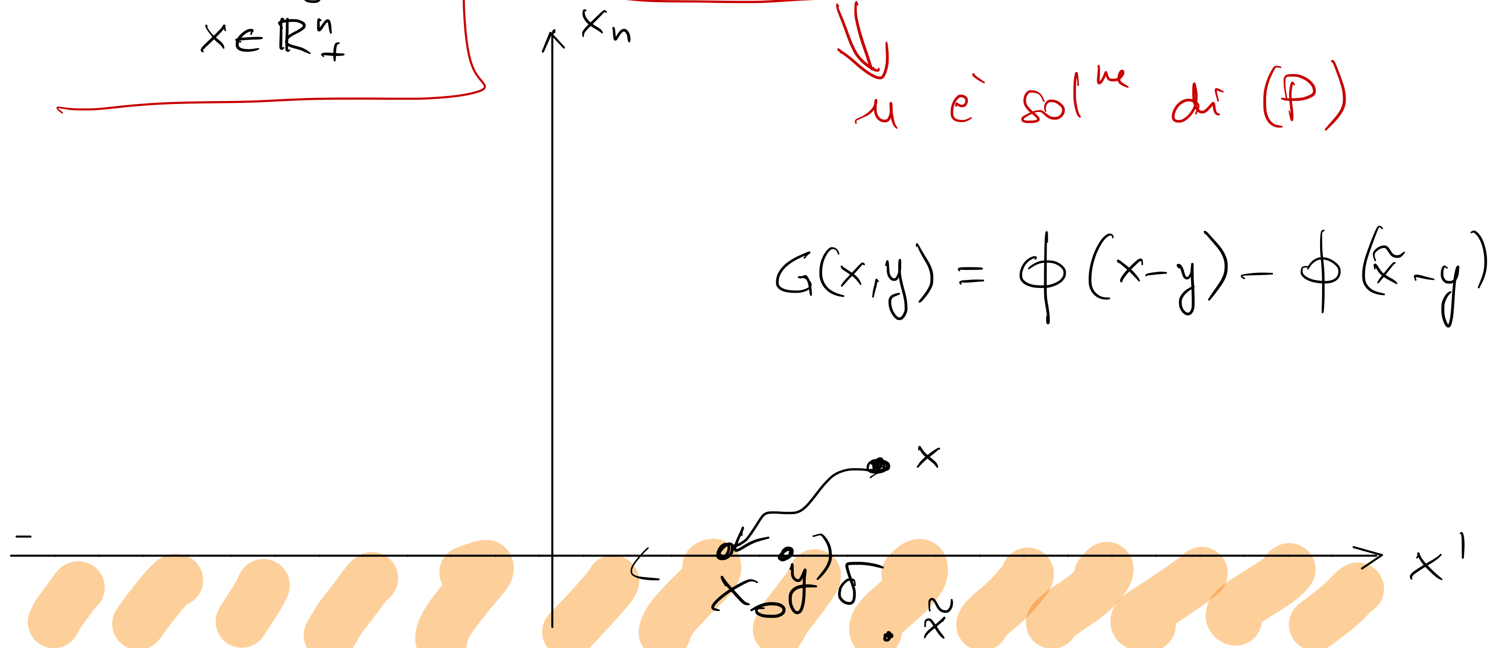
TEOREMA Supponiamo $g \in C(\mathbb{R}^{n-1}) \cap \underbrace{L^\infty(\mathbb{R}^{n-1})}_{\text{limitata}}$.

Sia $u(x)$ definita da $(*)$. Allora

1) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$
2) $\Delta u = 0$ in $\mathbb{R}^n_+$
3) $\lim\limits_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x) \quad \forall x_0 \in \partial\mathbb{R}^n_+$

$\Rightarrow$ u è sol^{ne} di (P)

$$G(x,y) = \phi(x-y) - \phi(\tilde{x}-y)$$



OSS 1 L'integrale (*) ha senso, converge, perché l'integrando all'infinito si comporta come $|y|^{u}$, $u > n-1$.

funzione di Green.

$$K(x,y) = -\frac{\partial G}{\partial \nu_e}(x,y) = \frac{\partial G}{\partial y_n}(x,y)$$

OSS 2 $G(x,y)$ armonica per $y \neq x, \tilde{x}$ $\quad \Delta_x G = 0$

$\frac{\partial G}{\partial y_n}(x,y)$ armonica in x per $y \neq x, \tilde{x}$.

$\Rightarrow K(x,y)$ armonica in x

OSS 3. $\displaystyle\int_{\partial \mathbb{R}^n_+} K(x,y)\,dy = 1 \quad \forall x \in \mathbb{R}^n_+$.

Dim $\qquad \underline{x} = (x', x_n)$

$$\int_{\partial \mathbb{R}^n_+} K(x,y)\,dy = \frac{2x_n}{n\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{dy}{|x-y|^n} =$$

$$= \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dy}{(|x'-y|^2 + x_n^2)^{n/2}} = \qquad z = \frac{x'-y}{x_n}$$

$$dz = \frac{dy}{x_n^{n-1}}$$

$$= \frac{2\cancel{x_n}}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dz \quad \cancel{x_n^{n-1}}}{(|z|^2 \cancel{x_n^2} + \underset{1}{\cancel{x_n^2}})^{n/2}} \quad \cancel{x_n^n} =$$

$$\int_{\partial \mathbb{R}^n_+} K(x,y)\,dy = \frac{2x_n}{n\,\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{dy}{|x-y|^n} =$$

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$$z = \frac{x'-y}{x_n}$$

$$dz = \frac{dy}{x_n^{n-1}}$$

$$= \frac{2\cancel{x_n}}{n\,\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dz \quad \cancel{x_n^{n-1}}}{\left(|z|^2 \cancel{x_n^2} + \cancel{x_n^2}\right)^{n/2}} \qquad \cancel{x_n^n} =$$

$$1$$

$$= \frac{2}{n\,\omega_n} \int_{\mathbb{R}^{n-1}} \frac{dz}{(1+|z|^2)^{n/2}} = 1.$$

Lo dim
per $n = 2$ e $n=3$.

$n=2$:

$$\frac{\cancel{2}}{\cancel{2}\,\pi} \int_{\mathbb{R}} \frac{dz}{1+z^2} = \frac{1}{\pi} \operatorname{arctg} z \Big|_{-\infty}^{+\infty} = 1$$

$n=3$

$$\frac{2}{\cancel{3}\cdot\frac{4}{\cancel{3}}\pi} \int_{\mathbb{R}^2} \frac{dz}{(1+|z|^2)^{3/2}} = \frac{1}{\cancel{4}2\pi} \int_0^{2\pi} d\theta \int_0^{+\infty} d\rho\, \frac{2\rho}{(1+\rho^2)^{3/2}}$$

$$= \frac{1}{\cancel{4\pi}} (-\cancel{2}) \cdot \cancel{2\pi} \, (1+\rho^2)^{-1/2} \Big|_0^{+\infty} = 1$$

Dim. 1) u è regolare

Per es., calcoliamo $\frac{\partial u}{\partial x_i}$ per semplicità $i \neq n$.

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial}{\partial x_i}\left(\frac{2x_n}{n\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{g(y)\, dy}{|x-y|^n} \right) = \quad \text{da giustificare}$$

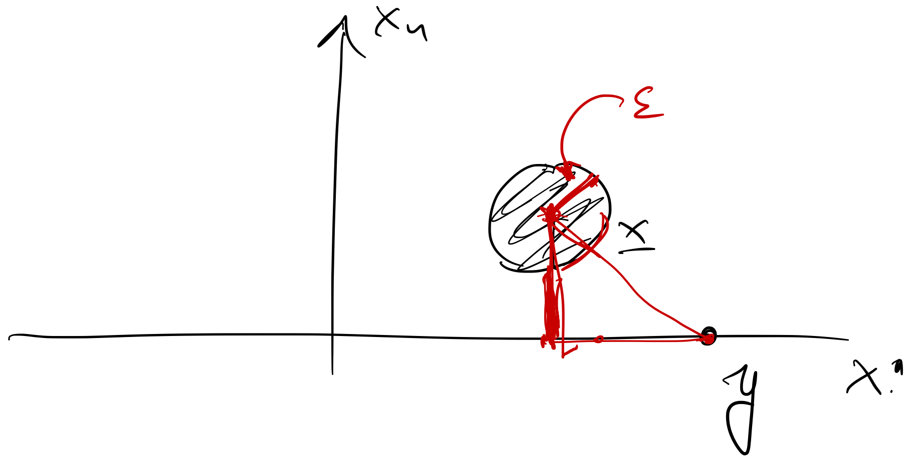
$$= \frac{2x_n}{n\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{\partial}{\partial x_i} \left(\frac{g(y)}{|x-y|^n} \right) dy =$$

$$= -\cancel{n} \frac{2x_n}{\cancel{n}\omega_n} \int_{\partial \mathbb{R}^n_+} g(y) \frac{1}{|x-y|^{n+1}} \frac{x_i - y_i}{|x-y|} =$$

$$= -\frac{2x_n}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{g(y)(x_i - y_i)}{|x-y|^{n+2}}$$

Per giustificare la derivazione sotto il segno di $\int$, devo mostrare che $\exists\, h(y)$ integrabile in $\partial \mathbb{R}^n_+$ t.c.

$$\left| \frac{g(y)(x_i - y_i)}{|x-y|^{n+2}} \right| \leq h(y) \qquad \forall x \text{ in un intorno di } \bar{x} \in \mathbb{R}^n_+ \text{ fissato}$$



$$\left|\frac{g(y)\,(x_i-y_i)}{|x-y|^{n+2}}\right| \le \sup|g(y)|\,\frac{1}{|x-y|^{n+1}} \le$$

$$\le \sup|g| \; \overbrace{\frac{1}{\max\left\{\frac{\bar{x}_n}{2}, |y|-|\bar{x}|-1\right\}^{n+1}}}^{h(y)}$$

$$|x-y| \ge \frac{\bar{x}_n}{2}$$

se scelgo $\varepsilon < \frac{\bar{x}_n}{2}$

$$|x-y| \ge \big||x|-|y|\big| \ge |y|-|x| \ge |y|-|\bar{x}|-1$$

$\varepsilon < 1$ $\qquad |x| \le |\bar{x}|+1$

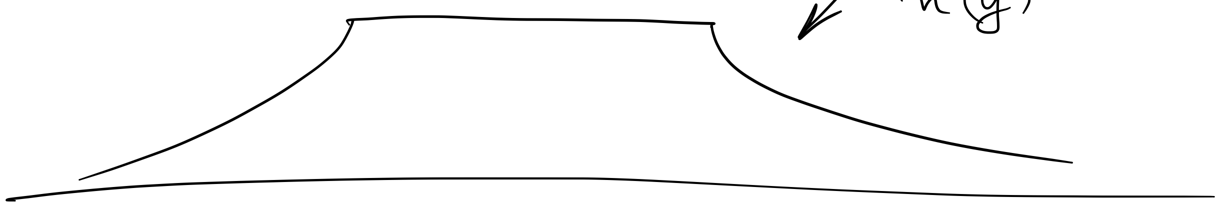
dis. triangolare per la differenza

$$\left.\begin{array}{l} |x| \le |x-y|+|y| \\ |y| \le |x-y|+|x| \end{array}\right\} \Longrightarrow \begin{array}{l} |x-y| \ge |x|-|y| \\ |x-y| \ge |y|-|x| \end{array}$$

$$|x-y| \ge \max\left\{\frac{\bar{x}_n}{2},\ |y|-|\bar{x}|-1\right\}$$

$$\frac{1}{|x-y|^{n+1}} \le \frac{1}{\max\left\{\frac{\bar{x}_n}{2}, |y|-|\bar{x}|-1\right\}^{n+1}}$$

$h(y)$



In questa maniera si mostra che tutte le derivate parziali di qualsiasi ordine passano "dentro" l'integrale e si scaricano sulla $K(x,y)$

Abbiamo provato che u è regolare.

Inoltre u è limitata

$$|u(x)| = \int_{\partial\mathbb{R}^n_+} K(x,y)\,|g(y)|\,dy \leq \sup|g| \underbrace{\int_{\partial\mathbb{R}^n_+} K(x,y)\,dy}_{1} =$$

$$= \sup|g|$$

1) OK!

2) $\Delta_x u = 0$

$$\Delta_x u(x) = \int_{\partial\mathbb{R}^n_+} g(y) \underbrace{\Delta_x K(x,y)}_{\overset{\shortparallel}{0}}\,dy = 0$$

3) Fissiamo $x_0 \in \partial\mathbb{R}^n_+$, $\varepsilon > 0$

Per la continuità di g, $\exists \delta > 0$ t.c.

$y \in \partial\mathbb{R}^n_+$, $|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \varepsilon$

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$$|u(x) - g(x_0)| = \left| \int_{\partial\mathbb{R}^n_+} g(y) k(x,y)\,dy - g(x_0) \int_{\partial\mathbb{R}^n_+} k(x,y)\,dy \right| =$$

$$= \left| \int_{\partial\mathbb{R}^n_+} [g(y) - g(x_0)]\, k(x,y)\,dy \right| \leq$$

dis. triangolare

$$\leq \int_{\partial\mathbb{R}^n_+} |g(y) - g(x_0)|\, k(x,y)\,dy =$$

$$= \underbrace{\int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} k(x,y) \underbrace{|g(y) - g(x_0)|}_{\leq \varepsilon} dy}_{\leq} + \underbrace{\int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} k(x,y) |g(y) - g(x_0)|\, dy}_{J}$$

$$\varepsilon \underbrace{\int k(x,y)\,dy}_{\leq \varepsilon}$$

qui $|y - x_0| \geq \delta$

J se $|x - x_0| \leq \frac{\delta}{2}$ e $|y - x_0| \geq \delta$

$$|y - x_0| \leq |y - x| + \underbrace{|x - x_0|}_{\leq \delta/2} \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{|y - x_0|}{2}$$

$$\Rightarrow |y - x| \geq \frac{|y - x_0|}{2}$$

$$J = \int_{|y-x_0|\geq\delta} k(x,y)\, \overbrace{|g(y)-g(x_0)|}^{\leq |g(y)|+|g(x_0)| \leq 2\sup|g|}\, dy =$$

abbiamo dim. che qui, se

se impongo $|x-x_0| \leq \frac{\delta}{2}$ $\qquad |x-y| \geq \frac{|y-x_0|}{2}$

$$\underset{\text{scrivo chi è } k(x,y)}{\leq} c\, x_n \int_{|y-x_0|\geq\delta} \frac{dy}{|x-y|^n} \leq c\, x_n \underbrace{\int_{|y-x_0|\geq\delta} \frac{dy}{|y-x_0|^n}}_{\overset{\wedge|}{c}}$$

$$\leq c\, x_n \longrightarrow 0$$

se $x_n \to 0^+$

Abbiamo provato che

$$|u(x) - g(x_0)| \leq \varepsilon + \underbrace{c\, x_n}_{\downarrow\, 0}$$

ε arbitrario

$$\Rightarrow |u(x) - g(x_0)| \xrightarrow[x \to x_0]{} 0$$

Principio di Dirichlet

$\Omega \subset \mathbb{R}^n$ aperto limitato e regolare

$$(P) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{su } \partial\Omega \\ u \in C^2(\overline{\Omega}) \end{cases}$$

Considero il seguente funzionale

energia associata a (P)

$$I[w] = \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \int_\Omega f\, w\, dx$$

$$w \in \mathcal{A} = \{ w \in C^2(\overline{\Omega}) : w = g \text{ su } \partial\Omega \}$$

$$I : \mathcal{A} \to \mathbb{R}$$

TEOREMA

u è sol^{ne} di (P) $\iff$ u minimizza I in $\mathcal{A}$, cioè

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Dim $\boxed{\Longrightarrow}$

u sol^{ne} di (P). Moltiplico l'eq^{ne} diff. per $u - w$, $w \in \mathcal{A}$

e integro su Ω.

$$\int_\Omega (-\Delta u)(u-w)\,dx = \int_\Omega f(u-w)\,dx$$

$$\parallel$$

$$\int_\Omega \nabla u\cdot(\nabla u-\nabla w)\,dx - \cancel{\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\underbrace{(u-w)}_{=0}\,d\sigma}$$

$$\parallel$$

$$\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u\cdot\nabla w\,dx \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \cancel{\frac{1}{2}\int_\Omega|\nabla u|^2} - \frac{1}{2}\int_\Omega|\nabla w|^2$$

$$\nabla u\cdot\nabla w \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2$$

$$\left(0\leq |\nabla u-\nabla w|^2 = (\nabla u-\nabla w,\nabla u-\nabla w) = |\nabla u|^2+|\nabla w|^2 - 2\nabla u\cdot\nabla w\right)$$

$$\Longrightarrow \underbrace{\frac{1}{2}\int_\Omega|\nabla u|^2 - \int_\Omega f\,u}_{I[u]} \leq \underbrace{\frac{1}{2}\int_\Omega|\nabla w|^2 - \int_\Omega f\,w}_{I[w]}$$

Dim $\Leftarrow$

Sia $u \in \mathcal{A}$ t.c. $I[u] \leq I[w] \ \forall w \in \mathcal{A}$.

In particolare, posso prendere $w = u + t\varphi \in \mathcal{A}$.
dove $t \in \mathbb{R}$ $\varphi \in C^2_c(\Omega)$

supporto compatto.

So che $I[u] \leq I[u + t\varphi] \ \forall t \in \mathbb{R}$.

In altre parole la funzione reale

$\eta(t) = I[u + t\varphi]$ ha minimo assoluto in $t = 0$.

$\underset{\text{Fermat}}{\Longrightarrow}$ se $\eta'(0)$ esiste, $\eta'(0) = 0$

$$\eta(t) = I[u + t\varphi] = \frac{1}{2}\int_\Omega |\nabla(u + t\varphi)|^2 - \int_\Omega f(u + t\varphi) = (*)$$

$$|\nabla(u + t\varphi)|^2 = |\nabla u + t\nabla\varphi|^2 = (\nabla u + t\nabla\varphi, \nabla u + t\nabla\varphi) =$$

$$= |\nabla u|^2 + 2t(\nabla u, \nabla\varphi) + t^2|\nabla\varphi|^2$$

$$(*) = \frac{1}{2}\int_\Omega |\nabla u|^2 + t\int_\Omega (\nabla u, \nabla\varphi) + \frac{t^2}{2}\int_\Omega |\nabla\varphi|^2 - \int_\Omega f u - t\int_\Omega f\varphi$$

$$\eta'(t) = \int_\Omega (\nabla u, \nabla\varphi) + t\int_\Omega |\nabla\varphi|^2 - \int_\Omega f\varphi$$

$$\eta'(0) = \int_\Omega (\nabla u, \nabla\varphi) - \int_\Omega f\varphi = 0$$

$$\int \nabla u \cdot \nabla \varphi = \int f \varphi$$

$\Downarrow$ integro per parti

$$\int_\Omega (-\Delta u)\varphi + \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi \, d\sigma}_{\substack{\Downarrow \\ 0}}$$

$$\Longrightarrow \int_\Omega (-\Delta u - f)\varphi \, dx = 0 \qquad \forall \varphi \in C_c^2(\Omega)$$

$$\Longrightarrow -\Delta u = f \qquad \forall x \in \Omega. \qquad \square$$