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DEF. $C^0[a,b] = \{f : [a,b] \to \mathbb{R} \text{ continue}\}$

$$(f+g)(x) := f(x)+g(x) \qquad (\lambda f)(x) = \lambda f(x)$$
$$f,g \in C^0[a,b],\ \lambda \in \mathbb{R}$$

$$\|f\|_{C^0[a,b]} = \|f\|_{\infty,[a,b]} = \|f\|_\infty := \max_{x\in[a,b]} |f(x)| = \sup_{x\in[a,b]} |f(x)|$$

PROP. $\|\cdot\|_\infty$ è una norma su $C^0[a,b]$.

- f tale che $\|f\|_\infty = 0 \Rightarrow \max|f(x)| = 0 \geq |f(x)|\ \forall x\in[a,b]$
$$\Rightarrow |f(x)| = 0 \ \forall x\in[a,b] \Rightarrow f(x) = 0 \ \forall x$$

- $|\lambda f(x)| = |\lambda|\,|f(x)| \qquad \forall f \in C^0[a,b],\ \forall \lambda\in\mathbb{R},\ \forall x\in[a,b]$

$$\Rightarrow \|\lambda f\|_\infty = \max |\lambda f(x)| = \max_{x\in[a,b]} |\lambda|\,|f(x)| = |\lambda| \max|f(x)| = |\lambda|\,\|f\|_\infty$$

- $\|f+g\|_\infty = \max|f(x)+g(x)| = |f(x_0)+g(x_0)| \leq |f(x_0)|+|g(x_0)|$
$$\leq \max_{x\in[a,b]}|f(x)| + \max_{x\in[a,b]}|g(x)| = \|f\|_\infty + \|g\|_\infty$$

TEOR. $(C^0[a,b], \|\cdot\|_\infty)$ è uno spazio di Banach

Sia $\{f_k\} \subseteq C^0[a,b]$ una successione di Cauchy, cioè

$\forall \varepsilon > 0\ \exists k_0 = k_0(\varepsilon)$ tale che $\|f_k - f_{k+p}\|_\infty = \max_{x\in[a,b]} |f_k(x) - f_{k+p}(x)| \leq \varepsilon$

per $k \geq k_0(\varepsilon)$ (p qualsiasi)

$\forall x_0 \in [a,b]$

$$\Rightarrow |f_k(x_0) - f_{k+p}(x_0)| \leq \|f_k - f_{k+p}\|_\infty \leq \varepsilon \qquad \forall k \geq k_0(\varepsilon)$$

$$\Rightarrow \underbrace{\{f_k(x_0)\}}_{= a_k} \text{ è una successione di Cauchy in } (\mathbb{R}, \|\cdot\|_2)$$

poiché $(\mathbb{R}, |\cdot|)$ è completo $\quad f_k(x_0) = a_k \longrightarrow L(x_0) =: f_\infty(x_0)$

$$f_\infty : [a,b] \longrightarrow \mathbb{R} \qquad x \longmapsto f_\infty(x)$$

¿ $f_\infty \in C^0[a,b]$?

$\forall \gamma > 0 \ \exists \delta = \delta(\gamma) > 0$ tale che $|f_\infty(x) - f_\infty(y)| \le \gamma$ se $|x-y| \le \delta$

$$|f_\infty(x) - f_\infty(y)| = |f_\infty(x) \pm f_k(x) \pm f_k(y) - f_\infty(y)|$$

$$\le |f_\infty(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f_\infty(y)|$$

$$\le \gamma + \gamma + \gamma \qquad \begin{array}{l} k \ge k_0^1 = k_0^1(\gamma, x) \\ k \ge k_0^2 = k_0^2(\gamma, y) \\ |x-y| \le \delta_k = \delta_k(k, \gamma) \end{array}$$

$\Rightarrow |f_\infty(x) - f_\infty(y)| \le 3\gamma$ se $|x-y| \le \delta_k$

$\Rightarrow f_\infty \in C^0[a,b]$

¿ $\forall \varepsilon > 0 \ \exists k_0(\varepsilon) \in \mathbb{N}$ tale che $\|f_k - f_\infty\|_\infty \le \varepsilon \ \forall k \ge k_0(\varepsilon)$?

$$|f_k(x) - f_\infty(x)| = \lim_{p \to +\infty} |f_k(x) - f_{k+p}(x)| \le \overline{\lim_{p\to+\infty}} \|f_k - f_{k+p}\|_\infty$$

$$\Rightarrow \quad |f_k(x) - f_\infty(x)| \le \overline{\lim_{p\to+\infty}} \|f_k - f_{k+p}\|_\infty \le \varepsilon \quad \forall k \ge k_0(\varepsilon) \ \forall x \in [a,b]$$

$\Rightarrow \|f_k - f_\infty\|_\infty \le \varepsilon \quad \forall k \ge k_0(\varepsilon)$

quindi $f_k \longrightarrow f_\infty \in C^0[a,b]$ ■

ESEMPI.

$$\left\{ f_k(x) = \frac{x^k}{2^k} \right\} \subseteq C^0[0,1]$$

fisso $x \in [0,1]$ $f_k(x) = \left(\frac{x}{2}\right)^k \xrightarrow[k \to +\infty]{?} 0 =: f_\infty(x)$

$$0 \leq \frac{x}{2} \leq \frac{1}{2} \quad \forall x \in [0,1]$$

$$\| f_k - f_\infty \|_\infty = \max_{[0,1]} \left| \frac{x^k}{2^k} - 0 \right| = \frac{1}{2^k} \xrightarrow[k \to +\infty]{} 0$$

$$\Rightarrow f_k \longrightarrow 0$$

- $\{f_k(x) = x^k\} \subseteq C^0[0,1]$

fisso $x \in [0,1]$ $f_k(x) = x^k \xrightarrow[k \to +\infty]{?} \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

$$\Rightarrow f_\infty(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases} \notin C^0[0,1]$$

$\Rightarrow \{x^k\}$ non è di Cauchy

$\| f_k - f_{k+p} \|_\infty =$

$$[x^k - x^{k+p}] = x^k(1 - x^p) = F(x)$$

$$F'(x) = kx^{k-1} - (k+p)x^{k+p-1} = x^{k-1}[k - (k+p)x^p]$$

$\Rightarrow \bar{x} = \left(\frac{k}{k+p}\right)^{\frac{1}{p}}$ punto di massimo di F

$$f_k(\bar{x}) - f_{k+p}(\bar{x}) = \left(\frac{k}{k+p}\right)^{\frac{k}{p}} - \left(\frac{k}{k+p}\right)^{\frac{k+p}{p}} = \left(\frac{k}{k+p}\right)^{\frac{k}{p}}\left[1 - \frac{k}{k+p}\right]$$

$$= \left(\frac{k}{k+p}\right)^{\frac{k}{p}} \cdot \frac{p}{k+p} = \| f_k - f_{k+p} \|_\infty$$

non è vero che $\left(\frac{k}{k+p}\right)^{\frac{k}{p}} \frac{p}{k+p} \leq \varepsilon$ $\forall k \geq k_0(\varepsilon)$ $\forall p \in \mathbb{N}$

DEF. $\ell^2 = \{ a : \mathbb{N} \to \mathbb{R} \ ; \ \sum_{k=0}^{+\infty} |a(k)|^2 < +\infty \}$

$a(k) = \frac{1}{2^k} \in \ell^2 \qquad a(k) = \frac{1}{k+1} \in \ell^2 \qquad a(k) = \frac{1}{\sqrt{k+1}} \notin \ell^2$

$e_i = \begin{cases} 1 & \text{se } k = i \\ 0 & \text{se } k \neq i \end{cases} \qquad e_i(k) = (0, 0, \dots, \underset{\substack{\uparrow \\ i\text{-esimo}}}{1}, 0, 0, \dots)$

$$\sum_{k \geq 0} \left(\frac{1}{\sqrt{k+1}} \right)^2 = \sum_{k \geq 0} \frac{1}{k+1} = +\infty$$

$a, b \in \ell^2 \qquad (a|b)_{\ell^2} := \sum_{k=0}^{+\infty} a(k) b(k)$

¿ $|(a|b)_{\ell^2}| < +\infty$?

$$\left| \sum_{k=0}^{N} a(k) b(k) \right| \leq \sum_{k=0}^{N} |a(k) b(k)| \leq \frac{1}{2} \sum_{k=0}^{N} (a^2(k) + b^2(k))$$

$$= \frac{1}{2} \sum_{k=0}^{N} |a(k)|^2 + \frac{1}{2} \sum_{k=0}^{N} |b(k)|^2$$

$|AB| \leq \frac{1}{2}(A^2 + B^2) \quad \forall A, B \in \mathbb{R}$

$2 |A| \cdot |B| \leq |A|^2 + |B|^2 \Leftrightarrow (|A| - |B|)^2 \geq 0$

se $N \to +\infty \Rightarrow \left| \sum_{k=0}^{+\infty} a(k) b(k) \right| \leq \frac{1}{2} \sum_{k=0}^{+\infty} |a(k)|^2 + \frac{1}{2} \sum_{k=0}^{+\infty} |b(k)|^2 < +\infty$

linearità di $(\cdot|\cdot)_{\ell^2}$: $\quad a, b, c \in \ell^2$

¿ $(a + b | c)_{\ell^2} = (a|c)_{\ell^2} + (b|c)_{\ell^2}$?

$$\sum_{k=0}^{N} (a(k) + b(k)) c(k) = \sum_{k=0}^{N} a(k) c(k) + \sum_{k=0}^{N} b(k) c(k) \qquad \forall N \in \mathbb{N}$$

DEF. $(H, (\cdot|\cdot)_H)$ spazio vettoriale dotato di prodotto scalare si dice spazio di Hilbert se $(H, \|\cdot\|_H)$ è completo
$(\|u\|_H^2 = (u|u)_H \quad \forall u \in H)$

TEOR. $(\ell^2, (\cdot|\cdot)_{\ell^2})$ è uno spazio di Hilbert.

$\{a_j\} \subset \ell^2 \qquad a_j = (a_j(0), a_j(1), \dots, a_j(k), \dots) \in \ell^2 \quad \forall j \in \mathbb{N}$

$\{a_j\}$ successione di Cauchy in ℓ^2, quindi:

$\forall \varepsilon > 0 \;\exists J_0 = J_0(\varepsilon)$ tale che $\|a_j - a_{j+p}\|_{\ell^2} \leq \varepsilon \quad \forall j \geq J_0(\varepsilon) \; (\forall p \in \mathbb{N})$

$$|a_j(k_0) - a_{j+p}(k_0)| \leq \left[\sum_{k=0}^{+\infty} |a_j(k) - a_{j+p}(k)|^2\right]^{1/2} = \|a_j - a_{j+p}\|_{\ell^2} \leq \varepsilon$$

$\Rightarrow \{a_j(k_0)\}_{j \in \mathbb{N}}$ è di Cauchy in $(\mathbb{R}, \|\cdot\|_2)$

$\Rightarrow \quad a_j(k_0) \xrightarrow[j \to +\infty]{} a_\infty(k_0) \qquad \forall k_0 \in \mathbb{N}$

¿ $a_\infty \in \ell^2$?

$$\sum_{k=0}^{N} |a_\infty(k)|^2 = \lim_{j \to +\infty} \sum_{k=0}^{N} |a_j(k)|^2 \leq C_0 \qquad \forall N \in \mathbb{N}$$

PROP. $\{x_j\} \subseteq (X, \|\cdot\|_X)$ successione di Cauchy $\Rightarrow \exists C_0 > 0 ; \|x_j\|_X \leq C_0$

$$\|x_j\|_X = \|x_j - x_{J_0} + x_{J_0}\|_X \leq \|x_j - x_{J_0}\|_X + \|x_{J_0}\|_X \leq \varepsilon + \|x_{J_0}\|_X \quad \forall j \geq J_0$$

$$\|x_j\|_X \leq \max\{\varepsilon + \|x_{J_0}\|_X ; \|x_{J_0-1}\|_X ; \|x_{J_0-2}\|_X ; \dots ; \|x_0\|_X\} =: C_0$$

$$\sum_{k=0}^{N} |a_j(k) - a_{j+p}(k)|^2 \xrightarrow[p \to +\infty]{} \sum_{k=0}^{N} |a_j(k) - a_\infty(k)|^2$$

$$\sum_{k=0}^{N} |a_j(k) - a_\infty(k)|^2 \longleftarrow \sum_{k=0}^{N} |a_j(k) - a_{j+p}(k)|^2 \leq \sum_{k=0}^{+\infty} |a_j(k) - a_{j+p}(k)|^2$$

$$\shortparallel$$

$$\|a_j - a_{j+p}\|_{\ell^2}^2$$

$$\Rightarrow \|a_j - a_\infty\|_{\ell^2}^2 \leq \|a_j - a_{j+p}\|_{\ell^2}^2 \leq \varepsilon^2 \quad \forall j \geq J_0(\varepsilon)$$

$$\Rightarrow a_j \longrightarrow a_\infty \text{ in } (\ell^2, (\cdot|\cdot)_{\ell^2}) \ \blacksquare$$

ESEMPIO $\overline{B(0,1)} \subseteq (\ell^2, (\cdot|\cdot)_{\ell^2})$ non è compatta

$$\bar{B} = \overline{B(0,1)} = \{ a \in \ell^2 ; \|a\|_{\ell^2} \leq 1 \} = \{ a ; \sum_{k=0}^{+\infty} |a(k)|^2 \leq 1 \}$$

$$\{e_j\} \subseteq \bar{B} \subseteq \ell^2 \qquad e_j(k) = \begin{cases} 1 & \text{se } k = j \\ 0 & \text{se } k \neq j \end{cases}$$

se $\exists \{e_{j(m)}\}$ convergente allora $e_{j(m)}(k) \longrightarrow e(k) \quad \forall k \in \mathbb{N}$

fisso k $\quad e_{j(m)}(k) \xrightarrow[m \to +\infty]{} 0 := e_\infty(k)$

$$\|e_{j_1} - e_{j_2}\|_{\ell^2}^2 = 2$$

$\boxed{14/10/2025}$

ESERCIZIO 2. *Per* $x = (x_1, x_2) \in \mathbb{R}^2$ *si consideri*

$$\|x\|_1 := |x_1| + |x_2| \qquad \|x\|_2 := \left[|x_1|^2 + |x_2|^2\right]^{1/2} = \sqrt{x \cdot x} \qquad \|x\|_\infty := \max\{|x_1|, |x_2|\}$$

i. Si provi che tutte le norme precedenti rendono $\mathbb{R}^2$ *uno spazio metrico completo.*
ii. Si mostri che solo $\|\cdot\|_2$ *è invariante per rotazioni.*
ii. Si disegni $B(0,1) \subseteq \mathbb{R}^2$ *per le tre norme.*

1. $(\mathbb{R}^2, \|\cdot\|_1)$ è uno spazio di Banach

$\{x_k\}$ successione di Cauchy, cioè: $\quad \forall \varepsilon > 0 \quad \exists k_0 = k_0(\varepsilon) \in \mathbb{N}$ tale che

$$\|x_k - x_{k+p}\|_1 \leq \varepsilon \quad \text{se } k \geq k_0(\varepsilon)$$

$$|x_{k,1}-x_{k+p,1}|+|x_{k,2}-x_{k+p,2}| \le \varepsilon \quad \text{se } k \ge k_0(\varepsilon)$$
$$\vee\!/ \quad |x_{k,i}-x_{k+p,i}| \quad \text{per } i=1,2$$

$\Rightarrow \{x_{k,i}\}_{k\in\mathbb{N}}$ è una successione di Cauchy in $(\mathbb{R}, \|\cdot\|_2)$

$$\Rightarrow x_{k,i} \longrightarrow x_{\infty,i} \quad \text{per } i=1,2$$
$$x_\infty=(x_{\infty,1}; x_{\infty,2})$$

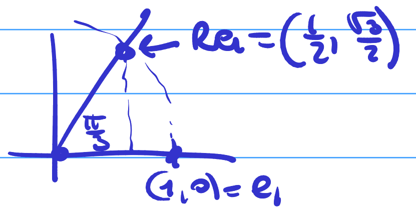
$$\|x_k-x_{k+p}\|_1=|x_{k,1}-x_{k+p,1}|+|x_{k,2}-x_{k+p,2}| \xrightarrow[p\to+\infty]{} |x_{k,1}-x_{\infty,1}|+|x_{k,2}-x_{\infty,2}|$$

$\forall \varepsilon>0 \quad \exists k_0=k_0(\varepsilon)\in\mathbb{N}$ tale che

$$\|x_k-x_\infty\|_1 \xleftarrow[p\to+\infty]{} \|x_k-x_{k+p}\|_1 \le \varepsilon \qquad \forall k \ge k_0(\varepsilon)$$

$$\Rightarrow \quad x_k \to x_\infty \quad \text{in } (\mathbb{R}^2, \|\cdot\|_1)$$

II. $e_1=(1,0)$



$$\|Re_1\|_1=\tfrac{1}{2}(1+\sqrt{3}) \neq 1$$
$$\|Re_1\|_\infty=\frac{\sqrt{3}}{2} \neq 1$$
$$\|e_1\|_1=\|e_1\|_\infty=1$$

$x\in(\mathbb{R}^2, \|\cdot\|_2)$ R rotazione $\Rightarrow \|Rx\|_2=\|x\|_2$

$$R=R_\vartheta=\begin{pmatrix}\cos(\vartheta) & -\sin(\vartheta)\\ \sin(\vartheta) & \cos(\vartheta)\end{pmatrix} \qquad R_\vartheta^T=R_{-\vartheta}$$

$$\|R_\vartheta x\|_2^2=(R_\vartheta x\cdot R_\vartheta x)=(R_\vartheta^T R_\vartheta x\cdot x)=(R_{-\vartheta}R_\vartheta x\cdot x)=(x\cdot x)=\|x\|_2^2$$

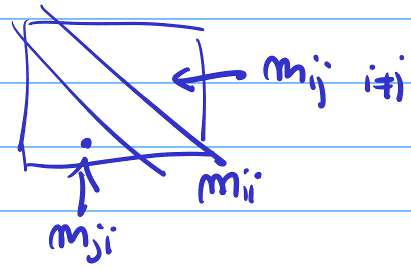
$$R_\vartheta^T=\begin{pmatrix}\cos(\vartheta) & \sin(\vartheta)\\ -\sin(\vartheta) & \cos(\vartheta)\end{pmatrix} \qquad R_\vartheta^T=R_{-\vartheta} \qquad R_{-\vartheta}R_\vartheta=\mathbb{1}_2$$

$(x\cdot My) = (M^T x\cdot y)$ $\qquad x=(x_1,\dots,x_n)$, $\quad y=(y_1,\dots,y_n)$ $\qquad M=(m_{ij})$

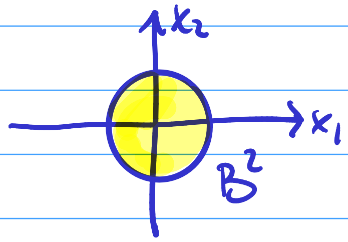
$$My = \left(\sum_{j=1}^n m_{ij} y_j\right)_{i=1\dots n} \Rightarrow x\cdot(My) = \sum_{i=1}^n x_i \left(\sum_{j=1}^n m_{ij}y_j\right) = \sum_{ij=1}^n m_{ij}x_i y_j$$

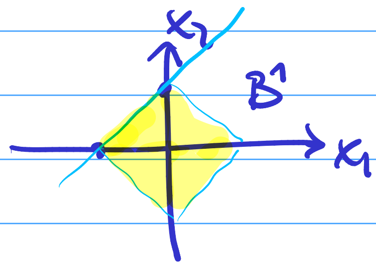
$$= \sum_{j=1}^n y_j\left(\sum_{i=1}^n m_{ij}x_i\right) = y\cdot M^T x = (M^T x)\cdot y$$

$$M=(m_{ij}) \Rightarrow M^T=(m_{ji})$$



iii $\quad B^2(0,1)=\{x\in\mathbb{R}^2 : \|x\|_2<1\} \Rightarrow \{x_1^2+x_2^2<1\}=B^2$

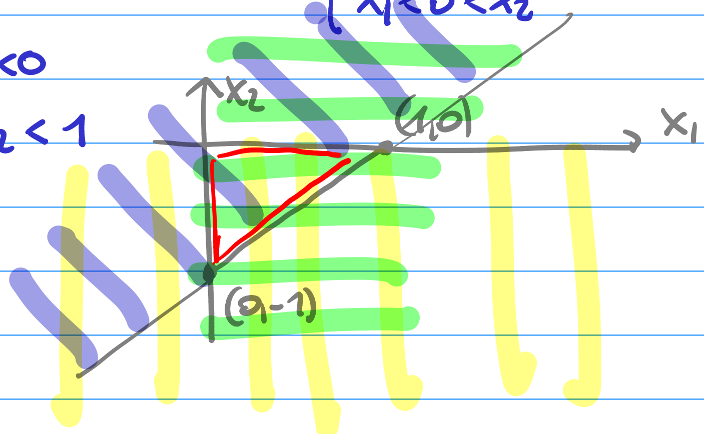




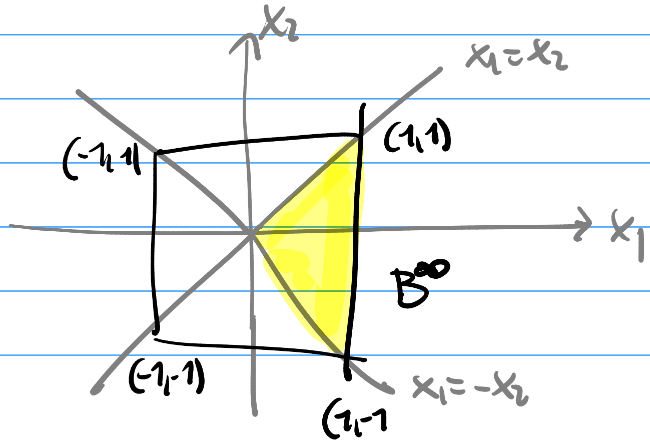
$B^1(0,1) = \{|x_1|+|x_2|<1\}$ $\qquad$ nel II quadrante: $\begin{cases} -x_1+x_2<1 \\ x_1<0<x_2\end{cases}$

nel IV quadrante $\quad x_1>0 \quad x_2<0$

$|x_1|+|x_2| = x_1-x_2<1$



$B^\infty = \{\max\{|x_1|,|x_2|\}<1\}$



$B^1 \subsetneq B^2 \subsetneq B^\infty$

ESERCIZIO 3. *Si provi che non esiste un prodotto scalare su $\mathbb{R}^2$ che induce la norma $\|\cdot\|_1$.*

se $\|x\|_1^2 = (x \cdot x)_1$ $\forall x \in \mathbb{R}^2$, allora

$$\frac{1}{2}\left(\|x\|_1^2 + \|y\|_1^2\right) = \|x+y\|_1^2 + \|x-y\|_1^2 \quad \forall x,y \in \mathbb{R}^2$$

scelgo $x=(1,1)$ e $y=(1,-1)$

$$\Rightarrow \frac{1}{2}\Big(\underset{\substack{\downarrow\\4}}{\|x\|_1^2} + \underset{\substack{\downarrow\\4}}{\|y\|_1^2}\Big) = 4 \neq \underset{\substack{\downarrow\\4}}{\|x+y\|_1^2} + \underset{\substack{\downarrow\\4}}{\|x-y\|_1^2} = 8$$

ESERCIZIO 9. *Siano A_1 e A_2 due sottoinsiemi aperti di $(\mathbb{R}^n, \|\cdot\|_2)$, si mostri che gli insiemi $A_1 \cap A_2$ e $A_1 \cup A_2$ sono aperti.*
Si concluda l'esercizio provando quali tra gli aggettivi chiuso, convesso e compatto possono sostituire l'aggettivo aperto nell'affermazione della prima parte del quesito.

DEF. $A \subseteq \mathbb{R}^n$ non vuoto si dice convesso se $\forall\, x,y \in A$ il segmento $\overline{xy} \subseteq A$.

$A_1, A_2 \subseteq \mathbb{R}^n$ $\quad A := A_1 \cup A_2$ è aperto

sia $p \in A$ allora $p \in A_i$ aperto $\Rightarrow \exists\, r>0$ tale che $B(p,r) \subseteq A_i \subseteq A_1 \cup A_2 = A$.

$B := A_1 \cap A_2$ è aperto

sia $p \in B$ allora $p \in A_1$ e $p \in A_2$ aperti

$\Rightarrow \exists\, r_1, r_2 > 0$ tali che $B(p,r_1) \subseteq A_1$ e $B(p,r_2) \subseteq A_2$

$r := \min\{r_1, r_2\} > 0$ $\quad B(p,r) \subseteq B(p,r_1), B(p,r_2) \subseteq A_1, A_2$

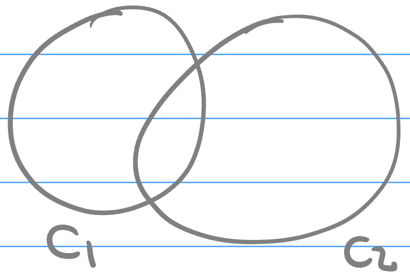
$\Rightarrow B(p,r) \subseteq A_1 \cap A_2 = B$.

TEOR. $\{A_i\}_{i\in I} \subseteq (X, \|\cdot\|_X)$ famiglia di aperti di uno spazio di Banach

- $\left(\bigcup_{i\in I} A_i\right)$ è aperto
- $(A_1 \cap \dots \cap A_N)$ è aperto

$C_1, C_2 \subseteq \mathbb{R}^n$ chiusi $\quad C = C_1 \cup C_2$

C_1^c, C_2^c aperti



$(C_1^c \cap C_2^c)$ è un aperto

$C_1 \cup C_2 = (C_1^c \cap C_2^c)^c$ è chiuso

$C_1 \cap C_2 = (C_1^c \cup C_2^c)^c$ è chiuso

PROP. $C \subseteq (X, \|\cdot\|_X)$ insieme in uno spazio di Banach, sono equivalenti

i. C è chiuso

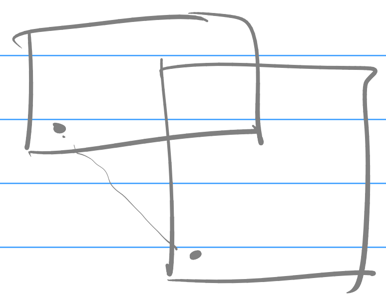
ii. $\partial C \subseteq C$

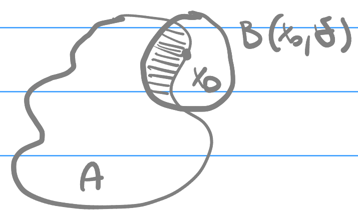
iii. $\{x_k\} \subseteq C$ convergente allora $x_k \to \bar{x} \in C$

$K_1, K_2 \subseteq \mathbb{R}^n$ compatti $\quad \{x_k\} \subseteq W = K_1 \cup K_2$

$\exists \{x_{k(j)}\} \subseteq K_1$ (o K_2)

$\Rightarrow \exists \{x_{k(j(h))}\} \subseteq \underset{W}{\underset{\cap}{K_1}}$ convergente $\Rightarrow W$ è compatto





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DEF. $f: \overset{\text{aperto}}{A} \subseteq \mathbb{R}^n \to \mathbb{R}$, x_0 punto di accumulazione per A, diciamo che

$f(x) \xrightarrow[\substack{\text{per } x \to x_0 \\ A}]{} L$ ove $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, x_0) > 0$ tale che

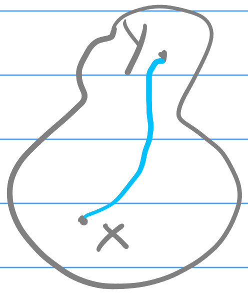
$\underset{x \in A}{\|x - x_0\|_2 \le \delta} \Rightarrow |f(x) - L| \le \varepsilon$

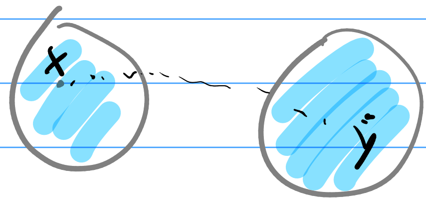
$x \in B(x_0, \delta) \cap A$

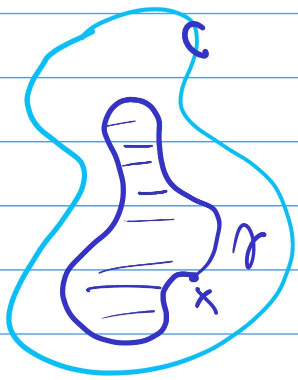
OSS. se $f: \underset{\text{aperto}}{A \subseteq \mathbb{R}^n} \to \mathbb{R}$ è continua in $x_0 \in A$

allora $\forall \{x_k\} \subseteq A$ con $x_k \to x_0 \Rightarrow f(x_k) \to f(x_0)$

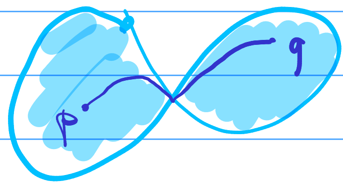
DEF. $A \subseteq \mathbb{R}^n$ si dice connesso se $\forall\, x, y \in A$ $\exists\, r : [a,b] \to A$ curva regolare a tratti tale che $r(a) = x$, $r(b) = y$ e $r(s) \in A$ $\forall s \in [a,b]$







$\gamma = \partial A \subseteq C$



DEF. $f: \underset{\text{aperto}}{A \subseteq \mathbb{R}^n} \to \mathbb{R}$ $\quad x_0 \in A$, $\quad w \in \mathbb{R}^n$ $(w \neq 0)$ $\quad \|w\|_2 = 1$

se $\underset{\text{finito}}{\exists}$ $\displaystyle\lim_{t \to 0} \frac{1}{t}\left[f(x_0 + tw) - f(x_0)\right] =: \partial_w f(x_0)$ $\quad$ DERIVATA DIREZIONALE

$$\left(\partial_w f(x_0) = \frac{\partial f}{\partial w}(x_0) = D_w f(x_0) = f_w(x_0)\right)$$

DERIVATE PARZIALI $\quad \partial_{e_i} f(x_0) =: \partial_i f(x_0) \left(= \partial_{x_i} f(x_0)\right)$

ESEMPIO. $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ $\quad f(x) = f(x_1,x_2) = \dfrac{x_1 x_2}{x_1^2+x_2^2}$

$$\tilde f(x) = \begin{cases} f(x) & \text{se } x \in \mathbb{R}^2\setminus\{(0,0)\} \\ 0 & \text{se } x=(0,0)\end{cases}$$

$\tilde f$ non è continua in $(0,0)$ $\qquad x_k = \left(\frac1k,\frac1k\right)$ $\quad k\geq 1$

$$\tilde f(x_k) = \tilde f\left(\frac1k,\frac1k\right) = \frac{k^{-2}}{k^{-2}+k^{-2}} = \frac12 \not\to 0 = \tilde f(0,0)$$

$$\frac1t\left[\tilde f(t,0) - \tilde f(0,0)\right] = 0 \xrightarrow[t\to0]{} 0 := \partial_1 \tilde f(0,0)$$

$$\| $$

$$\frac1t\left[\tilde f\big((0,0)+t\underbrace{(1,0)}_{e_1}\big) - \tilde f(0,0)\right]$$

$$\partial_1 \tilde f(x) = \partial_1\left(\frac{x_1x_2}{x_1^2+x_2^2}\right) = \frac{x_2(x_1^2+x_2^2) - x_1x_2\cdot 2x_1}{(x_1^2+x_2^2)^2} = \frac{x_2^3 - x_1^2x_2}{(x_1^2+x_2^2)^2}$$

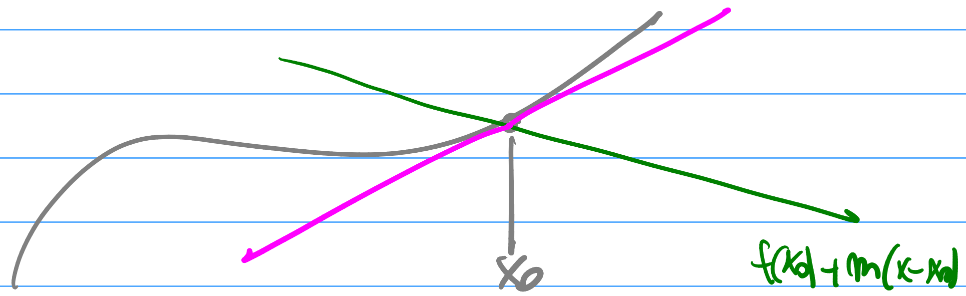
OSS. $f:\mathbb{R}\to\mathbb{R}$ $\quad x_0\in\mathbb{R}$ $\quad$ derivabile

$$\lim_{t\to0}\frac{f(x_0+t)-f(x_0)}{t} = f'(x_0)$$

$$\lim_{t\to0}\frac1t\left[f(x_0+t) - \big(f(x_0)+f'(x_0)t\big)\right] = 0$$

$t = x - x_0$

$$\lim_{\substack{x\to x_0\\(x-x_0\to0)}} \frac{f(x)-\big(f(x_0)+\cancel{f'(x_0)}^{\,m}(x-x_0)\big)}{(x-x_0)} = 0 \quad \big(= f'(x_0) - m\big)$$



DEF. $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ (aperto) $x_0 \in A$ diciamo che f è differenziabile in x_0 se $\exists\, W \in \mathbb{R}^n$ tale che

$$\lim_{x \to x_0} \frac{f(x) - [f(x_0) + W \cdot (x - x_0)]}{\|x - x_0\|_2} = 0$$

iperpiano tangente $x_{n+1} = f(x_0) + W \cdot (x - x_0)$

$$x_{n+1} = [f(x_0) - W \cdot x_0] + W_1 x_1 + W_2 x_2 + \dots + W_n x_n$$

$$W \cdot (x - x_0) = \sum_{i=1}^{n} W_i (x - x_0)_i = \sum_{i=1}^{n} (W_i x_i - W_i x_{0i}) = \dots$$

PROP. $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ (aperto), $x_0 \in A$, f differenziabile in x_0 _

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \Big(\underbrace{\|x - x_0\|_2}_{\to 0} \cdot \underbrace{\frac{f(x) - f(x_0)}{\|x - x_0\|_2}}_{\text{limitato}} \Big) = 0$$

$$\frac{f(x) - f(x_0) - W \cdot (x - x_0)}{\|x - x_0\|_2} \to 0$$

$$\frac{|f(x) - f(x_0) - W \cdot (x - x_0)|}{\|x - x_0\|_2} \geq \frac{|f(x) - f(x_0)|}{\|x - x_0\|_2} - \frac{\|W\|_2 \|x - x_0\|_2}{\|x - x_0\|_2}$$

$$\Rightarrow \frac{|f(x) - f(x_0)|}{\|x - x_0\|_2} \leq \|W\|_2 + \underbrace{\frac{|f(x) - f(x_0) - W \cdot (x - x_0)|}{\|x - x_0\|_2}}_{\to 0} \leq \|W\|_2 + 1$$

① f differenziabile $\Rightarrow$ f continua

$x = x_0 + t e_k \qquad k = 1, \dots, n$

$$\frac{f(x) - f(x_0) - w\cdot(x - x_0)}{\|x - x_0\|_2} = \frac{f(x_0 + te_k) - f(x_0) - w\cdot te_k}{\|te_k\|_2}$$

$$= \frac{f(x_0+te_k) - f(x_0)}{|t|} - \frac{t\, w_k}{|t|} \begin{cases} \xrightarrow{t>0} \quad \dfrac{f(x_0+te_k)-f(x_0)}{t} - w_k \\ \xrightarrow{t<0} \quad -\left(\dfrac{f(x_0+te_k)-f(x_0)}{t} - w_k\right) \end{cases}$$

$$\Rightarrow \quad \partial_k f(x_0) - w_k = 0 \Rightarrow w_k = \partial_k f(x_0) \qquad k = 1, \dots, n$$

$$\Rightarrow \quad w = \nabla f(x_0) = (\partial_1 f(x_0), \partial_2 f(x_0), \dots, \partial_n f(x_0)) \qquad (2)$$

gradiente di f in x_0

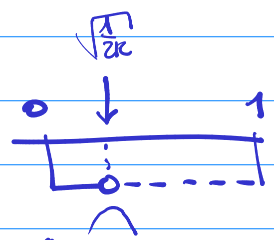
(3) $\quad \partial_w f(x_0) = w \cdot \nabla f(x_0) \qquad \forall\, w \in \mathbb{R}^n,\ w \neq 0$

ESEMPIO. $\quad f_k(x) = kx\, e^{-kx^2} \qquad x \in [0,1],\ k \in \mathbb{N}$

$$f_k(x) \xrightarrow[k\to+\infty]{} \begin{cases} 0 & \text{se } x = 0 \\ 0 & \text{se } x \in (0,1] \end{cases} = f_\infty(x)$$

$$\|f_k - f_\infty\|_\infty = \max_{[0,1]} |f_k(x)|$$

$$f_k'(x) = k e^{-kx^2}\,[1 - 2kx^2] \geq 0$$



$$\|f_k\|_\infty = \|f_k - f_\infty\|_\infty = f_k\!\left(\frac{1}{\sqrt{2k}}\right) = \frac{k}{\sqrt{2k}}\, e^{-1/2} = \sqrt{\frac{k}{2}} \xrightarrow[k\to+\infty]{} +\infty$$