Esercizio 1

1

$$A = |1\rangle\langle 2| + |2\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B = \exp\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \exp\left[\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\right] =$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1/e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{2}\left(e + \frac{1}{e}\right)\mathbb{1} + \frac{1}{2}\left(e - \frac{1}{e}\right)A$$

$$e^{|2\rangle\langle 1|}\, e^{|1\rangle\langle 2|} = (\mathbb{1} + |2\rangle\langle 1|)(\mathbb{1} + |1\rangle\langle 2|)$$
$$= \mathbb{1} + A + |2\rangle\langle 2|$$

Quindi

$$C = \left[e^A, e^{|2\rangle\langle 1|} e^{|1\rangle\langle 2|}\right] =$$
$$= \left[\frac{1}{2}\left(e + \frac{1}{e}\right)\mathbb{1} + \frac{1}{2}\left(e - \frac{1}{e}\right)A,\ \mathbb{1} + A + |2\rangle\langle 2|\right]$$
$$= \frac{1}{2}\left(e - \frac{1}{e}\right)\left[A, |2\rangle\langle 2|\right] = \frac{1}{2}\left(e - \frac{1}{e}\right)\left(|1\rangle\langle 2| - |2\rangle\langle 1|\right)$$

2

$$a = -be^{-i\phi} \qquad b = \sqrt{\frac{2}{3}}$$

3

$$H = \epsilon|E_+\rangle\langle E_+| - \epsilon|E_-\rangle\langle E_-| =$$
$$= \epsilon\left(\frac{1}{3} - b^2\right)|1\rangle\langle 1| + \epsilon\left(|a|^2 - \frac{1}{3}\right)|2\rangle\langle 2|$$
$$+ |1\rangle\langle 2|\,\epsilon\left(\frac{a^*}{\sqrt{3}} - \frac{be^{i\phi}}{\sqrt{3}}\right) + |2\rangle\langle 1|\,\epsilon\left(\frac{a}{\sqrt{3}} - \frac{be^{-i\phi}}{\sqrt{3}}\right)$$
$$= -\frac{\epsilon}{3}|1\rangle\langle 1| + \frac{\epsilon}{3}|2\rangle\langle 2|$$
$$- \frac{2\sqrt{2}}{3}\epsilon\left(e^{i\phi}|2\rangle\langle 1| + e^{-i\phi}|1\rangle\langle 2|\right)$$

②

1) Spettro

$E = -\frac{5}{6} E_I \quad n=1 \quad S_x = -\frac{\hbar}{2} \quad$ deg. 1

$E = -\frac{1}{2} E_I \quad n=1 \quad S_x = \frac{\hbar}{2} \quad$ deg. 1

$E = -\frac{1}{12} E_I \quad n=2 \quad S_x = -\frac{\hbar}{2} \quad$ deg. 4

$E = \frac{1}{18} E_I \quad n=3 \quad S_x = -\frac{\hbar}{2} \quad$ deg. 9 $\longrightarrow$ STATO SUCCESSIVO HA $n=4$

2) Se $|s_x\rangle_x$ sono le autofunzioni di S_x

$$\psi_0 = \frac{1}{\sqrt{2}} |2\,1\,1\rangle \left(|\tfrac{1}{2}\rangle_x - |\tfrac{-1}{2}\rangle_x \right)$$

$$\psi_0(t) = \frac{1}{\sqrt{2}} |2\,1\,1\rangle \left(e^{-i\alpha t} |\tfrac{1}{2}\rangle_x - e^{i\alpha t} |\tfrac{-1}{2}\rangle_x \right) \qquad \alpha = \frac{E_I}{6\hbar} \left[\begin{array}{l}\text{modulo}\\ \text{fase}\\ \text{globale}\end{array}\right]$$

$$S_x \psi_0(t) = \frac{1}{\sqrt{2}} \frac{\hbar}{2} |2\,1\,1\rangle \left(e^{-i\alpha t} |\tfrac{1}{2}\rangle_x + e^{+i\alpha t} |\tfrac{-1}{2}\rangle_x \right)$$

$$L_x \psi_0(t) = \frac{1}{2} (L_- + \cancel{L_+}) \psi_0 = \frac{1}{2} \frac{1}{\sqrt{2}} L_- |2\,1\,1\rangle \left(e^{-i\alpha t} |\tfrac{1}{2}\rangle_x - e^{i\alpha t} |\tfrac{-1}{2}\rangle_x \right)$$

$$= \frac{\hbar}{2} |2\,1\,0\rangle \left(e^{-i\alpha t} |\tfrac{1}{2}\rangle_x - e^{i\alpha t} |\tfrac{-1}{2}\rangle_x \right)$$

$$J_x \psi_0(t) = \frac{\hbar}{2\sqrt{2}} |2\,1\,1\rangle \left(e^{-i\alpha t} |\tfrac{1}{2}\rangle_x + e^{i\alpha t} |\tfrac{-1}{2}\rangle_x \right)$$

$$+ \frac{\hbar}{2} |2\,1\,0\rangle \left(e^{-i\alpha t} |\tfrac{1}{2}\rangle_x - e^{+i\alpha t} |\tfrac{-1}{2}\rangle_x \right)$$

$$\langle \psi_0 | J_x^2 | \psi_0 \rangle = \left(\frac{\hbar}{2\sqrt{2}}\right)^2 + \left(\frac{\hbar}{2\sqrt{2}}\right)^2 + \left(\frac{\hbar}{2}\right)^2 + \left(\frac{\hbar}{2}\right)^2 = \frac{3\hbar^2}{4}$$

st fond : $|1\,0\,0\rangle\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_x$

$$\langle\psi_f|\,\frac{\lambda}{r^3}\,r_x^2\,S_x^2\,|\psi_f\rangle =$$

$$= \lambda\,\langle 1\,0\,0|\,\frac{r_x^2}{r^3}\,\frac{\hbar^2}{4}\,|1\,0\,0\rangle$$

$$= \frac{\lambda\hbar^2}{4}\cdot\frac{1}{3}\langle 1\,0\,0|\,\frac{(r_x^2+r_y^2+r_z^2)}{r^3}\,|1\,0\,0\rangle$$ per simmetria dello stato fondamentale

$$= \frac{\lambda\hbar^2}{12}\,\langle 1\,0\,0|\,\frac{1}{r}\,|1\,0\,0\rangle$$

$$= \frac{\lambda\hbar^2}{12}\,\frac{1}{\pi a_0^3}\,4\pi\int_0^\infty r\,dr\,e^{-2r/a_0} = \frac{\lambda\hbar^2}{12a_0}$$