1

a)

$$[H, J^2] = \frac{\alpha}{\hbar}[L_z + 2S_z, J^2] = \frac{\alpha}{\hbar}[S_z, J^2]$$

$$[L_z, J^2] = [L_z, 2\bar{L}\cdot\bar{S}] = -2i\hbar(S_y L_x - S_x L_y)$$

$$[S_z, J^2] = [S_z, 2\bar{L}\cdot\bar{S}] = -2i\hbar(L_y S_x - L_x S_y)$$

Quindi $[H, J^2] = 2i\alpha(L_x S_y - L_y S_x)$

b)

$[H, J_x] = i\alpha(L_y + 2S_y)$ $\quad [H, J_y] = -i\alpha(L_x + 2S_x)$ $\quad [H, J_z] = 0$

2)

Se $\chi^{\pm}$ sono gli autovettori di S_z con autoval. $\pm\hbar/2$

$$\psi = \frac{1}{2}(|1\,-1\rangle + i|1\,1\rangle)(\chi^+ + \chi^-)$$

Nella base $|J\,J_z\rangle_J$ abbiamo

$$\psi = \frac{1}{2}\left[\sqrt{\frac{1}{3}}\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle_J - \sqrt{\frac{2}{3}}\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle_J\right] + \frac{i}{2}\left|\frac{3}{2}\,\frac{3}{2}\right\rangle_J$$

$$+\frac{1}{2}\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle_J + \frac{i}{2}\left[\sqrt{\frac{1}{3}}\left|\frac{3}{2}\,\frac{1}{2}\right\rangle_J + \sqrt{\frac{2}{3}}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle_J\right]$$

(a)

$$P\left(J^2 = \frac{15}{4}\hbar^2\right) = \frac{1}{12} + \frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{2}{3}$$

$$P\left(J^2 = \frac{3\hbar^2}{4}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

(b)

$P\left(J_z = \frac{3\hbar}{2}\right) = \frac{1}{4}$ $\qquad P\left(J_z = \frac{\hbar}{2}\right) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$

$P\left(J_z = -\frac{\hbar}{2}\right) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$ $\qquad P\left(J_z = -\frac{3\hbar}{2}\right) = \frac{1}{4}$

(c) $\psi(t) = \frac{1}{2}|-1\ \tfrac{1}{2}\rangle + \frac{i}{2}e^{-i\omega t}|1\ \tfrac{1}{2}\rangle + \frac{1}{2}e^{i\omega t}|-1\ -\tfrac{1}{2}\rangle + \frac{i}{2}|1\ -\tfrac{1}{2}\rangle$

$L_x|\psi t\rangle = \frac{1}{2}(L_+ + L_-)|\psi t\rangle =$

QUI $\omega = 2\alpha/\hbar$

$|\ell_z\ s_z\rangle$ = base

$$= \frac{\sqrt{2}\hbar}{2}\Big[\frac{1}{2}|0\ \tfrac{1}{2}\rangle + \frac{i}{2}e^{-i\omega t}|0\ \tfrac{1}{2}\rangle$$
$$+ \frac{1}{2}e^{i\omega t}|0\ -\tfrac{1}{2}\rangle + \frac{i}{2}|0\ -\tfrac{1}{2}\rangle\Big]$$

$$= \frac{\sqrt{2}\hbar}{4}\Big[(1+ie^{-i\omega t})|0\ \tfrac{1}{2}\rangle + (i+e^{i\omega t})|0\ -\tfrac{1}{2}\rangle\Big]$$

$$\langle\psi t|L_x^2|\psi t\rangle = \frac{\hbar^2}{8}\Big[|1+ie^{-i\omega t}|^2 + |i+e^{i\omega t}|^2\Big]$$
$$= \frac{\hbar^2}{2}(1+\sin\omega t) = \frac{\hbar^1}{2}\left(1+\sin\frac{2\alpha t}{\hbar}\right)$$

(3)

$\Psi_{fond} = \Psi_{100}\,\chi^-$ $\qquad \Psi_{100}$ st. fond. atomo idrogeno

Se $\varphi_\pm$ sono le autofuntioni di S_x con autoval $\pm\hbar/2$

$$\Psi_{fond} = \frac{1}{\sqrt{2}}\Psi_{100}(\varphi_+ - \varphi_-)$$

(a)

$$e^{-i\pi J_x/4\hbar}\Psi_{fond} = e^{-i\pi L_x/4\hbar}\,e^{-i\pi S_x/4\hbar}\,\Psi_{fond}$$
$$= \frac{1}{\sqrt{2}}\Psi_{100}\Big[e^{-i\pi S_x/4\hbar}(\varphi_+ - \varphi_-)\Big]$$
$$= \frac{1}{\sqrt{2}}\Psi_{100}\Big[e^{-i\pi/8}\varphi_+ - e^{i\pi/8}\varphi_-\Big]$$

Quindi

$$\langle 0|e^{-i\pi J_x/4\hbar}|0\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}e^{-i\pi/8} + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}e^{i\pi/8}\right) = \cos\frac{\pi}{8}$$

b)

$$\langle 0|p^2|0\rangle = \langle \psi_{100}|p^2|\psi_{100}\rangle \leftarrow$$ Hamiltoniana puramente Coulombiana

$$\psi_{100} = \frac{1}{a_B^{3/2}\sqrt{\pi}}\, e^{-r/a_B}$$ a_B = raggio di Bohr

$$\vec{p}|\psi_{100}\rangle = -i\hbar\nabla\psi_{100} = -i\hbar\,\hat{u}_r\left(-\frac{1}{a_B}\right)\psi_{100} = \frac{i\hbar}{a_B}\hat{u}_r\psi_{100}$$

$$\langle\psi_{100}|p^2|\psi_{100}\rangle = \left|\vec{p}|\psi_{100}\rangle\right|^2 = \frac{\hbar^2}{a_B^2}\langle\psi_{100}|\psi_{100}\rangle = \frac{\hbar^2}{a_B^2}$$

Alternativamente $\vec{p}^{\,2} = -\hbar^2\nabla^2 = -\hbar^2\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{L^2}{\hbar^2 r^2}\right)$

$$p^2\psi_{100} = \frac{1}{\sqrt{\pi}\,a_B^{3/2}}(-\hbar^2)\left(\frac{1}{a_B^2} - \frac{2}{a_B r}\right)e^{-r/a_B}$$

Quindi

$$\langle\psi_{100}|p^2|\psi_{100}\rangle = \int r^2 dr\, d\Omega\left(-\frac{\hbar^2}{\pi a_B^3}\right)\left(\frac{1}{a_B^2} - \frac{2}{a_B r}\right)e^{-2r/a_B}$$

$$= \frac{\hbar^2}{a_B^2}$$