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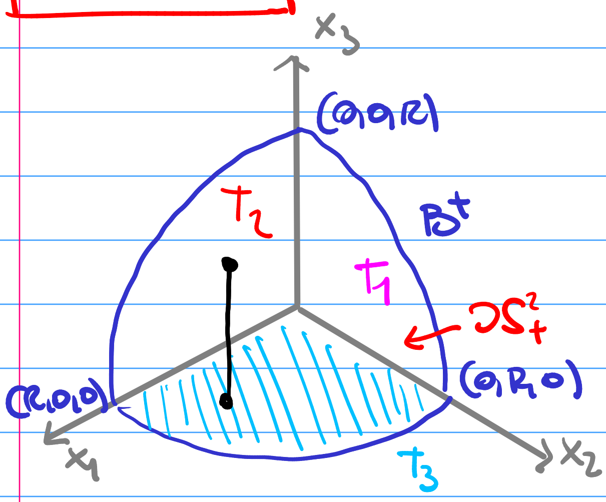


$F \in C^1(A, \mathbb{R}^3) \quad A \subseteq \mathbb{R}^3$ aperto

$B^+ \subseteq A \quad (B^+ = B(0,R) \cap \{x_1, x_2, x_3 > 0\})$

$$\operatorname{div}(F)(x) = (\nabla \cdot F)(x) = \partial_1 F_1(x) + \partial_2 F_2(x) + \partial_3 F_3(x)$$

$$\int_{B^+} \operatorname{div}(F)(x)\,dx = \int_{B^+} (\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3)(x)\,dx$$

$$= \int_{T_1} \left(\int \partial_1 F_1 \, dx_1 \right) dx_2\, dx_3 + \int_{T_2} \left(\int \partial_2 F_2 \, dx_2 \right) dx_1\, dx_3 + \int_{T_3} \left(\int \partial_3 F_3 \, dx_3 \right) dx_1\, dx_2$$

$$\int_{T_3} \left(\int \partial_3 F_3 \, dx_3 \right) dx_1\, dx_2 = \int_{\substack{\{x_1^2 + x_2^2 < R^2\} \\ x_3 = 0}} \left[\int_0^{\sqrt{R^2 - x_1^2 - x_2^2}} \partial_3 F_3(x)\, dx_3 \right] dx_1\, dx_2$$

$$= \int_{\{x_1^2 + x_2^2 < R^2\}} \left[F_3 \left(x_1, x_2, \sqrt{R^2 - x_1^2 - x_2^2} \right) - F_3(x_1, x_2, 0) \right] dx_1\, dx_2$$

$$\longrightarrow \int_{\substack{\{x_1^2 + x_2^2 < R^2\} \\ x_1, x_2 > 0}} -F_3(x_1, x_2, 0)\, dx_1\, dx_2 = \int_{T_3} F(x) \cdot \underbrace{N(x)}_{\| \atop (-e_3)}\, d\sigma$$

$\bar{x}(u) = (u_1, u_2, 0)$ $\qquad \partial_1 \bar{x} = (1, 0, 0)$

$(u_1, u_2) \in \{u_1^2 + u_2^2 < R^2,\ u_1, u_2 > 0\}$ $\qquad \partial_2 \bar{x} = (0, 1, 0)$

$N = \partial_1 \bar{x} \wedge \partial_2 \bar{x} = e_3$

$$y(u) = \left(u_1, u_2, \sqrt{R^2-u_1^2-u_2^2}\right) \qquad (u_1,u_2) \in \underbrace{\{u_1^2+u_2^2 \leq R^2 ; u_1, u_2 \geq 0\}}_{T_3}$$

$$\partial_1 y(u) = \left(1, 0, -\frac{u_1}{\sqrt{R^2-u_1^2-u_2^2}}\right)$$

$$\partial_2 y = \left(0, 1, -\frac{u_2}{\sqrt{R^2-u_1^2-u_2^2}}\right)$$

$$(\partial_1 y \wedge \partial_2 y)(u) = \left(\frac{u_1}{\sqrt{R^2-u_1^2-u_2^2}} ; \frac{u_2}{\sqrt{R^2-u_1^2-u_2^2}} ; 1\right)$$

$$(0,0,\underset{\nearrow}{F_3(x)})\cdot \underset{\uparrow}{N(x)}\, d\sigma \longleftarrow \|\partial_1 y(u) \wedge \partial_2 y(u)\|_2\, du$$

$$F_3(y(u)) \qquad \frac{\partial_1 y(u) \wedge \partial_2 y(u)}{\|\partial_1 y(u) \wedge \partial_2 y(u)\|_2}$$

$$\int_{T_3} (0,0,F_3(y(u)))\cdot\left[\partial_1 y(u) \wedge \partial_2 y(u)\right] du$$

$$= \int_{T_3} F_3\left(u_1, u_2, \sqrt{R^2-u_1^2-u_2^2}\right) du_1\, du_2$$

$$\Rightarrow \int_{B^+} \partial_3 F_3(x)\, dx = \int_{T_3} -F_3(x)\cdot e_3\, dx + \int_{\partial S^2_+} (0,0,F_3)\cdot N\, d\sigma$$

$$= \int_{T_3} (0,0,F_3)\cdot \overset{-e_3}{\overset{\shortparallel}{N}}\, d\sigma + \int_{T_1} (0,0,F_3)\cdot \underset{-e_1}{\underset{\shortparallel}{N}}\, d\sigma + \int_{T_2} (0,0,F_3)\cdot \underset{-e_2}{\underset{\shortparallel}{N}}\, d\sigma$$

$$+ \int_{\partial S^2_+} (0,0,F_3)\cdot N\, d\sigma$$

$$\Rightarrow \int_{B^+} \operatorname{div}(F)\, dx = \int_{\partial B^+} F\cdot N\, d\sigma$$

TEOREMA DELLA DIVERGENZA (GAUSS)

$F \in C^1(A, \mathbb{R}^3)$, $A \subseteq \mathbb{R}^3$ aperto

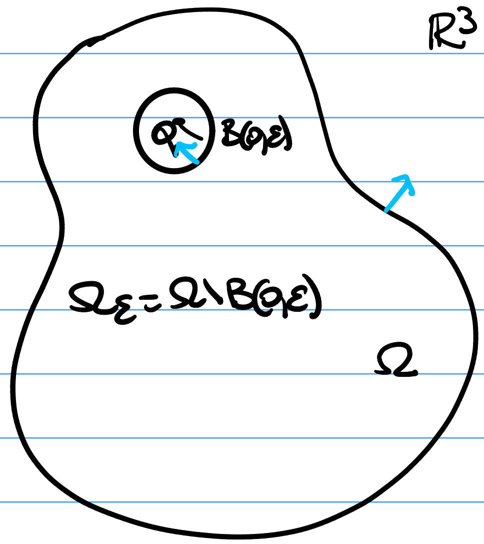
$\Omega \subseteq A$ aperto con bordo regolare a tratti

allora

$$\int_{\Omega} \mathrm{div}(F)(x)\,dx = \int_{\Omega} \nabla \cdot F(x)\,dx = \int_{\partial\Omega} F(x)\cdot N(x)\,d\sigma = \Phi_{\partial\Omega}(F)$$

(N normale uscente a $\partial\Omega$)

OSS. $E(x) = k \dfrac{x}{\|x\|_2^3}$ $\quad x \in \mathbb{R}^3,\ x \neq 0$



$$\Phi_{\partial B(q,\varepsilon)}(E) = 4\pi k$$

$$\int_{\Omega_\varepsilon} \mathrm{div}(E)\,dx = \Phi_{\partial\Omega_\varepsilon}(E)$$

$$= \Phi_{\partial\Omega}(E) - \Phi_{\partial B(q,\varepsilon)}(E)$$

$$\mathrm{div}(E) = k\left[\partial_1 \frac{x_1}{\|x\|_2^3} + \partial_2 \frac{x_2}{\|x\|_2^3} + \partial_3 \frac{x_3}{\|x\|_2^3}\right]$$

$$\partial_i \frac{x_i}{\|x\|_2^3} = \partial_i \frac{x_i}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \frac{(x_1^2+x_2^2+x_3^2)^{3/2} - x_i \cdot \frac{3}{2}(x_1^2+x_2^2+x_3^2)^{1/2}\cdot 2x_i}{(x_1^2+x_2^2+x_3^2)^3}$$

$$= \frac{x_1^2+x_2^2+x_3^2-3x_i^2}{\|x\|_2^5} = \frac{x_j^2+x_k^2-2x_i^2}{\|x\|_2^5}$$

$i,k,j \in \{1,2,3\}$
$i\neq k,\ i \neq j,\ k \neq j$

$$\nabla E = k\left[\frac{1}{\|x\|_2^5}\left(\underbrace{x_2^2+x_3^2-2x_1^2}_{\partial_1 E_1} + \underbrace{x_1^2+x_3^2-2x_2^2}_{\partial_2 E_2} + \underbrace{x_1^2+x_2^2-2x_3^2}_{\partial_3 E_3}\right)\right] = 0$$

$\Rightarrow \quad \Phi_{\partial\Omega}(E) = 4\pi k$ chiunque sia Ω !

ESERCIZIO 8. *Calcolare l'area dell'insieme D delimitato dalla curva*

$$\gamma: \quad \{x(t) = (x_1(t), x_2(t)) = \big(\cos(t)\sin^2(t), \sin(t)\cos(t)\big), t \in [0, \pi/2]\}$$

¿ $m_2(D)$? $\quad m_2(D) = \int_D \underset{\substack{\uparrow ?\\ \operatorname{div}(F)}}{1}\, dx_1 dx_2 \;\overset{\uparrow}{=}\; \int_{\underset{\substack{\shortparallel\\ \gamma}}{\partial D}} (F\cdot N)\, ds$

$F(x) = (x_1, 0) \longrightarrow \operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2 = \partial_1 x_1 + \partial_2 0 = 1$

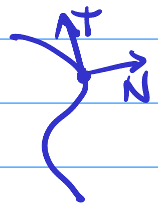
$F(x) = (0, x_2) \longrightarrow \operatorname{div}(F) = 1$

$F(x) = \frac{1}{2}(x_1, x_2) \longrightarrow \operatorname{div}(F) = 1$

$x'(t) = \big(-\operatorname{sen}^3(t) + 2\operatorname{sen}(t)\cos^2(t), -\operatorname{sen}^2(t) + \cos^2(t)\big)$

$\big(-\operatorname{sen}(t) + 3\operatorname{sen}(t)\cos^2(t), \cos^2(t) - \operatorname{sen}^2(t)\big)$

$-\operatorname{sen}^3(t) = -\operatorname{sen}(t)(1-\cos^2(t)) = -\operatorname{sen}(t) + \operatorname{sen}(t)\cos^2(t)$



$N(t) = \big(\operatorname{sen}^2(t) - \cos^2(t), -\operatorname{sen}(t) + 3\operatorname{sen}(t)\cos^2(t)\big) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x'(t)$

$\int_\gamma (F\cdot N)\, ds = \int_0^{\pi/2} \big(\underset{\substack{\downarrow}}{\overset{F(x(t))}{\cos(t)\operatorname{sen}^2(t), 0}}\big)\cdot\big(\operatorname{sen}^2(t) - \cos^2(t), -\operatorname{sen}(t) + 3\operatorname{sen}(t)\cos^2(t)\big)\, dt$

$= \int_0^{\pi/2} \big[\cos(t)\operatorname{sen}^4(t) - \operatorname{sen}^2(t)\cos^3(t)\big]\, dt =$

$\int (F\cdot N)\, ds = \int F(x(t)) \frac{N x'(t)}{\cancel{\|N x'(t)\|_2}} \underset{\substack{\shortparallel \\ \|x'(t)\|_2}}{\cancel{\|N x'(t)\|_2}}\, dt$

$$= \int_0^{\pi/2} \left[\cos(t)\,\text{sen}^4(t) - \text{sen}^2(t)\cos(t)\left(1-\text{sen}^2(t)\right)\right] dt$$

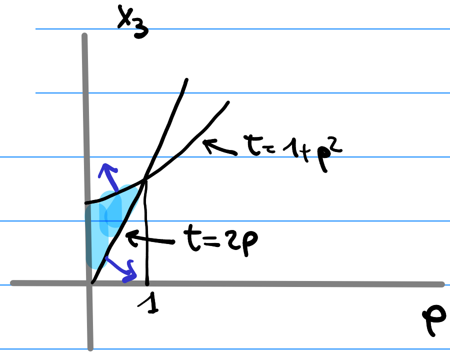
$$= \int_0^{\pi/2} \left[2\cos(t)\,\text{sen}^4(t) - \cos(t)\,\text{sen}^2(t)\right] dt$$

$$= \left[\frac{2}{5}\text{sen}^5(t) - \frac{1}{3}\text{sen}^3(t)\right]_0^{\pi/2} = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$$

ESERCIZIO 11. *Calcolare il flusso di* $F = (x_3, x_1^2 x_2, x_2^2 x_3)$ *uscente dalla superficie che delimita il solido*

$$E = \{x \in \mathbb{R}^3 : 2\sqrt{x_1^2 + x_2^2} \leq x_3 \leq 1 + x_1^2 + x_2^2\}$$

$x_3 = 2\rho$ $\qquad$ $x_3 = 1 + \rho^2$



$$\Phi_{\partial E}(F) = \int_{\partial E} [F(x)\cdot N(x)]\,d\sigma \overset{\text{GAUSS}}{=} \int_E [\nabla \cdot F(x)]\,dx$$

$$\text{div}(F) = \nabla\cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 = 0 + x_1^2 + x_2^2 = x_1^2 + x_2^2$$

$$\int_E \text{div}(F)\,dx \overset{\text{C.C.}}{=} \int_0^{2\pi}\int_0^1\int_{2\rho}^{1+\rho^2} \rho^2\cdot\rho\;dt\,d\rho\,d\theta = 2\pi\int_0^1 \left[\rho^3 t\right]_{2\rho}^{1+\rho^2} d\rho$$

$$= 2\pi\int_0^1 (\rho^3 + \rho^5 - 2\rho^4)\,d\rho = 2\pi\left[\frac{1}{4}\rho^4 + \frac{1}{6}\rho^6 - \frac{2}{5}\rho^5\right]_0^1$$

$$= 2\pi\,\frac{15+10-24}{60} = \frac{\pi}{30}$$

$\cancel{x(u) = (u_1, u_2, 2\sqrt{u_1^2+u_2^2}) \qquad (u_1,u_2) \in B(0,1)\setminus B(0,\varepsilon)}$

$\bar{x}(s) = (s_1\cos(s_2), s_1\,\text{sen}(s_2), 2s_1) \qquad s_1 \in [0,1],\ s_2 \in [0, 2\pi]$

$\qquad s \in [0,1]\times[0,2\pi]$

$y(u) = (u_1, u_2, 1 + u_1^2 + u_2^2) \qquad u \in B(0,1)$

$\partial_1 \bar{x}(s) = (\cos(s_2), \text{sen}(s_2), 2)$

$\partial_2 \bar{x}(s) = (-s_1\,\text{sen}(s_2), s_1\cos(s_2), 0)$

$$(\partial_1\bar{x}\wedge\partial_2\bar{x})(s)=(-2s_1\cos(s_2);\ -2s_1\,\text{sen}(s_2);\ s_1) = -v(s)$$

entrante in E!

$$\partial_1 y(s) = (1, 0, 2u_1)$$
$$\partial_2 y(s) = (0, 1, 2u_2)$$
$$(\partial_1 y\wedge\partial_2 y) = (-2u_1, -2u_2, 1) \leftarrow \text{uscente da } E$$

$$\Phi_{\partial E_1}(F) = \int_{\text{cono}} (F\cdot N)\,d\sigma = \int_{[0,1]\times[0,2\pi]} (x_3, x_1^2 x_2, x_2^2 x_3)\cdot(-\partial_1\bar{x}\wedge\partial_2\bar{x})\,ds$$

$$= \int_{[0,1]\times[0,2\pi]} (2s_1, s_1^2\cos^2(s_2)\cdot s_1\,\text{sen}(s_2), s_1^2\,\text{sen}^2(s_2)\cdot 2s_1)\cdot (2\cos(s_2)s_1, 2s_1\,\text{sen}(s_2), -s_1)\,ds$$

$$= \int_{[0,1]\times[0,2\pi]} [\underbrace{\cancel{4s_1^2\cos(s_2)}}_{=0} + 2s_1^4\cos^2(s_2)\,\text{sen}^2(s_2) - 2s_1^4\,\text{sen}^2(s_2)]\,ds_1 ds_2$$

$$= 2\Big[\underbrace{\int_0^1 s_1^4\,ds_1}_{=\frac{1}{5}}\Big]\cdot\Big[\underbrace{\int_0^{2\pi}\cos^2(s_2)\,\text{sen}^2(s_2)\,ds_2}_{=\int_0^{2\pi}\frac{1}{4}\text{sen}^2(2s) = \pi/4}\Big] - 2\Big[\underbrace{\int_0^1 s_1^4\,ds_1}_{=\frac{1}{5}}\Big]\cdot\Big[\underbrace{\int_0^{2\pi}\text{sen}^2(s_2)\,ds_2}_{=\pi}\Big]$$

$$= 2\cdot\frac{1}{5}\cdot\frac{\pi}{4} - 2\cdot\frac{1}{5}\cdot\pi = -\frac{3\pi}{10}$$

$$\Phi_{\partial E_2}(F) = \int_{\text{paraboloide}} F(x)\cdot N(x)\,d\sigma = \int_{B(0,1)} (1+u_1^2+u_2^2, u_1^2 u_2, u_2^2(1+u_1^2+u_2^2))\cdot (-2u_1, -2u_2, 1)\,du_1 du_2$$

$$= \int_{B(0,1)} [\underbrace{\cancel{-2u_1}}_{=0} \underbrace{\cancel{-2u_1^3}}_{=0} \underbrace{\cancel{-2u_1 u_2^2}}_{=0} - 2u_1^2 u_2^2 + u_2^2 + u_1^2 u_2^2 + u_2^4]\,du_1 du_2$$

$$\underset{\text{C.P.}}{=} \int_0^1\int_0^{2\pi} [-\rho^5\cos^2(\vartheta)\,\text{sen}^2(\vartheta) + \rho^3\,\text{sen}^2(\vartheta) + \rho^5\underbrace{\text{sen}^4(\vartheta)}_{\text{sen}^2(\vartheta)(1-\cos^2(\vartheta))}]\,d\vartheta\, d\rho$$

$$= \int_0^1 \int_0^{2\pi} \left[\rho^3 \operatorname{sen}^2(\theta) + \rho^5 \operatorname{sen}^2(\theta) - 2\rho^5 \operatorname{sen}^2(\theta)\cos^2(\theta)\right] d\theta\, d\rho$$

$$= \left[\int_0^1 \rho^3 d\rho\right] \cdot \overbrace{\left[\int_0^{2\pi} \operatorname{sen}^2(\theta) d\theta\right]}^{=\pi} + \left[\int_0^1 \rho^5 d\rho\right] \overbrace{\left[\int_0^{2\pi} \operatorname{sen}^2(\theta) d\theta\right]}^{=\pi}$$

$$-2 \left[\int_0^1 \rho^5 d\rho\right]\left[\frac{1}{4}\underbrace{\int_0^{2\pi} \operatorname{sen}^2(2\theta) d\theta}_{=\pi}\right]$$

$$= \frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3}$$

$$\Phi_{\partial E}(F) = \frac{\pi}{3} - \frac{3}{10}\pi = \frac{10-9}{30}\pi = \frac{\pi}{30}$$

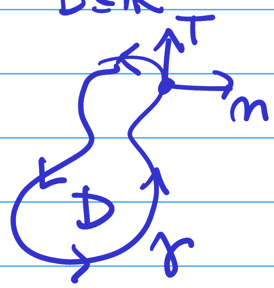
$$\int_0^{k\pi} \operatorname{sen}^2(\theta) d\theta = \int_0^{k\pi} \frac{1}{2}(1 - \underbrace{\cos(2\theta)}_{\pi\text{- periodica, di media nulla}}) d\theta = \int_0^{k\pi} \frac{1}{2} d\theta = \frac{k\pi}{2} \quad , k \in \mathbb{Z}$$

$$\int_0^{k\pi} \operatorname{sen}^2(\theta) d\theta = \frac{1}{2}\int_0^{k\pi} (\operatorname{sen}^2(\theta) + \cos^2(\theta)) d\theta = \frac{1}{2}\int_0^{k\pi} d\theta = \frac{k\pi}{2}$$

OSSERVAZIONI

$$\int_D \operatorname{div}(F)(x) dx = \int_{\partial D} F(x) \cdot n(x) ds$$

$D \subseteq \mathbb{R}^2$



$$\int_D \left[\partial_1 F_1(x) + \partial_2 F_2(x)\right] dx_1 dx_2$$

$$= \int_{\partial D} \left[F_1(x) \cdot n_1(x) + F_2(x) \cdot n_2(x)\right] ds$$

$$= \int_\gamma \left[F_1(x) \cdot T_2(x) + F_2(x) \cdot (-T_1(x))\right] ds$$

scelgo $F(x) = (A(x), 0)$

$$\Rightarrow \int_D \partial_1 A(x) dx_1 dx_2 = \int_{\partial D^+} A(x) \cdot T_2(x) ds = \int_{\partial D^+} A(x)\, dx_2$$

se $F(x) = (0, A(x))$ otteniamo

$$\int_D \partial_2 A(x)\, dx_1 dx_2 = -\int_{\partial D^+} A(x) \cdot T_1(x)\, ds = \int_{\partial D^+} -A(x)\, dx_1$$

FORMULE DI GAUSS-GREEN

$$\int_D (\partial_1 F_1(x) + \partial_2 F_2(x))\, dx_1 dx_2 = \int_{\partial D^+} [F_1(x) T_2(x) - F_2(x) T_1(x)]\, ds$$

$$= \int_{\partial D^+} \underbrace{(-F_2(x), F_1(x))}_{H(x) = (H_1(x), H_2(x))} \cdot \underbrace{(T_1(x), T_2(x))}_{T(x)}\, ds$$

$$\Rightarrow \int_{\partial D^+} (H(x) \cdot T(x))\, ds = \int_D [\partial_1 H_2(x) - \partial_2 H_1(x)]\, dx_1 dx_2$$

$$= \int_D [\mathrm{rot}(H)]_3\, dx_1 dx_2$$

TEOREMA DEL ROTORE (STOKES)

$H \in C^1(A, \mathbb{R}^3)$, $\Sigma \subseteq A \subseteq \mathbb{R}^3$, $\Sigma = x(D)$, (x, D) superficie regolare, allora

$$\int_\Sigma \mathrm{rot}(H) \cdot N(x)\, d\sigma = \int_{\partial \Sigma^+} H(x) \cdot T(x)\, ds$$