TEOREMA $f(x,t)$ continua in $[a,b]\times[c,d]$

$\frac{\partial f}{\partial t}(x,t)$ " " " "

Allora la funzione $g(t)=\int_a^b f(x,t)\,dx$ è di classe $C^1([c,d])$ e

$$g'(t)=\int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$$

$$\frac{d}{dt}\left[\int_a^b f(x,t)\,dx\right]=\int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$$

DIM.

$$\frac{1}{h}\left[g(t+h)-g(t)\right]=\frac{1}{h}\int_a^b \left[f(x,t+h)-f(x,t)\right]dx=(*)$$

OSS $\varphi(t+h)-\varphi(t)=\int_t^{t+h}\varphi'(\tau)\,d\tau \quad \forall \varphi\in C^1$

$$(*)=\frac{1}{h}\int_a^b dx\int_t^{t+h}d\tau\,\frac{\partial f}{\partial t}(x,\tau)=\frac{1}{h}\int_t^{t+h}d\tau\underbrace{\left[\int_a^b dx\,\frac{\partial f}{\partial t}(x,\tau)\right]}_{r(\tau)}=$$

$$=\frac{1}{h}\int_t^{t+h}d\tau\; r(\tau)\overset{\text{teorema della media}}{=}$$

$$=r(t+\theta h)\qquad \theta\in[0,1].$$

OSS per il precedente teorema $r(\tau)$ è continua in tutto $[c,d]$. (in quanto $\frac{\partial f}{\partial t}$ è continua)

$$\lim_{h\to 0} \frac{1}{h}\left[g(t+h)-g(t)\right] = \lim_{h\to 0} r(\underbrace{t+\theta h}_{\to t}) = r(t)$$

$$\theta(h) = \theta \in [0,1] \quad , \quad r(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$$

$\square$

Esempio

$$\frac{d}{dt}\int_0^1 e^{x^2t^2}\sqrt{x}\,dx = \int_0^1 2x^2t\,e^{x^2t}\sqrt{x}\,dx$$

$\forall t \in \mathbb{R}.$

Esempio:

$$\frac{d}{dt}\left[\int_{t^2}^{e^t} \operatorname{sen}(x^2 t^3)\,dx\right] =$$

$$= \frac{d}{dt} F(t, a(t), b(t)) \quad \text{dove}$$

$$F(t, u, v) = \int_u^v \operatorname{sen}(x^2 t^3)\,dx$$

$$a(t) = t^2 \,, \qquad b(t) = e^t .$$

OSS

$$F_t(t,u,v) = \int_u^v 3x^2 t^2 \cos(x^2 t^3)\,dx$$ per il thm. precedente

$$F_v(t,u,v) = \operatorname{sen}(v^2 t^3)$$ per il thm. fond. calc. int.

$$F_u(t,u,v) = -\operatorname{sen}(u^2 t^3)$$

Quindi

$$\frac{d}{dt}\left[\int_{t^2}^{e^t} \operatorname{sen}(x^2 t^3)\,dx\right] = \frac{d}{dt} F(t, a(t), b(t))$$

$$= F_t(t, a(t), b(t)) + F_u(t, a(t), b(t))\, a'(t) + F_v(t, a(t), b(t))\, b'(t)$$

$$= \int_{t^2}^{e^t} 3t^2 x^2 \cos(x^2 t^3)\,dx - \operatorname{sen}(t^4 t^3)\, 2t + \operatorname{sen}(e^{2t} t^3)\, e^t$$

TEOREMA Sia $f(x,t)$ continua in $I \times J$ (intervalli)

$\frac{\partial f}{\partial t}(x,t)$ " " " "

$a(t), b(t) : J \longrightarrow I$ di classe C^1.

Allora la funzione $g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$ è di classe $C^1(J)$ e

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx + f(b(t),t)\,b'(t) + \\ - f(a(t),t)\,a'(t)$$

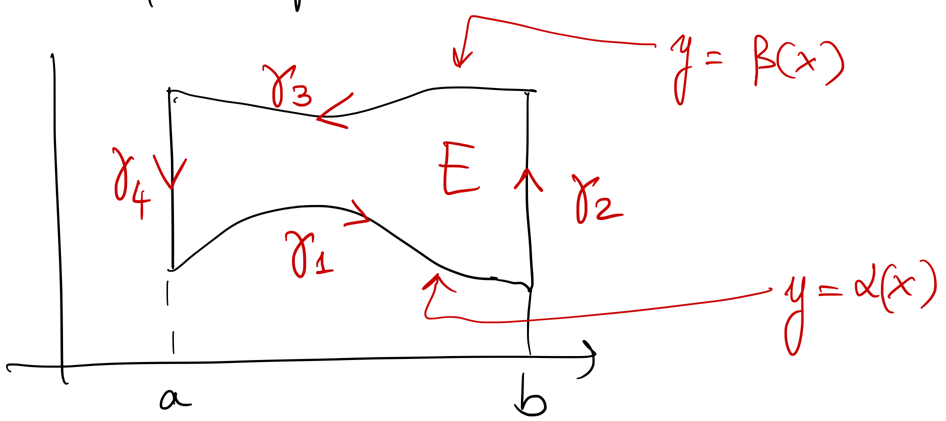
Applicazione. Dim. della prima formula di Gauss-Green

$$\iint_E \frac{\partial f}{\partial x}\,dx\,dy = \int_{\partial^+ E} f\,dy \quad \forall f(x,y) \in C^1(E).$$

per un dominio normale regolare della forma

$$E = \{(x,y) : a \le x \le b,\ \alpha(x) \le y \le \beta(x)\}$$

$\alpha(x), \beta(x)$ funz. di classe $C^1([a,b])$ t.c. $\alpha(x) < \beta(x)$



$$\int_{\partial^+ E} f(x,y)\,dy = \int_{\gamma_1} f\,dy + \int_{\gamma_2} \cdots + \int_{\gamma_3} \cdots + \int_{\gamma_4} \cdots$$

$\gamma_1 : \begin{cases} x = x \\ y = \alpha(x) \end{cases}$ $x \in [a,b] \Rightarrow \int_{\gamma_1} f\,dy = \int_a^b dx\, f(x, \alpha(x))\, \alpha'(x)$

$\gamma_2 : \begin{cases} x = b \\ y = y \end{cases}$ $y \in [\alpha(b), \beta(b)] \Rightarrow \int_{\gamma_2} f\,dy = \int_{\alpha(b)}^{\beta(b)} f(b,y)\,dy$

$$\int_{\gamma_3} f\,dy = -\int_a^b f(x, \beta(x))\, \beta'(x)\,dx$$

$$\int_{\gamma_4} f\,dy = -\int_{\alpha(a)}^{\beta(a)} f(a,y)\,dy$$

$$\Rightarrow \int_{\partial^+ E} f\,dy = \int_a^b dx\, f(x,\alpha(x))\,\alpha'(x) + \int_{\alpha(b)}^{\beta(b)} f(b,y)\,dy$$
I $\quad$ II

$$- \int_a^b f(x,\beta(x))\,\beta'(x)\,dx - \int_{\alpha(a)}^{\beta(a)} f(a,y)\,dy$$
III $\quad$ IV

$$\iint_E \frac{\partial f}{\partial x}\,dx\,dy = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} dy\, \frac{\partial f}{\partial x}(x,y) =$$ [per il preced. teorema]

$$= \int_a^b dx \left\{ \underbrace{\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right]}_{\text{si usa il teorema fond.}} - f(x,\beta(x))\,\beta'(x) + f(x,\alpha(x))\,\alpha'(x) \right\}$$

$$= \int_{\alpha(b)}^{\beta(b)} f(b,y)\,dy - \int_{\alpha(a)}^{\beta(a)} f(a,y)\,dy - \int_a^b f(x,\beta(x))\,\beta'(x)\,dx + \int_a^b f(x,\alpha(x))\,\alpha'(x)\,dx$$
II $\quad$ IV $\quad$ III $\quad$ I