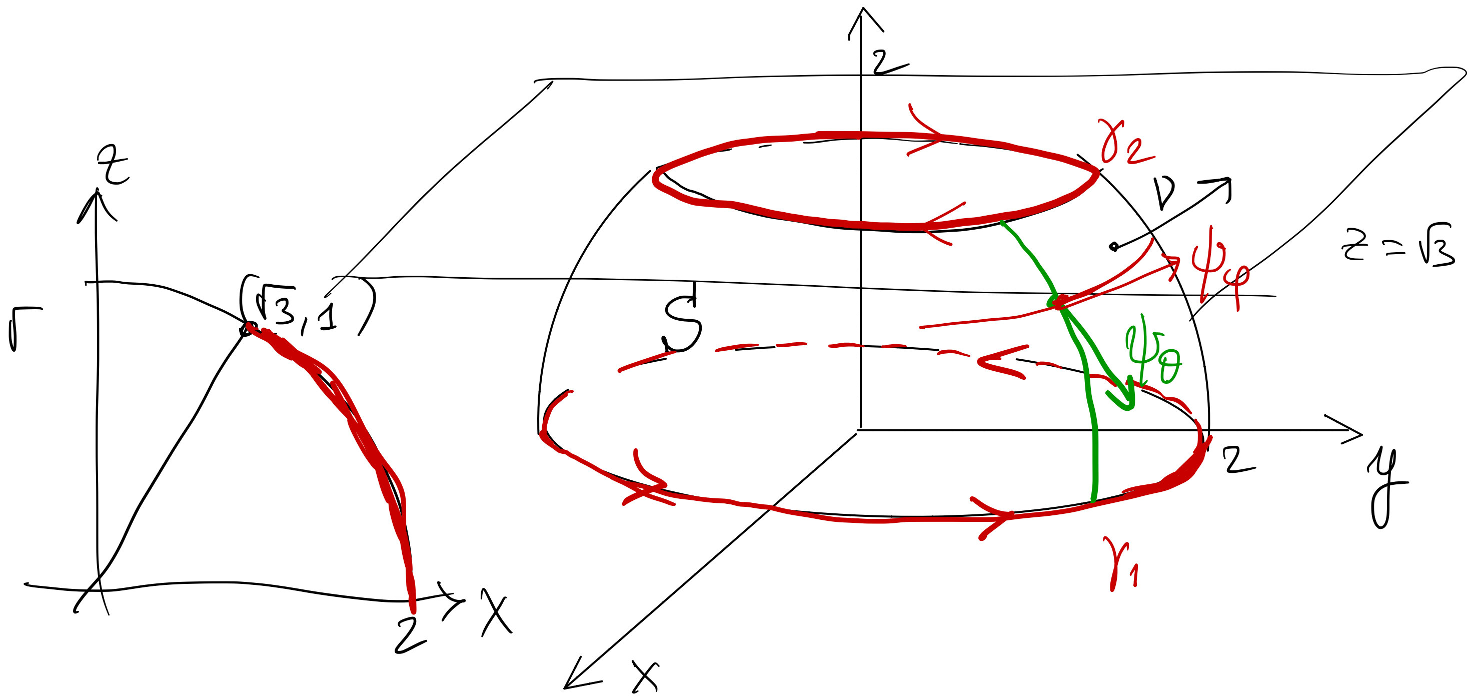
Esercizio. $\underline{F}(x,y,z) = (yz, -xz, xy)$,

determinare $$\iint_S \operatorname{rot} F \cdot \nu \, d\sigma$$

dove S è la parte della superficie sferica $x^2+y^2+z^2=4$ compresa tra i piani $\{z=0\}$ e $\{z=\sqrt{3}\}$, dove ν è il versore normale uscente dalla sfera



<u>OSS</u> S è una superficie avente per bordo le due circonferenze
$$\gamma_1 = \{(x,y,z) : z=0, \quad x^2+y^2=4\}$$
percorsa in verso antiorario (vista dall'alto)
$$\gamma_2 = \{(x,y,z) : z=\sqrt{3}; \quad x^2+y^2=1\}$$
percorsa in verso orario (vista dall'alto).

Teorema di Stokes:

$$\iint_S \text{rot}\, F \cdot \nu \, d\sigma = \int_{b^+ S} F \cdot T \, ds$$

$\gamma_1 \quad \begin{cases} x = 2\cos\theta \\ y = 2\,\text{sen}\,\theta \\ z = 0 \end{cases}$ $\longrightarrow$ $\gamma_1'(\theta) = (-2\,\text{sen}\,\theta, 2\cos\theta, 0)$

$\theta \in [0, 2\pi]$

verso corretto!

$\gamma_2^- \quad \begin{cases} x = \cos\theta \\ y = \text{sen}\,\theta \\ z = \sqrt{3} \end{cases}$ $\longrightarrow$ $(\gamma_2^-)'(\theta) = (-\text{sen}\,\theta, \cos\theta, 0)$

$\theta \in [0, 2\pi]$

N.B. va percorsa nel verso opposto.

$$\int_{b^+ S} F \cdot T \, ds = \int_{\gamma_1} F \cdot T \, ds - \int_{\gamma_2^-} F \cdot T \, ds =$$

$$= \underbrace{\int_0^{2\pi} d\theta \left[0 \cdot (-2\,\text{sen}\,\theta) + 0 \cdot (2\cos\theta) + (- - -) \cdot 0 \right] d\theta}_{= 0} +$$

$$- \int_0^{2\pi} d\theta \left[(\text{sen}\,\theta \cdot \sqrt{3}) \cdot (-\text{sen}\,\theta) + (-\sqrt{3}\cos\theta) \cos\theta + (\;) \cdot 0 \right] d\theta$$

$$= +\sqrt{3} \int_0^{2\pi} d\theta = 2\sqrt{3}\,\pi.$$

Oppure, senza usare Stokes, facciamo il calcolo diretto

$$\underline{F}(x,y,z) = (yz, -xz, xy),$$

$$\text{rot } \underline{F} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & xy \end{pmatrix} = (2x, 0, -2z)$$

Parametrizziamo S.

$$\psi \begin{cases} x = 2\,\text{sen}\theta\cos\varphi \\ y = 2\,\text{sen}\theta\,\text{sen}\varphi \\ z = 2\cos\theta \end{cases} \qquad (\theta,\varphi) \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \times [0, 2\pi]$$

$$\psi_\theta = (2\cos\theta\cos\varphi,\ 2\cos\theta\,\text{sen}\varphi,\ -2\,\text{sen}\theta)$$

$$\psi_\varphi = (-2\,\text{sen}\theta\,\text{sen}\varphi,\ 2\,\text{sen}\theta\cos\varphi,\ 0)$$

$A(\theta,\varphi) = 4\,\text{sen}^2\theta\cos\varphi$

$B(\theta,\varphi) =$ inutile (perché $(\text{rot } F)_2 = 0$)

$C(\theta,\varphi) = 4\,\text{sen}\theta\cos\theta$

L'orientazione è corretta? sì, perché $C(\theta,\varphi) > 0$.

$$\iint_S (\text{rot}\, F \cdot \underline{\nu})\, d\sigma =$$

$$= \int_{\pi/6}^{\pi/2} d\theta \int_0^{2\pi} d\varphi \left[\underbrace{4\,\text{sen}\,\theta \cos\varphi}_{(\text{rot}\, F)_1} \cdot \underbrace{4\,\text{sen}^2\theta \cos\varphi}_{A} + \underbrace{(-4\cos\theta)}_{(\text{rot}\, F)_3} \underbrace{(4\,\text{sen}\,\theta\cos\theta)}_{C} \right] =$$

$$= 16 \underbrace{\int_{\pi/6}^{\pi/2} d\theta\, \text{sen}^3\theta}_{\int (1-\cos^2\theta)\,\text{sen}\,\theta} \underbrace{\left[\int_0^{2\pi} d\varphi \cos^2\varphi\right]}_{\pi} - 16 \int_{\pi/6}^{\pi/2} \cos^2\theta\, \text{sen}\,\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\int (1-\cos^2\theta)\,\text{sen}\,\theta = -\cos\theta + \frac{\cos^3\theta}{3}\Bigg|_{\pi/6}^{\pi/2} = \frac{\sqrt{3}}{2} - \frac{\cancel{3}\sqrt{3}}{8\cdot\cancel{3}} = \frac{3\sqrt{3}}{8}$$

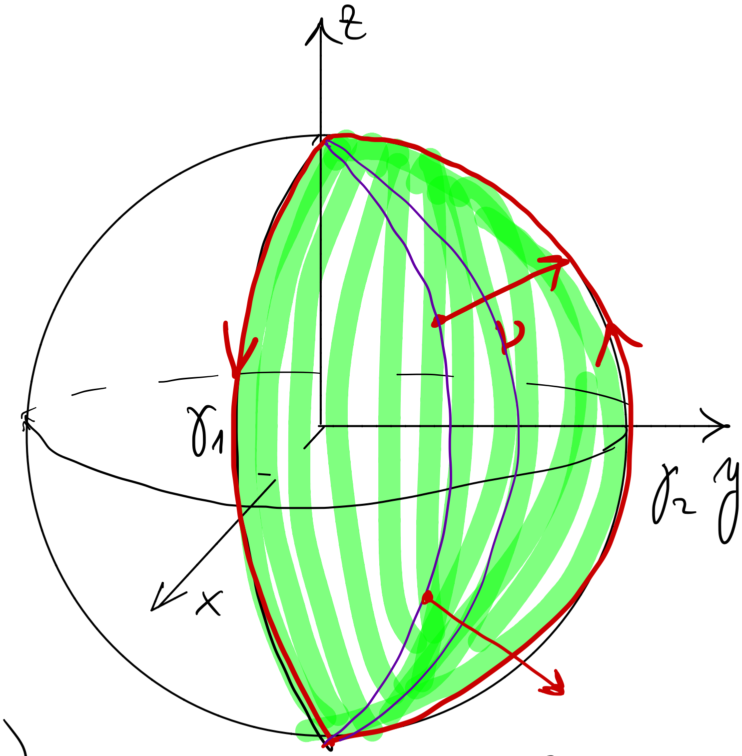
Esercizio Calcolare la circuitazione di

$$\underline{F}(x,y,z) = (xz^2 + y^2,\ -y,\ x^2 z)$$

lungo il bordo dello "spicchio" di sfera

$$S = \{ (x,y,z) : \quad x^2+y^2+z^2 = 9,\ x \geq 0,\ y \geq 0 \}.$$

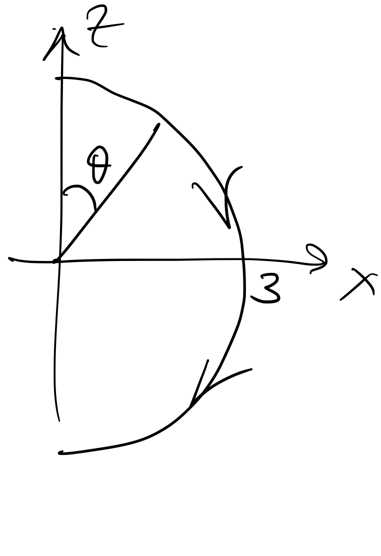
con l'orientazione indotta dal versore normale esterno.



1° modo (calcolo diretto)

$$b^+ S = \gamma_1 \cup \gamma_2$$

$$\gamma_1: \begin{cases} x = 3\,\mathrm{sen}\,\theta \\ y = 0 \\ z = 3\cos\theta \end{cases} \qquad x'(\theta) = 3\cos\theta \qquad \theta \in [0,\pi]. \qquad z'(\theta) = -3\,\mathrm{sen}\,\theta$$



$$\gamma_2^-: \begin{cases} x = 0 \\ y = 3\,\mathrm{sen}\,\theta \\ z = 3\cos\theta \end{cases} \Rightarrow \quad y'(\theta) = 3\cos\theta \quad \text{oppure} \quad z'(\theta) = -3\,\mathrm{sen}\,\theta \qquad \gamma_2^+ \begin{cases} x = 0 \\ y = -3\,\mathrm{sen}\,\theta \\ z = 3\cos\theta \end{cases}$$

$$\int_{\gamma_1} F \cdot T \, ds \; - \int_{\gamma_2^-} F \cdot T \, ds =$$

$$= \int_0^{\pi} d\theta \left[27 \operatorname{sen}\theta \cos^2\theta \cdot 3\cos\theta + 27 \operatorname{sen}^2\theta \cos\theta \left(-3\operatorname{sen}\theta\right) \right] d\theta \; +$$

$$81 \operatorname{sen}\theta \cos\theta \left[\cos^2\theta - \operatorname{sen}^2\theta \right]$$

$$\frac{81}{2} \operatorname{sen}(2\theta) \quad \cos(2\theta) = \frac{81}{4} \operatorname{sen}(4\theta)$$

$$- \int_0^{\pi} d\theta \left[\underbrace{(-3\operatorname{sen}\theta) \cdot (3\cos\theta)}_{-\frac{9}{2} \operatorname{sen}(2\theta)} \right]$$

2) Usando Stokes $\underline{F}(x,y,z) = (xz^2+y^2, -y, x^2z)$

$$\text{rot } F = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ xz^2+y^2 & -y & x^2z \end{pmatrix} = (0, 2xz - 2xz, -2y) =$$

$$= (0, 0, -2y)$$

Parametrizziamo S.

$$\psi \begin{cases} x = 3 \operatorname{sen}\theta \cos\varphi \\ y = 3 \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = 3 \cos\theta \end{cases} \qquad (\theta, \varphi) \in [0, \pi] \times \left[0, \frac{\pi}{2}\right]$$

verso corretto.

$$\psi_\theta = (3\cos\theta\cos\varphi, \ 3\cos\theta \operatorname{sen}\varphi, -3\operatorname{sen}\theta)$$

$$\psi_\varphi = (-3\operatorname{sen}\theta\operatorname{sen}\varphi, \ 3\operatorname{sen}\theta\cos\varphi, \ 0)$$

$A(\theta, \varphi) =$ inutile

$B(\theta, \varphi) =$ inutile (perché $(\text{rot } F)_2 = 0$)

$C(\theta, \varphi) = 9 \operatorname{sen}\theta \cos\theta$

$$\iint_S \text{rot } F \cdot \nu \, d\sigma = \int_0^\pi d\theta \int_0^{\pi/2} d\varphi \, (-6\operatorname{sen}\theta\operatorname{sen}\varphi)(9\operatorname{sen}\theta\cos\theta) =$$

$$= -54 \underbrace{\int_0^\pi d\theta \operatorname{sen}^2\theta\cos\theta}_{\overset{!!}{0}} \cdot \underbrace{\int_0^{\pi/2} \operatorname{sen}\varphi \, d\varphi}_{1}$$

Integrali dipendenti da parametri

$$g(t) = \int_a^b f(x,t)\,dx$$

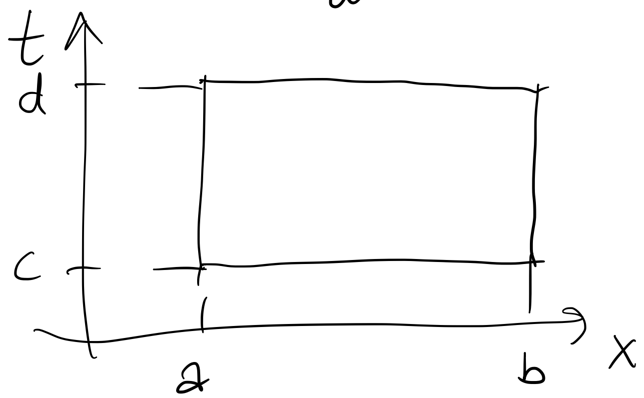
Domande tipiche:

1) $g(t)$ è continua

2) $g(t)$ è derivabile?
   Quanto vale $g'(t)$?

TEOREMA Sia $f(x,t)$ continua in $[a,b]\times[c,d]$.

Allora $g(t) = \int_a^b f(x,t)\,dx$ è continua in $[c,d]$.



Esempio: $\int_0^{\pi} \operatorname{sen}^4(t^2x^6)\,dx = g(t)$

$g(t)$ continua $\forall t\in\mathbb{R}$ (perché posso applicare il teorema $\forall$ intervallo $[c,d]$).

TEOREMA Sia $f(x,t)$ continua in $[a,b]\times[c,d]$.

Allora $g(t) = \int_a^b f(x,t)\,dx$ è continua in $[c,d]$.

Dim. Nell'ipotesi supplementare che $f \in C^1([a,b]\times[c,d])$

In realtà ho usato solo che
$f \in C([a,b]\times[c,d])$
$f_t \in C([a,b]\times[c,d])$

Sia $t_0 \in [c,d]$. Voglio provare che

$$\lim_{t\to t_0} g(t) \overset{?}{=} g(t_0)$$

$$\lim_{t\to t_0} |g(t)-g(t_0)| \overset{?}{=} 0$$

$$0 \le |g(t)-g(t_0)| = \left|\int_a^b f(x,t)\,dx - \int_a^b f(x,t_0)\,dx\right| =$$

$$= \left|\int_a^b [f(x,t)-f(x,t_0)]\,dx\right| =$$

$$= \left|\int_a^b (t-t_0)\, f_t(x,\xi)\,dx\right| \le \quad \text{dove } \xi \text{ compreso tra } t \text{ e } t_0$$

$$\le \int_a^b |t-t_0|\, \underbrace{|f_t(x,\xi)|}_{\le M \text{ per Weierstrass}}\,dx$$

$$\le |t-t_0|\, M\,(b-a) \xrightarrow[t\to t_0]{} 0. \qquad \square$$