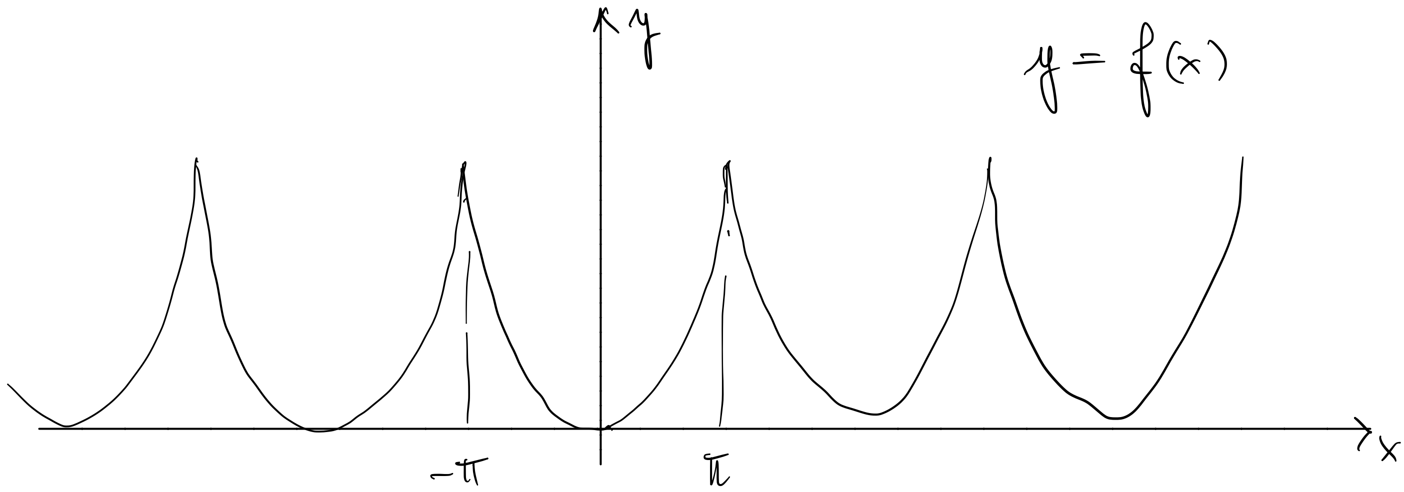
Calcolare la serie di Fourier della funzione definita da

$f(x) = x^2$ per $x \in [-\pi, \pi]$, prolungata per periodicità a tutto $\mathbb{R}$, e dire quanto vale la sua somma.

Calcolare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$



<u>oss</u> f è pari $\Rightarrow b_k = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\,\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos kx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos kx\,dx =$$

$$= \frac{2}{k\pi}\,\cancel{x^2 \operatorname{sen} kx\Big|_0^{\pi}} - \frac{4}{k\pi}\int_0^{\pi} x \operatorname{sen} kx\,dx =$$

$$= \frac{4}{k^2\pi}\,x\cos kx\Big|_0^{\pi} - \cancel{\frac{4}{k^2\pi}\int_0^{\pi}\cos kx\,dx} =$$

$$a_k = \frac{4}{k^2\pi} x\cos kx \Big|_0^{\pi} = \frac{4}{k^2\pi}\pi\cos(k\pi) = (-1)^k\frac{4}{k^2}$$

Serie di Fourier:

$$\frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx)$$

La sua somma (f regolare a tratti e continua) vale $f(x)$

In particolare per $x \in [-\pi, \pi]$

$$\frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx) = x^2$$

Prendo $x = \pi \Rightarrow$

$$\frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{\overbrace{(-1)^k(-1)^k}^{=1}}{k^2} = \pi^2$$

$$\Rightarrow 4\sum_{k=1}^{\infty}\frac{1}{k^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2}{3}\pi^2$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}}$$

$x = 0$

$$\Rightarrow \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = 0$$

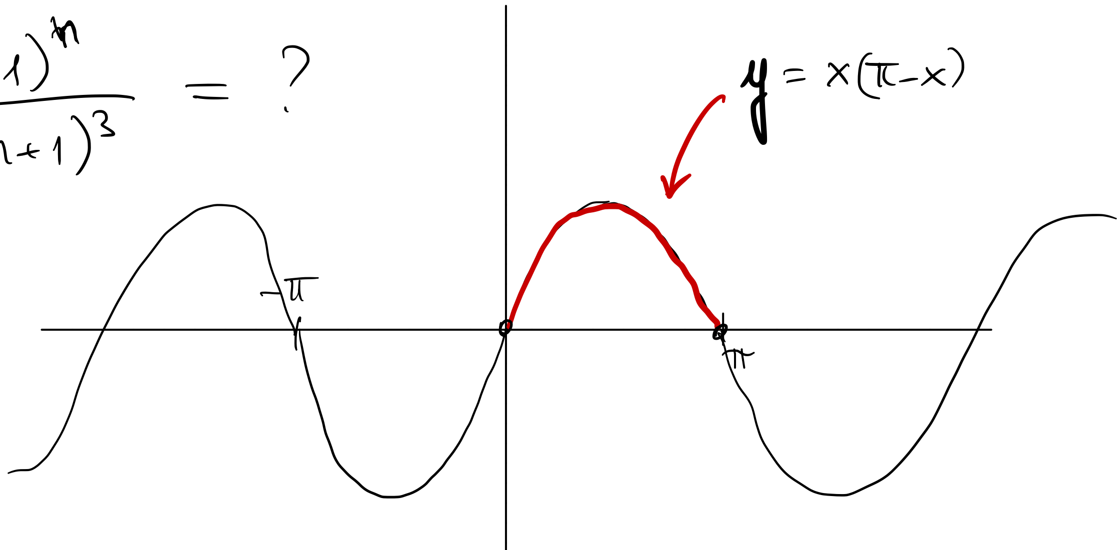
$$\boxed{\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}}$$

$$f(x) = x\,(\pi - |x|) \qquad x \in (-\pi, \pi].$$

+ estesa per 2π-periodicità.

- Calcolare Σ di Fourier.
- Dire a cosa converge
- $$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = ?$$

f è dispari



$\Rightarrow a_k = 0 \quad \forall k = 0, 1, 2 \ldots$

$$b_k = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \operatorname{sen} kx \, dx =$$

$$= \cancel{-\frac{2}{\pi k} (\pi x - x^2) \cos kx \Big|_0^{\pi}} + \frac{2}{\pi k} \int_0^{\pi} (\pi - 2x) \cos kx \, dx =$$

$$= \cancel{\frac{2}{\pi k^2} (\pi - 2x) \operatorname{sen} kx \Big|_0^{\pi}} + \frac{4}{\pi k^2} \int_0^{\pi} \operatorname{sen} kx \, dx =$$

$$= -\frac{4}{\pi k^3} \cos kx \Big|_0^{\pi} = -\frac{4}{\pi k^3} \left(\cos(k\pi) - 1 \right) =$$

$$= \frac{4}{\pi k^3} \left[1 - (-1)^k \right] = \begin{cases} 0 & k \text{ pari} \\ \dfrac{8}{\pi k^3} & k \text{ dispari} \end{cases}$$

OSS anche in questo caso f è continua, quindi

$$f(x) = \sum_{k=1}^{\infty} b_k \operatorname{sen} kx = \sum_{n=0}^{\infty} \frac{8}{\pi(2n+1)^3} \operatorname{sen}(2n+1)x$$

Prendo $x = \frac{\pi}{2}$

$$\operatorname{sen}\left((2n+1)\frac{\pi}{2}\right) = (-1)^n$$

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

$$\shortparallel$$

$$\frac{\pi}{2}\left(\frac{\pi}{2}\right)$$

$$\shortparallel$$

$$\frac{\pi^2}{4} \qquad \Longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

Serie di Fourier con notazione complessa

Sappiamo che $e^{ix} = \cos x + i \operatorname{sen} x$

Ora, sia $f(x): \mathbb{R} \to \mathbb{R}$ 2π-periodica e regolare a tratti.

(per semplicità f continua)

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \operatorname{sen} kx \right)$$

$$e^{ix} = \cos x + i \operatorname{sen} x$$

$$e^{-ix} = \cos x - i \operatorname{sen} x$$

$$\Rightarrow \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \left(= \operatorname{ch}(ix) \right)$$

$$\operatorname{sen} x = \frac{e^{ix} - e^{-ix}}{2i} \quad \left(= -i \operatorname{sh}(ix) \right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \frac{e^{ikx} + e^{-ikx}}{2} - i b_k \left(\frac{e^{ikx} - e^{-ikx}}{2} \right) \right]$$

$$= \frac{a_0}{2} e^{i0x} + \sum_{k=1}^{\infty} \left[\left(\frac{a_k - i b_k}{2} \right) e^{ikx} + \left(\frac{a_k + i b_k}{2} \right) e^{-ikx} \right] =$$

$$= \sum_{k=-\infty}^{+\infty} c_k e^{ikx}$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \frac{e^{ikx} + e^{-ikx}}{2} - i\, b_k \left(\frac{e^{ikx} - e^{-ikx}}{2} \right) \right]$$

$$= \frac{a_0}{2} e^{i0x} + \sum_{k=1}^{\infty} \left[\left(\frac{a_k - ib_k}{2} \right) e^{ikx} + \left(\frac{a_k + ib_k}{2} \right) e^{-ikx} \right] =$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad c_k = ?$$

Se $k > 0 \Rightarrow c_k = \frac{1}{2}(a_k - ib_k) =$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\cos kx - i \operatorname{sen} kx] \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}$$

se $k < 0$

$$c_k = \frac{1}{2}(a_{-k} + i b_{-k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\underbrace{\cos(-kx)}_{\cos kx} \underbrace{+ i \operatorname{sen}(-kx)}_{-i \operatorname{sen} kx}] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}$$

se $k = 0 \Rightarrow \quad c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{i0x} dx = \frac{a_0}{2}$

$$f(x) = \underbrace{\sum_{k=-\infty}^{\infty} c_k e^{ikx}}_{\lim\limits_{n \to +\infty} \sum_{k=-n}^{n} c_k e^{ikx}} \quad \text{dove} \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

$\forall k \in \mathbb{Z}$.

Lo sviluppo è valido anche se $f : \mathbb{R} \to \mathbb{C}$

AVVISO: martedì 31/5

Lezione ore 12:00 → 13:30 Aula 8

Tutoraggio ore 14:00 → 15:30 Aula 15

Teorema di Stokes: esercizi

Sia $\underline{F}(x,y,z) = (2y^3, -x^3, 2z^3)$

Calcolare il lavoro di $\underline{F}$ lungo γ intersezione del cilindro $x^2+y^2=4$ con il piano $x+y+z=1$, orientata in senso antiorario (se vista "dall'alto")

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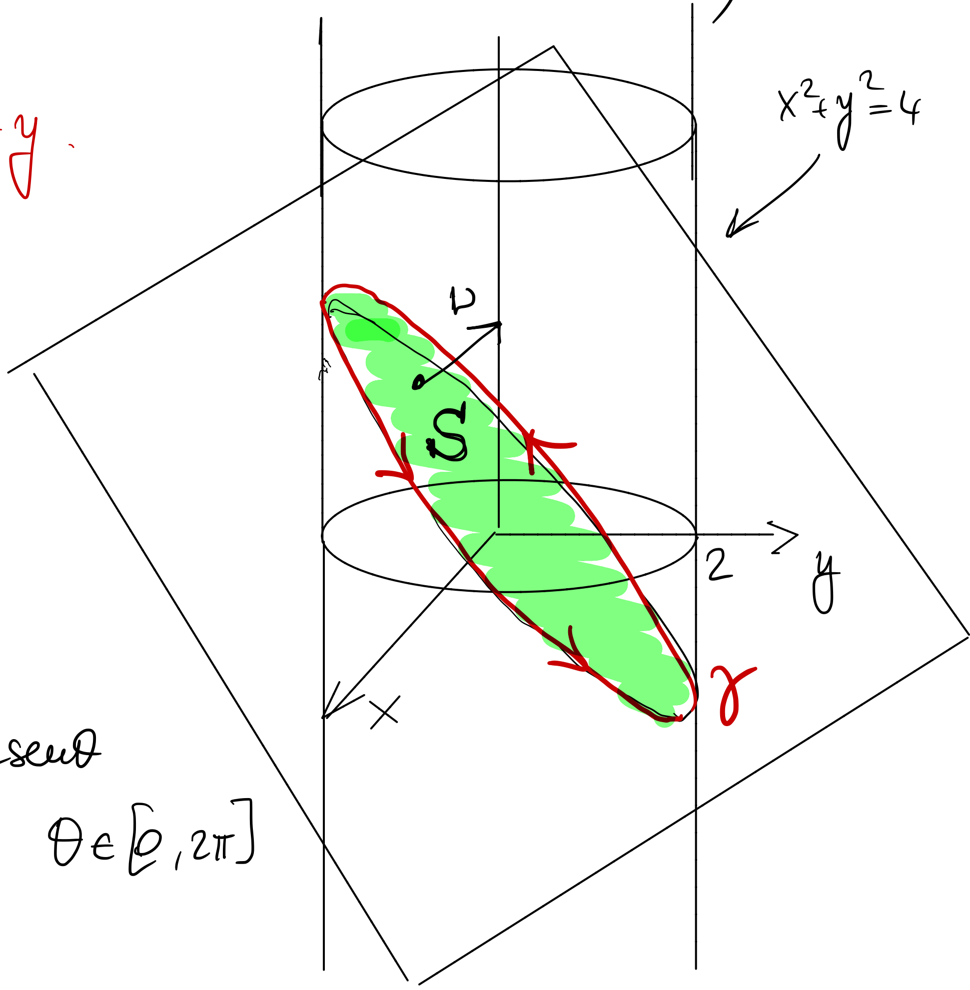
$z = 1-x-y.$



1° calcolo diretto

$$\gamma : \begin{cases} x = 2\cos\theta \\ y = 2\operatorname{sen}\theta \\ z = 1-2\cos\theta-2\operatorname{sen}\theta \end{cases}$$

verso giusto! $\qquad \theta \in [0, 2\pi]$

$$\int_\gamma \underline{F}\cdot\underline{T}\,ds = \int_0^{2\pi}\Big[2\,(2\operatorname{sen}\theta)^3(-2\operatorname{sen}\theta) - (2\cos\theta)^3\,2\cos\theta + 2\,(1-2\cos\theta-2\operatorname{sen}\theta)^3\,(2\operatorname{sen}\theta-2\cos\theta)\Big]\,d\theta$$

$$= -32\int_0^{2\pi}\operatorname{sen}^4\theta\,d\theta - 16\int_0^{2\pi}\cos^4\theta\,d\theta +$$

$$+ \underbrace{2\,\frac{(1-2\cos\theta-2\operatorname{sen}\theta)^4}{4}\Bigg|_0^{2\pi}}_{\overset{!!}{0}} = \text{facile,}$$

2° Teorema di Stokes

OSS γ è il bordo della superficie S.

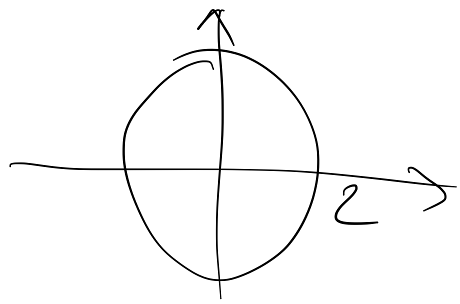
$$S = \{(x,y,z) : z = 1-x-y ; \quad x^2+y^2 \leq 4\}$$

Teorema di Stokes

$$\int_\gamma \underline{F} \cdot \underline{T}\, ds = \int_{b^+S} \underline{F} \cdot \underline{T}\, ds = \iint_S \text{rot} F \cdot \nu \, d\sigma = (*)$$

Parametrizzo S come grafico

$$\begin{cases} x = x \\ y = y \\ z = 1-x-y = f(x,y) \end{cases} \qquad (x,y) \in C = \{(x,y) : x^2+y^2 \leq 4\}$$



$$A(x,y) = -f_x(x,y) = 1$$

$$B(x,y) = -f_y(x,y) = 1$$

$$C(x,y) = 1$$ ok, orientato bene!

$$\text{rot} F = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2y^3 & -x^3 & 2z^3 \end{pmatrix} = (0, 0, -3x^2-6y^2)$$

$$(*) = \iint_C (-3x^2-6y^2) \cdot 1 \, dx\,dy.$$

Coord polari

$$= -3\int_0^{2\pi} d\theta \int_0^2 d\rho \, (\rho^2 \cos^2\theta + 2\rho^2 \sin^2\theta)\,\rho =$$

$$= -3\int_0^{2\pi} d\theta \int_0^{2} d\rho \left(\rho^2\cos^2\theta + 2\rho^2 \operatorname{sen}^2\theta\right)\rho =$$

$$= -3\cdot 4\int_0^{2\pi} d\theta \left(\cos^2\theta + 2\operatorname{sen}^2\theta\right) = -12\cdot 3\pi = -36\pi$$