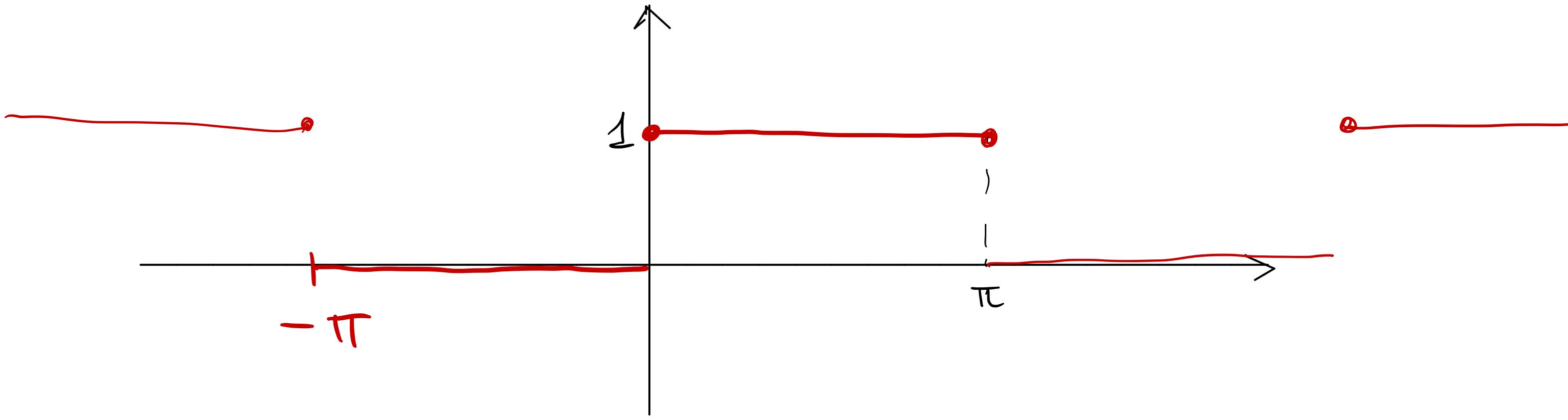
Calcolare la serie di Fourier della funzione 2π-periodica che in $(-\pi,\pi]$ vale

$$\begin{cases} 1 & \text{se} \quad x \in [0,\pi] \\ 0 & \text{se} \quad x \in (-\pi,0), \end{cases}$$



e dire a cosa converge la serie.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos kx\,dx = 0$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \operatorname{sen} kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_0^{\pi} =$$

$$= -\frac{1}{k\pi}\left(\cos(k\pi) - 1\right) = \frac{1}{k\pi}\left(1-(-1)^k\right) = \begin{cases} 0 & k \text{ pari} \\ \dfrac{2}{k\pi} & k \text{ dispari} \end{cases}$$

La serie di Fourier è data da

$$\frac{1}{2} + \sum_{h=0}^{\infty} \frac{2}{(2h+1)\pi}\cdot \operatorname{sen}\left((2h+1)x\right)$$

Questa serie converge a $f(x)$ per $x \neq k\pi$

se $x = k\pi$ converge a $\frac{1}{2}$.

Fissata $f(x)$ 2π-periodica e regolare a tratti, sia $n \in \mathbb{N}$ fissato.

Sia $s_n(x)$ il polinomio trigonometrico di Fourier di ordine n.

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \operatorname{sen} kx)$$

dove

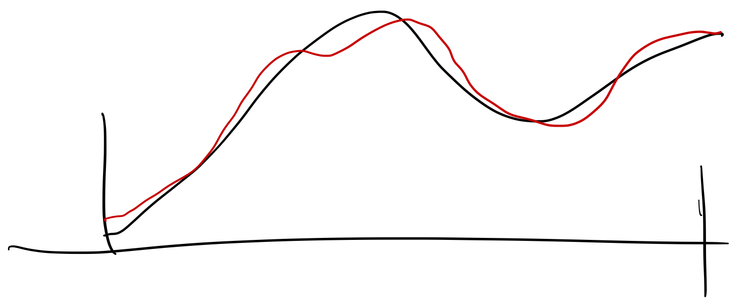
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \qquad k = 0, 1, 2, \dots$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen} kx \, dx \qquad k = 1, 2, \dots$$

TEOREMA $s_n(x)$ minimizza l'integrale

$$\int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx$$

tra tutti i polinomi trigonometrici di ordine n.



Dim Sia $t(x)$ un qualsiasi polinomio trigonometrico di ordine n, cioè

$$t(x) = \frac{\lambda_0}{2} + \sum_{k=1}^{n} (\lambda_k \cos kx + \mu_k \operatorname{sen} kx)$$

$\lambda_0, \lambda_1, \dots \lambda_n, \mu_1 \dots \mu_n$ sono numeri reali.

La funzione da minimizzare è una funzione di $2n+1$ variabili

$$F(\lambda_0, \lambda_1, \dots \lambda_n, \mu_1, \dots \mu_n) =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x) - t(x)|^2 dx =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx - 2 \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) t(x) dx}_{A} + \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} t(x)^2 dx}_{B}$$

$$A = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) t(x) dx =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \left[\frac{\lambda_0}{2} + \sum_{k=1}^{n} (\lambda_k \cos kx + \mu_k \operatorname{sen} kx) \right] dx =$$

$$= \frac{\lambda_0}{2} \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx}_{a_0} + \sum_{k=1}^{n} \left[\lambda_k \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx}_{a_k} + \mu_k \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen} kx}_{b_k} \right]$$

$$= \frac{\lambda_0 a_0}{2} + \sum_{k=1}^{n} (\lambda_k a_k + \mu_k b_k)$$

$$\text{(B)} = \frac{1}{\pi}\int_{-\pi}^{\pi} t(x)^2\,dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{\lambda_0}{2} + \sum_{k=1}^{n}(\lambda_k\cos kx + \mu_k \operatorname{sen} kx)\right]\left[\frac{\lambda_0}{2} + \sum_{h=1}^{n}(\lambda_h\cos hx + \mu_h \operatorname{sen} hx)\right] dx$$

OSS tutti i prodotti "misti" si annullano dopo integrazione.

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{\lambda_0^2}{4} + \sum_{k=1}^{n}\left[\lambda_k^2\cos^2 kx + \mu_k^2 \operatorname{sen}^2 kx\right]\right] dx$$

$$= \frac{1}{\pi}\left[\frac{\lambda_0^2}{4}2\pi + \sum_{k=1}^{n}\left[\lambda_k^2\pi + \mu_k^2\pi\right]\right] =$$

$$= \frac{\lambda_0^2}{2} + \sum_{k=1}^{n}(\lambda_k^2 + \mu_k^2)$$

$$F(\lambda_0 \ldots, \lambda_n, \mu_1, \ldots, \mu_n) =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx - 2\left[\frac{\lambda_0 a_0}{2} + \sum_{k=1}^{n}(\lambda_k a_k + \mu_k b_k)\right] +$$

$$+ \frac{\lambda_0^2}{2} + \sum_{k=1}^{n}(\lambda_k^2 + \mu_k^2) + \frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)$$

$$- \frac{a_0^2}{2} - \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

$$F(\lambda_0, \dots, \lambda_n, \mu_1, \dots, \mu_n) =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx - 2\left[\frac{\lambda_0 a_0}{2} + \sum_{k=1}^{n} (\lambda_k a_k + \mu_k b_k)\right] +$$

$$+ \frac{\lambda_0^2}{2} + \sum_{k=1}^{n} (\lambda_k^2 + \mu_k^2) + \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)$$

$$- \frac{a_0^2}{2} - \sum_{k=1}^{n} (a_k^2 + b_k^2) =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx + \frac{(a_0 - \lambda_0)^2}{2} + \sum_{k=1}^{n} \left[(a_k - \lambda_k)^2 + (b_k - \mu_k)^2\right] +$$

$$- \frac{a_0^2}{2} - \sum_{k=1}^{n} (a_k^2 + b_k^2)$$

Il minimo assoluto di F è assunto per

$$\lambda_0 = a_0, \quad \lambda_1 = a_1, \dots \quad \lambda_n = a_n$$

$$\mu_1 = b_1 \dots \quad \mu_n = a_n$$

cioè quando $t(x) = S_n(x)$ $\square$

COROLLARIO Scegliendo $t(x) = S_n(x)$, si ottiene

$$\underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x) - S_n(x)|^2 dx}_{\geq 0} = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx +$$

$$- \left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right]$$

$$\Longrightarrow \quad \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Mando $n \longrightarrow +\infty$

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

(disuguaglianza di Bessel).

In realtà questa disuguaglianza si può provare che è un'uguaglianza

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

(identità di Parseval)

Come fare l'approssimazione di Fourier quando $f(x)$ è T-periodica, con $T \neq 2\pi$.

$$f(x+T) = f(x) \qquad \forall x \in \mathbb{R} \qquad T>0$$

<u>Esempio</u> $f(x) = \{x\}$ parte frazionaria di x

$$\{x\} = x - [x]$$

parte intera di x.

$$[x] = \max\{n \in \mathbb{Z} \text{ t.c. } n \leq x\}.$$

$[1] = 1$

$[1,37] = 1$

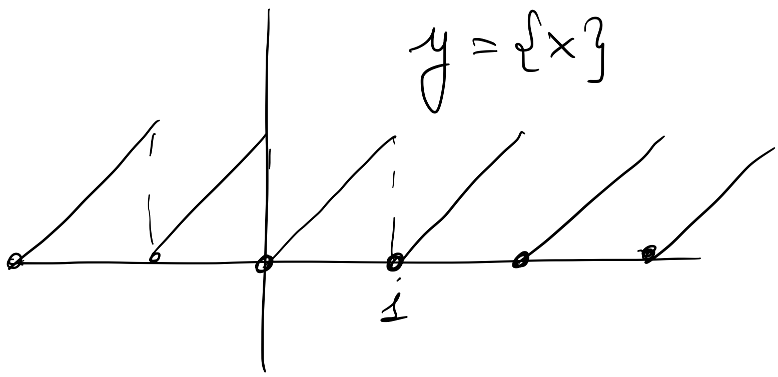
$[\pi] = 3$

$[-\pi] = -4$

$\{1,37\} = 0,37$

$\{\pi\} = 0,1415926\ldots$

$\{-1,37\} = 0,63$



$\{x\}$ ha periodo $T = 1$.

$f(x)$ T-periodica

$\Rightarrow$ mi riconduco al caso 2π-periodico

pongo $g(x) = f\left(\frac{xT}{2\pi}\right)$

f è T-periodica

$$g(x+2\pi) = f\left(\frac{(x+2\pi)T}{2\pi}\right) = f\left(\frac{xT}{2\pi} + T\right) = f\left(\frac{xT}{2\pi}\right) = g(x)$$

$\forall x$

$\Rightarrow$ g è 2π-periodica

La serie di Fourier di g è data da

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \operatorname{sen} kx)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)\,dx \qquad k = 0, 1, 2 \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\operatorname{sen}(kx)\,dx \qquad k = 1, 2 \ldots$$

Supponiamo per semplicità f continua (e reg. a tratti)

$\Rightarrow$ g continua (e reg. a tratti)

$$\Rightarrow \quad g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \operatorname{sen}(kx)\right]$$

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \operatorname{sen}(kx)\right]$$

$$f(x) = g\left(\frac{2\pi}{T} x\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos \frac{2\pi k x}{T} + b_k \operatorname{sen} \frac{2\pi k x}{T}\right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{xT}{2\pi}\right) dx$$

$$\boxed{\begin{gathered} \frac{xT}{2\pi} = y \\ dx = dy\,\frac{2\pi}{T} \end{gathered}}$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} f(y)\,dy = \frac{2}{T} \int_{0}^{T} f(y)\,dy.$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(kx)\,dx = \frac{2}{T} \int_{-T/2}^{T/2} f(y) \cos\left(\frac{2\pi k y}{T}\right) dy$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(y) \operatorname{sen}\left(\frac{2\pi k y}{T}\right) dy.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)\,dx = \frac{2}{T}\int_{-T/2}^{T/2} f(y)\cos\left(\frac{2\pi k y}{T}\right)dy$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(y)\ \text{sen}\left(\frac{2\pi k y}{T}\right)dy.$$

$$f(x) = \{x\} \qquad T = 1.$$

$$a_0 = 2\int_0^1 \{x\}\,dx = 2\int_0^1 x\,dx = 1$$

$$a_k = 2\int_0^1 x\cos(2\pi k x)\,dx =$$

$$= \frac{2}{2\pi k}\cancel{x\,\text{sen}(2\pi k x)\Big|_0^1} - \frac{1}{\pi k}\int_0^1 \text{sen}(2\pi k x)\,dx =$$

$$= \frac{1}{2\pi^2 k^2}\cos(2\pi k x)\Big|_0^1 = 0$$