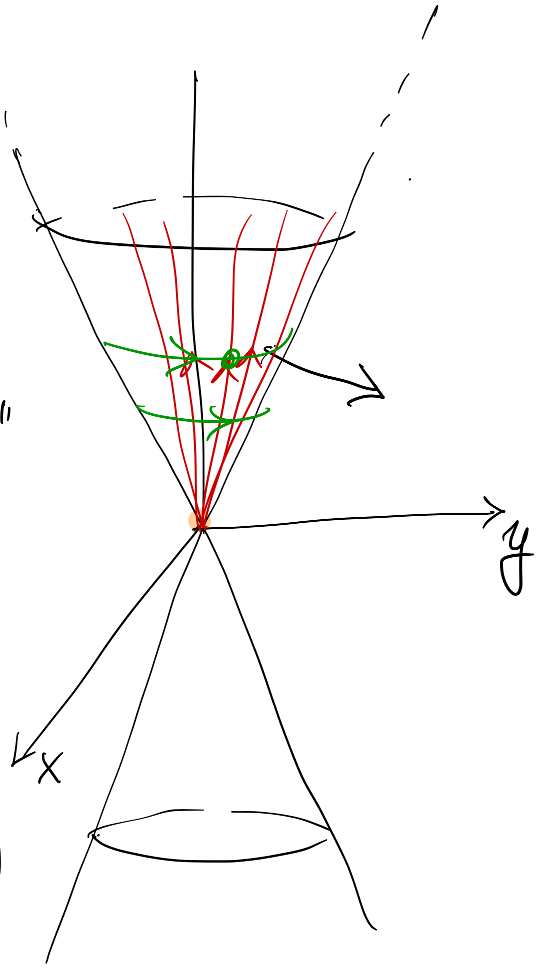
ESERCIZIO: Calcolare il versore normale <u>esterno</u> e il piano tangente al cono

$$z^2 = 9x^2 + 9y^2,$$

nel pto $P_0\left(-2\sqrt{3},\, 2, 12\right)$.



<u>1° modo</u> $\quad z = \pm 3\sqrt{x^2+y^2}$

Nel nostro caso scegliamo il "+"

$$\varphi \begin{cases} x = x \\ y = y \\ z = \underbrace{3\sqrt{x^2+y^2}}_{f(x,y)} \end{cases}$$

P_0 corrisponde a $x = -2\sqrt{3}$
$y = 2$.

$$\varphi_x(x,y) = \left(1,\ 0,\ \frac{3x}{\sqrt{x^2+y^2}}\right)$$

$$\varphi_y(x,y) = \left(0,\ 1,\ \frac{3y}{\sqrt{x^2+y^2}}\right)$$

$$A(x,y) = -\frac{3x}{\sqrt{x^2+y^2}} = -f_x(x,y)$$

$$B(x,y) = -\frac{3y}{\sqrt{x^2+y^2}} = -f_y(x,y)$$

$$C(x,y) = 1$$

Per $x = -2\sqrt{3}$ e $y = 2$ si ha

$$A = +\frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}\ ; \quad B = -\frac{6}{4} = -\frac{3}{2}$$

$$C = 1$$

$$A = +\frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2} \; ; \; B = -\frac{6}{4} = -\frac{3}{2}$$

$$C = 1$$

$$v = \frac{\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}, 1\right)}{\underbrace{\sqrt{\frac{27}{4} + \frac{9}{4} + 1}}_{\sqrt{10}}} = \frac{\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}, 1\right)}{\sqrt{10}}$$

NB il verso non è quello richiesto, devo cambiare segno

$$v = \frac{\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}, -1\right)}{\sqrt{10}}$$

Piano tangente $\quad A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

$$3\sqrt{3}\,(x + 2\sqrt{3}) - 3\,(y-2) + 2(z-12) = 0.$$

$$z = \underset{\overset{\parallel}{z_0}}{12} + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$x_0 = -2\sqrt{3}, \quad y_0 = 2.$$

2)

2) in coord. cilindriche

$$\varphi \begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = 3\rho \end{cases}$$

$(-2\sqrt{3}, 2, 12)$ corrisponde a.

$\boxed{\rho = 4}$

$$\begin{cases} -2\sqrt{3} = 4\cos\theta \\ 2 = 4\,\mathrm{sen}\,\theta \end{cases} \qquad \cos\theta = -\frac{\sqrt{3}}{2} \quad \mathrm{sen}\,\theta = \frac{1}{2}.$$

$\varphi_\rho = (\cos\theta, \mathrm{sen}\,\theta, 3)$

$\varphi_\theta = (-\rho\,\mathrm{sen}\,\theta, \rho\cos\theta, 0)$

$\boxed{\theta = \frac{5}{6}\pi.}$

$$\begin{aligned} A(\rho,\theta) &= -3\rho\cos\theta \\ B(\rho,\theta) &= -3\rho\,\mathrm{sen}\,\theta \\ C(\rho,\theta) &= \rho \end{aligned} \quad \Big| \quad \text{per } \rho = 4,\ \theta = \frac{5}{6} \Longrightarrow \quad \begin{aligned} A &= 6\sqrt{3} \\ B &= -6 \\ C &= 4. \end{aligned}$$

$$\nu = \frac{(3\sqrt{3}, -3, 2)}{\sqrt{27+9+4}} = \frac{(3\sqrt{3}, -3, 2)}{2\sqrt{10}}.$$

Anche in questo caso cambio segno.

Piano tangente $\Rightarrow$ uguale.

<u>OSS</u> Il versore normale è determinato a meno del segno. Cambiando l'ordine dei parametri, si cambia verso del versore normale

Area di una superficie

D dominio regolare di $\mathbb{R}^2$.

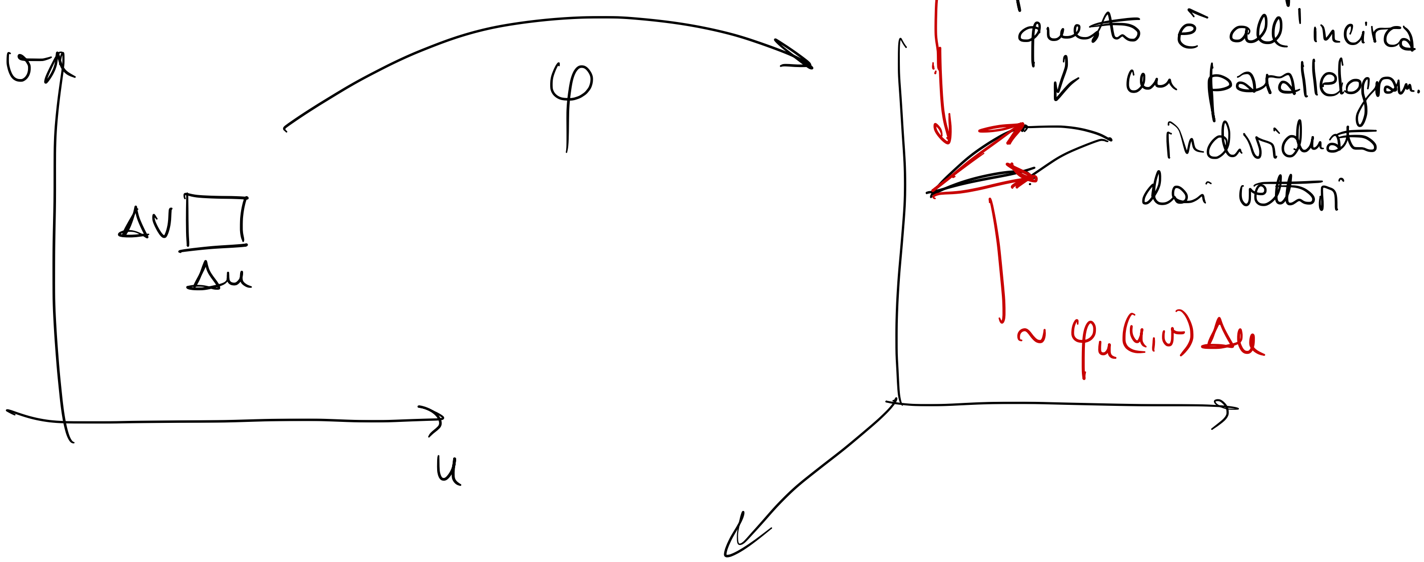
$\underline{\varphi} : D \to \mathbb{R}^3$ superficie regolare

$(u,v) \mapsto \varphi(u,v) = (x(u,v), y(u,v), z(u,v))$.

$$\text{Area di } \varphi = \iint_D \| \underline{\varphi}_u \wedge \underline{\varphi}_v \| \, du\,dv =$$

$$= \iint_D \sqrt{A(u,v)^2 + B(u,v)^2 + C(u,v)^2} \, du\,dv.$$

$$\underline{\varphi}_u \wedge \underline{\varphi}_v = (A(u,v), B(u,v), C(u,v)).$$



Area di una sfera.

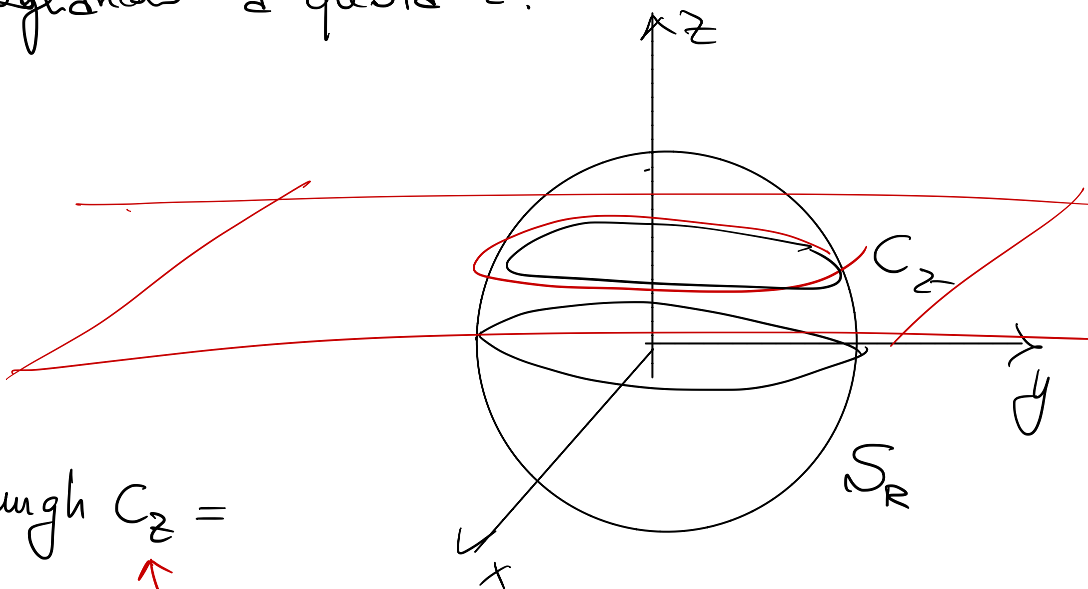
$$\psi \begin{cases} x = R \operatorname{sen}\theta \cos\varphi \\ y = R \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = R\cos\theta \end{cases}$$

$$(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$$

$$\| \psi_\theta \wedge \psi_\varphi \| = R^2 \operatorname{sen}\theta.$$

$$\text{Area } \psi = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, R^2 \operatorname{sen}\theta = R^2 \cdot 2\pi \cdot 2 = 4\pi R^2$$

OSS. Per il calcolo delle aree non vale il "principio di Cavalieri". Non posso, per esempio, calcolare l'area della sfera integrando la lunghezza delle circonferenze che si ottengono tagliando a quota z.



$$\text{Area } S_R = \int_{-R}^{R} dz \text{ lungh } C_z =$$

circ. di raggio $\sqrt{R^2-z^2}$ =

$$= \int_{-R}^{R} 2\pi \sqrt{R^2-z^2}\, dz = 2\pi \frac{\pi R^2}{2} = \pi^2 R^2 \quad ??$$

Sbagliato

Area di una superficie grafico

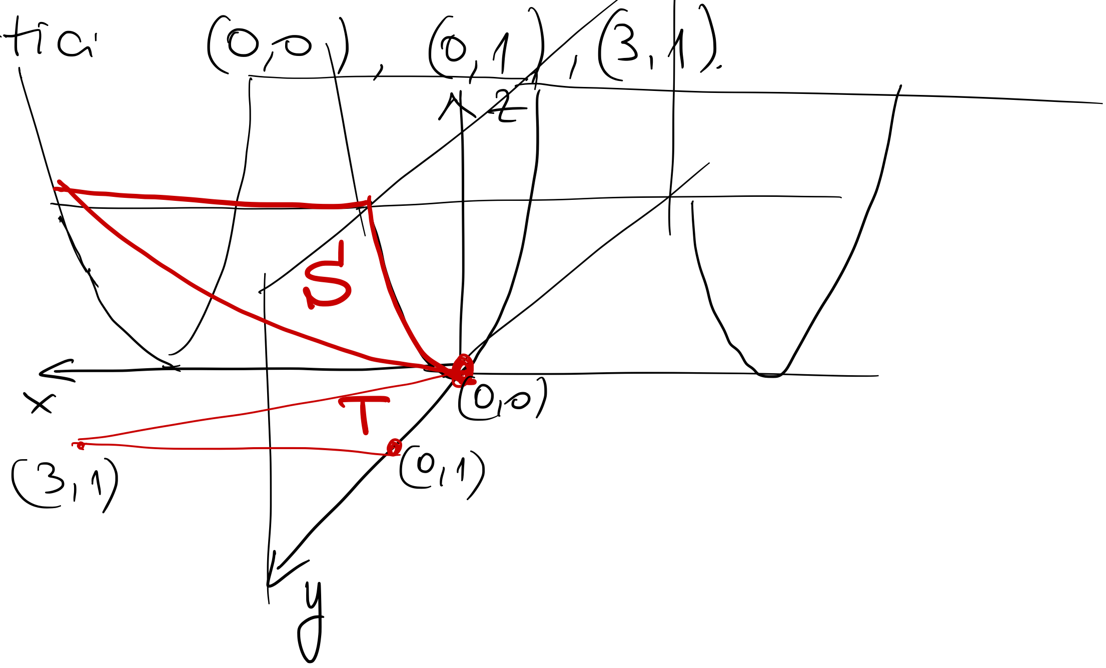
$$\varphi \qquad \begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases} \qquad (x,y) \in D$$

$$\varphi_x \wedge \varphi_y = (-f_x, -f_y, 1)$$

$$\|\varphi_x \wedge \varphi_y\| = \sqrt{1 + f_x^2 + f_y^2}$$

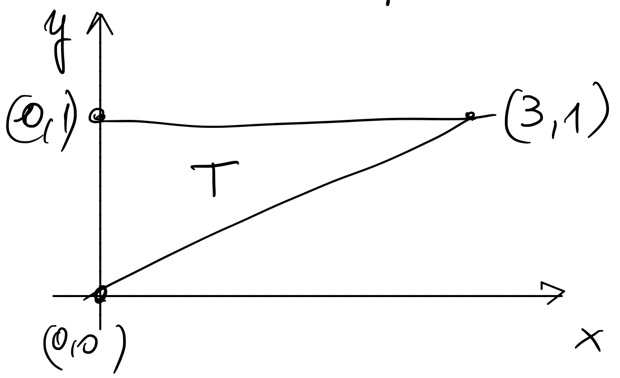
Per casa: ricalcolare l'area della sfera come 2 volte l'area della semisfera $z = \sqrt{R^2-x^2-y^2}$ $(x,y) \in C_R$

ESERCIZIO Calcolare l'area della parte di superficie $z = 2y^2$ che ha per proiezione sul piano xy il triangolo T di vertici $(0,0)$, $(0,1)$, $(3,1)$.



$$f(x,y) = 2y^2$$

$$\text{Area } S = \iint_T \sqrt{1+16y^2}\, dx\,dy = (*)$$



$$T = \{(x,y) : \; 0 \le y \le 1, \; 0 \le x \le 3y\}$$

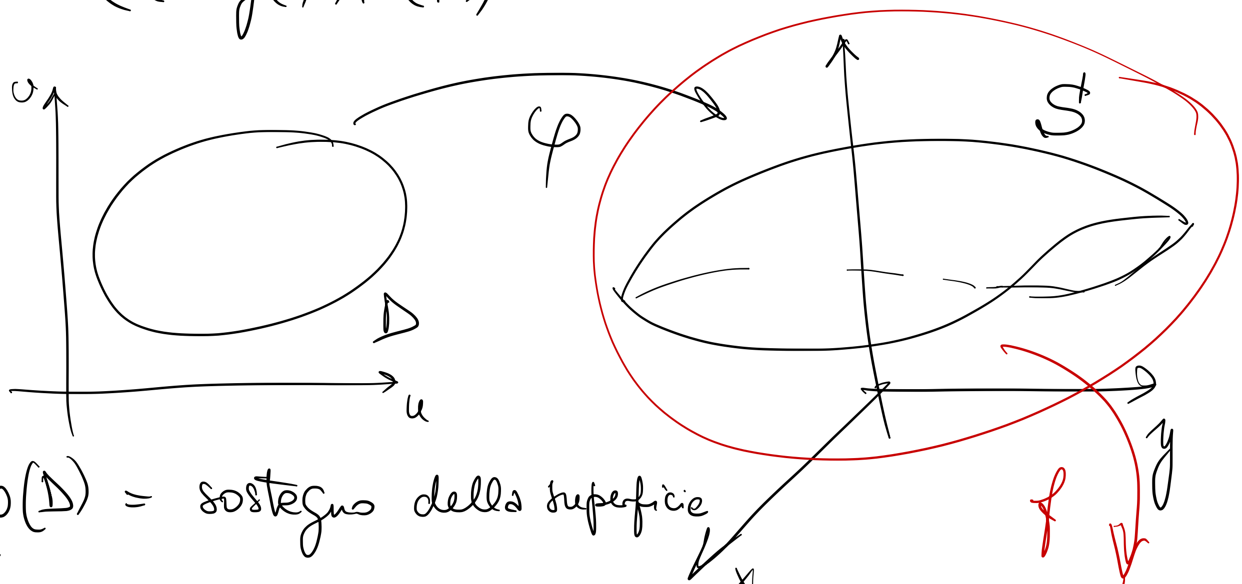
$$(*) = \int_0^1 dy \int_0^{3y} dx \sqrt{1+16y^2} = \int_0^1 dy \sqrt{1+16y^2} \cdot 3y =$$

$$= \text{facile} \ldots$$

Integrali di superficie.

$\varphi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ superficie regolare

$(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$



Sia $S = \varphi(D)$ = sostegno della superficie

Sia $f(x,y,z)$ una funzione continua definita in S.

Def

$$\iint_S f \, d\sigma = \iint_D f(x(u,v), y(u,v), z(u,v)) \underbrace{\|\varphi_u \wedge \varphi_v\|}_{= \sqrt{A^2+B^2+C^2}} \, du\,dv$$

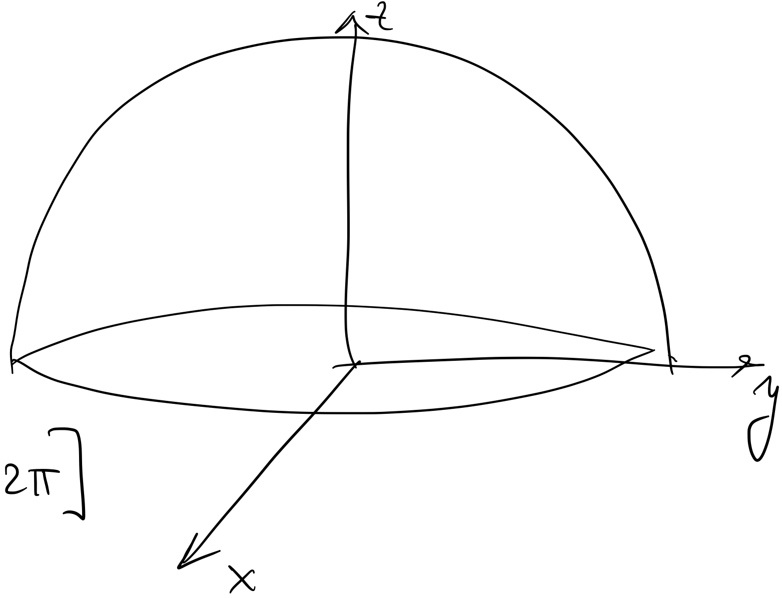
Esempio:

Baricentro di una semisfera

$B = (x_B, y_B, z_B)$.

$$x_B = \frac{1}{\text{area } S} \iint_S x \, d\sigma$$ e analogamente per y_B, z_B

$$\begin{cases} x = R \operatorname{sen}\theta \cos\varphi \\ y = R \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = R\cos\theta \end{cases}$$



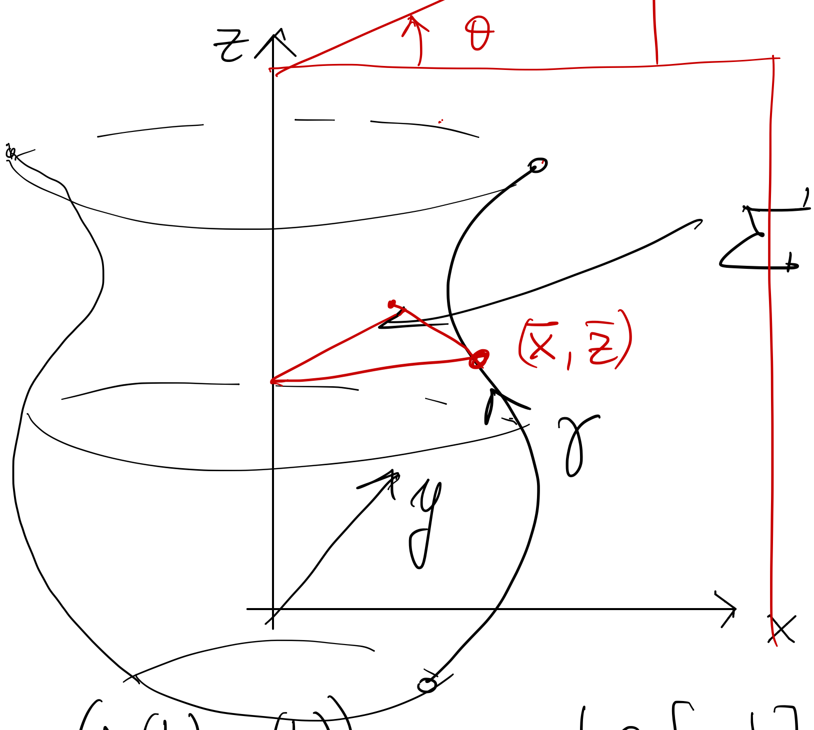
$$(\theta,\varphi) \in \left[0, \frac{\pi}{2}\right] \times [0, 2\pi]$$

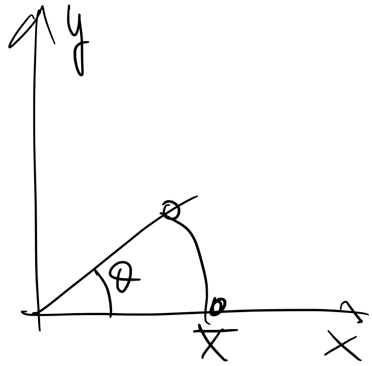
$$x_B = \frac{1}{\text{area } S} \iint_S x \, d\sigma = \dots = 0 = y_B$$

$$z_B = \frac{1}{\text{area } S} \iint_S z \, d\sigma = \frac{1}{2\pi R^2} \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \; R\cos\theta \, R^2 \operatorname{sen}\theta =$$

$$= R \int_0^{\pi/2} \cos\theta \operatorname{sen}\theta \, d\theta = \frac{R}{2}$$

Area di una superficie di rotazione





Sia $\gamma(t) = (x(t), z(t))$ $t \in [a,b]$ una curva regolare del semipiano xz, con $x > 0$.

Faccio ruotare la curva intorno all'asse z

$\Rightarrow$ ottengo una superficie

area Σ : parametrizziamola

$$\varphi \begin{cases} x = \bar{x}\cos\theta = x(t)\cos\theta \\ y = \bar{x}\,\mathrm{sen}\,\theta = x(t)\,\mathrm{sen}\,\theta \\ z = \bar{z} = z(t) \end{cases} \qquad (t, \theta) \in [a,b] \times [0, 2\pi]$$

$$\varphi_t = (x'(t)\cos\theta, \quad x'(t)\,\mathrm{sen}\,\theta, \quad z'(t))$$

$$\varphi_\theta = (-x(t)\,\mathrm{sen}\,\theta, \quad x(t)\cos\theta, \quad 0)$$

$$A(t,\theta) = -x(t)\,z'(t)\cos\theta$$

$$B(t,\theta) = -x(t)\,z'(t)\,\mathrm{sen}\,\theta$$

$$C(t,\theta) = x(t)\,x'(t)$$

$$A(t,\theta) = -x(t)\, z'(t) \cos\theta$$
$$B(t,\theta) = -x(t)\, z'(t)\, \operatorname{sen}\theta$$
$$C(t,\theta) = x(t)\, x'(t)$$

$$\sqrt{A^2+B^2+C^2} = \sqrt{x(t)^2 z'(t)^2 + x(t)^2 x'(t)^2} =$$
$$= x(t)\sqrt{x'(t)^2+z'(t)^2}$$

$$\text{Area } \Sigma_+ = \int_a^b dt \int_0^{2\pi} d\theta \; x(t)\sqrt{x'(t)^2+z'(t)^2} =$$
$$= 2\pi \int_a^b x(t) \underbrace{\sqrt{x'(t)^2+z'(t)^2}\,dt}_{ds} = 2\pi \int_\gamma x\, ds$$

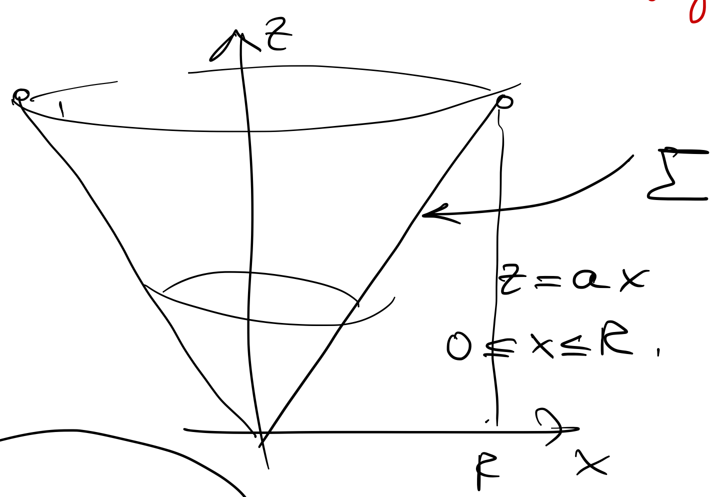
TEOREMA DI GULDINO PER LE AREE DELLE SUPERFICI DI ROTAZIONE.

$$\text{Area } \Sigma_+ = 2\pi \int_\gamma x\, ds = 2\pi\, L(\gamma) \boxed{\frac{1}{L(\gamma)} \int_\gamma x\, ds} =$$

ascissa del baricentro di γ.

$$= 2\pi x_B \, L(\gamma)$$

Area laterale del cono.



$$\text{Area}(\Sigma) = 2\pi \frac{R}{2} \cdot R \cdot \sqrt{1+a^2} =$$
$$= \pi R^2 \sqrt{1+a^2}$$