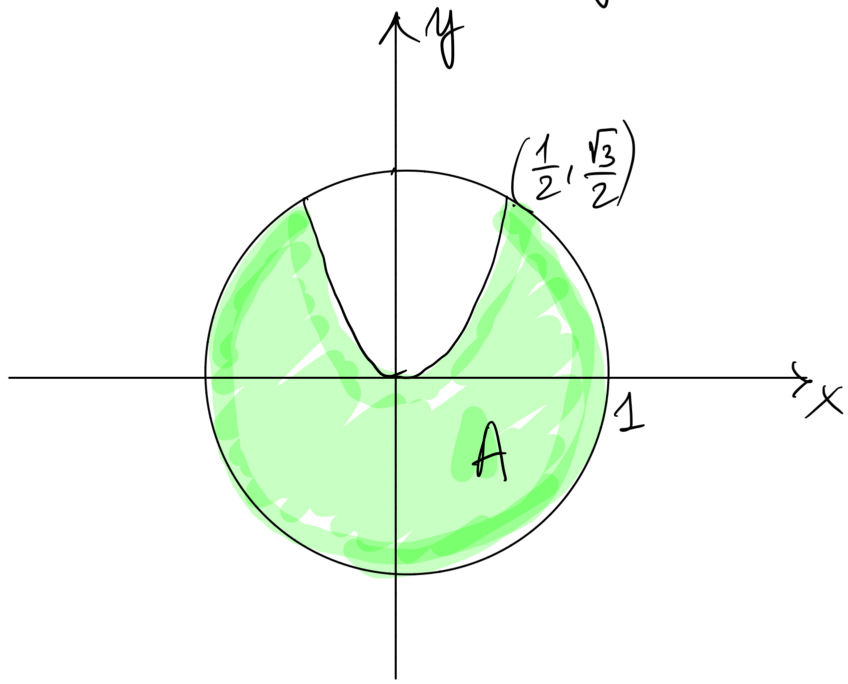
Lunedì 16/05 il ricevimento studenti è annullato per una riunione.

Lezione: regolare!

__Esercizio__ Calcolare $\iint_A \sqrt{x^2+y^2}\, dx dy$, dove

$$A = \{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 \leq 1, \quad y \leq 2\sqrt{3}\, x^2\}$$

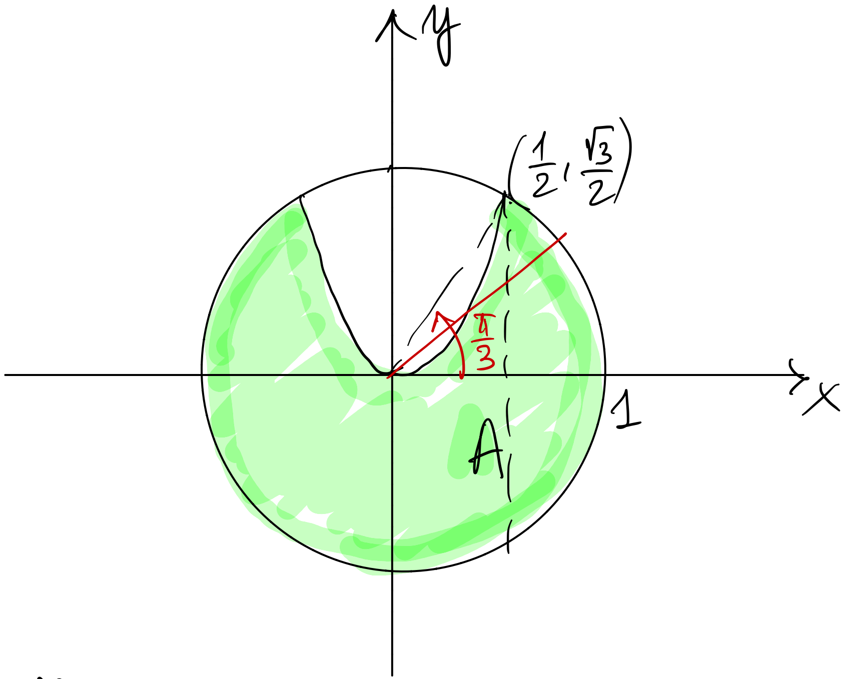


$$\begin{cases} x^2+y^2=1 \\ y = 2\sqrt{3}\, x^2 \end{cases} \Rightarrow \quad \frac{y}{2\sqrt{3}} + y^2 = 1 \quad \Rightarrow$$

$$\Rightarrow \quad 2\sqrt{3}\, y^2 + y - 2\sqrt{3} = 0$$

$$y = \frac{-1 \pm \sqrt{1+48}}{4\sqrt{3}} = \frac{-1 \pm 7}{4\sqrt{3}}$$

$$y = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad\qquad x = \pm \frac{1}{2}$$



$$y = 2\sqrt{3}\,x^2$$

$$\cancel{\rho}\,\text{sen}\,\theta = 2\sqrt{3}\,\rho^{\cancel{2}}\cos^2\theta$$

$$\boxed{\rho = \frac{\text{sen}\,\theta}{2\sqrt{3}\cos^2\theta}}$$

$$\iint_A \sqrt{x^2+y^2}\,dxdy = (*)$$

In coordinate cartesiane:

$$(*) = 2\left[\int_0^{1/2} dx \int_{-\sqrt{1-x^2}}^{2\sqrt{3}x^2} dy\,\sqrt{x^2+y^2} + 2\int_{1/2}^{1} dx \int_0^{\sqrt{1-x^2}} dy\,\sqrt{x^2+y^2}\right]$$

Ok ma non elementarissimo.

In coord. polari:

$$(*) = 2\left[\int_{-\pi/2}^{0} d\theta \int_0^1 d\rho\,\rho^2 + \int_0^{\pi/3} d\theta \int_{\frac{\text{sen}\,\theta}{2\sqrt{3}\cos^2\theta}}^{1} d\rho\,\rho^2\right] =$$

$$= 2\left[\frac{\pi}{2}\,\frac{1}{3} + \frac{1}{3}\int_0^{\pi/3} d\theta\left(1 - \frac{\text{sen}^3\theta}{8\cdot 3\sqrt{3}\cos^6\theta}\right)\right] =$$

$$\boxed{-\text{sen}\,\theta\, d\theta = dt} \qquad \boxed{\cos\theta = t}$$

$$= 2\left[\frac{\pi}{6} + \frac{\pi}{9} - \frac{1}{72\sqrt{3}}\int_0^{\pi/3} \frac{(1-\cos^2\theta)}{\cos^6\theta}\,\text{sen}\,\theta\, d\theta\right] =$$

$$= 2\left[\;''\quad '' - \frac{1}{72\sqrt{3}}\int_{1/2}^{1}\left(\frac{1}{t^6} - \frac{1}{t^4}\right)dt\right]$$

Esercizio. Calcolare il volume di

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + y^2 \leq \sqrt{z(1-z)}\}$$

OSS E' un solido di rotazione. Se consideriamo

$$\Omega' = \{(x,y,z) : x^2 + y^2 \leq \sqrt{z(1-z)}\}$$

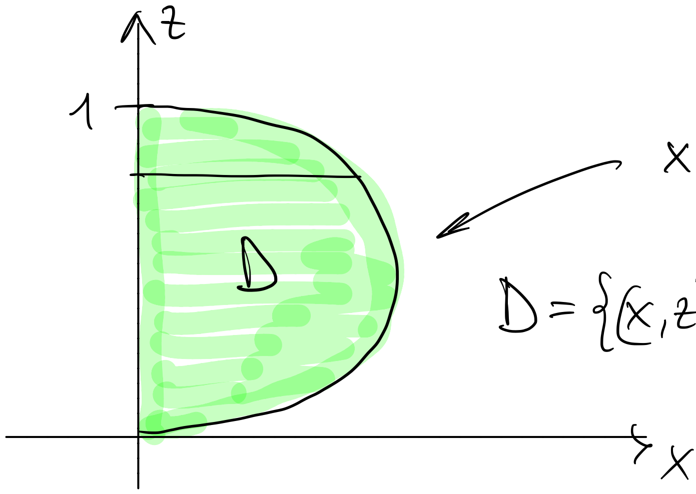
questo è di rotazione intorno all'asse z. (dipende solo da x^2+y^2.)

Ω è semplicemente il traslato di Ω' del vettore $(1,0,0)$.

Disegnamo Ω'. $\Rightarrow y=0, x>0$.

$$x^2 = \sqrt{z(1-z)}$$

$$\boxed{x = \sqrt[4]{z(1-z)}} \qquad 0 \leq z \leq 1$$



$x = \sqrt[4]{z(1-z)}$

$$D = \{(x,z) : 0 \leq z \leq 1, \; 0 \leq x \leq \sqrt[4]{z(1-z)}\}$$

Ω' è ottenuto ruotando D intorno all'asse z.

Ω è poi ottenuto traslando Ω' di 1 nella direz. dell'asse x

$$\mathrm{vol}(\Omega) = \mathrm{vol}(\Omega') \overset{\text{Guldino}}{=} 2\pi \iint_D x \, dx\, dz =$$

$$\boxed{D = \left\{ (x,z) : 0 \le z \le 1, \quad 0 \le x \le \sqrt[4]{z(1-z)} \right\}}$$

$$= 2\pi \int_0^1 dz \int_0^{\sqrt[4]{z(1-z)}} dx \quad x = \pi \int_0^1 dz \sqrt{z-z^2} = (*)$$

$$z - z^2 = \frac{1}{4} - \frac{1}{4} + z - z^2 = \frac{1}{4} - \left(z - \frac{1}{2}\right)^2$$

$$(*) = \pi \int_0^1 dz \sqrt{\frac{1}{4} - \left(\frac{2z-1}{2}\right)^2} = \frac{\pi}{2} \int_0^1 dz \sqrt{1-(2z-1)^2} =$$

$$\boxed{\begin{aligned} 2z-1 &= t \\ dz &= \frac{dt}{2} \end{aligned}}$$

$$= \frac{\pi}{4} \int_{-1}^{1} dt \sqrt{1-t^2} = \frac{\pi}{4} \cdot \frac{\pi}{2}$$

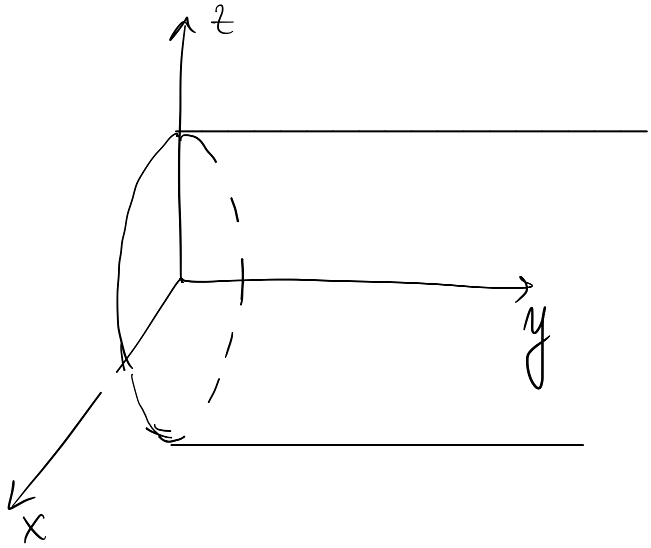
$$\int_{-1}^{1} \sqrt{1-t^2}\, dt = \cancel{t\sqrt{1-t^2}\Big|_{-1}^{1}} + \int_{-1}^{1} \frac{t^2 - 1 + 1}{\sqrt{1-t^2}}\, dt = -\int_{-1}^{1} \sqrt{1-t^2}\, dt + \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow \int_{-1}^{1} \sqrt{1-t^2}\, dt = \frac{1}{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2} \arcsin t \Big|_{-1}^{1} = \frac{\pi}{2}$$

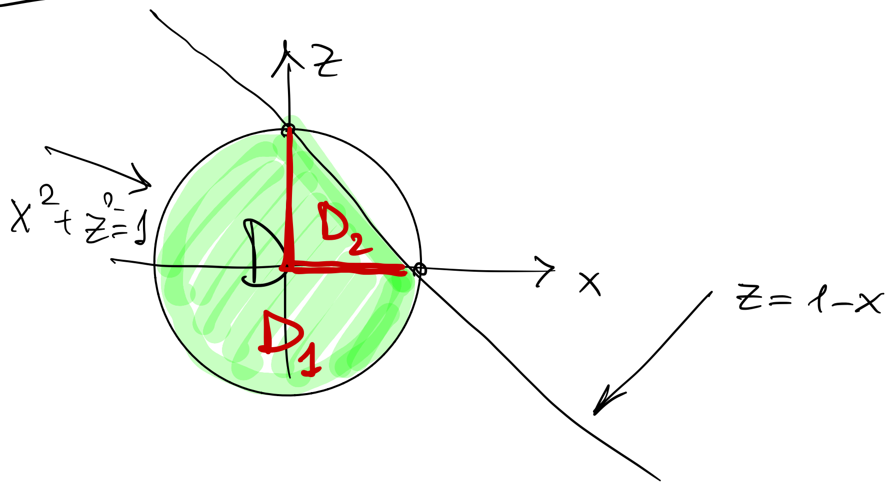
ESERCIZIO

$$\iiint_T x \, dx\,dy\,dz, \quad \text{dove}$$

$$T = \{(x,y,z): \; x^2+z^2 \le 1, \quad 0 \le y \le 1-x-z\}$$

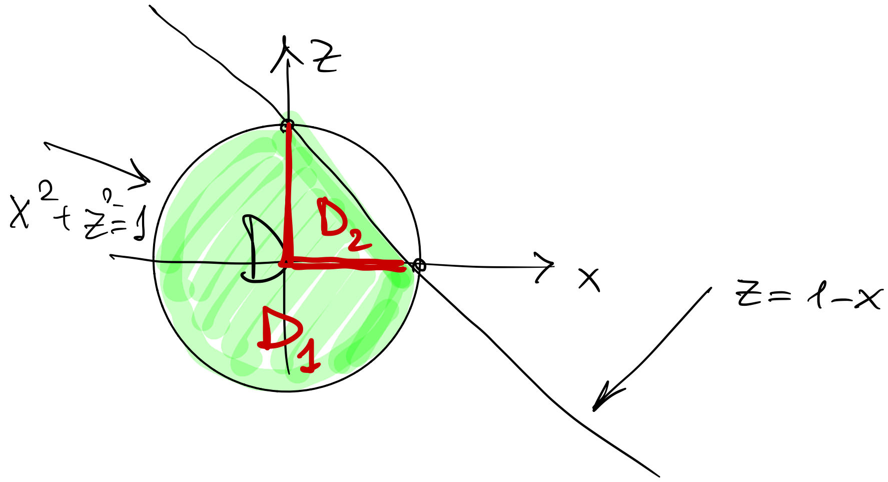


N.B. deve essere $\quad 0 \le 1-x-z \Rightarrow x+z \le 1.$



$$T = \{(x,y,z): \; (x,z) \in D, \quad 0 \le y \le 1-x-z\}$$

$$\iiint_T x \, dx\,dy\,dz = \iint_D dx\,dz \, x \int_0^{1-x-z} dy = \iint_{D_1} \cdots + \iint_{D_2} \cdots$$



$$T = \{(x,y,z) : (x,z)\in D, \quad 0 \le y \le 1-x-z\}$$

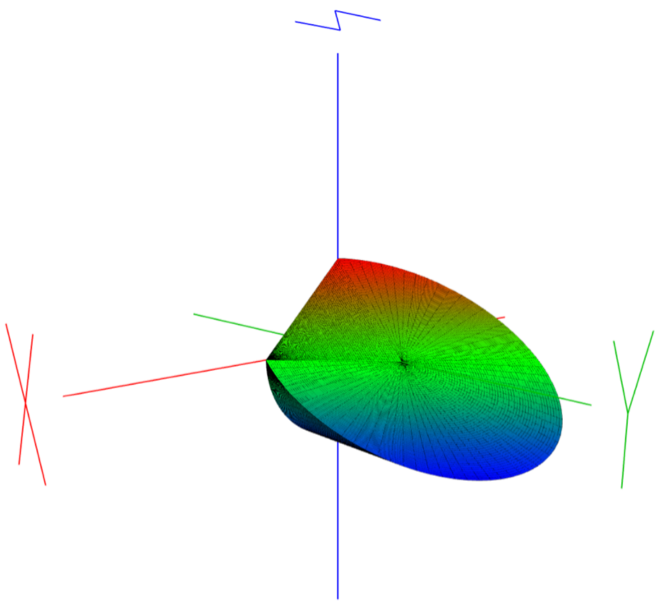
$$\iiint_T x \, dx\,dy\,dz = \iint_D dx\,dz \, x \int_0^{1-x-z} dy = \iint_D dx\,dz \; x(1-x-z) =$$

$$= \iint_{D_1} \cdots\cdots + \iint_{D_2} \cdots\cdots = \int_{\pi/2}^{2\pi} d\theta \int_0^1 d\rho \; \rho^2 \cos\theta \left(1-\rho\cos\theta-\rho\,\mathrm{sen}\,\theta\right)$$

$$+ \int_0^1 dx \int_0^{1-x} dz \; x(1-x-z) =$$

$$= \text{completare}\ldots$$

uso coord polari
$x = \rho\cos\theta$
$z = \rho\,\mathrm{sen}\,\theta$

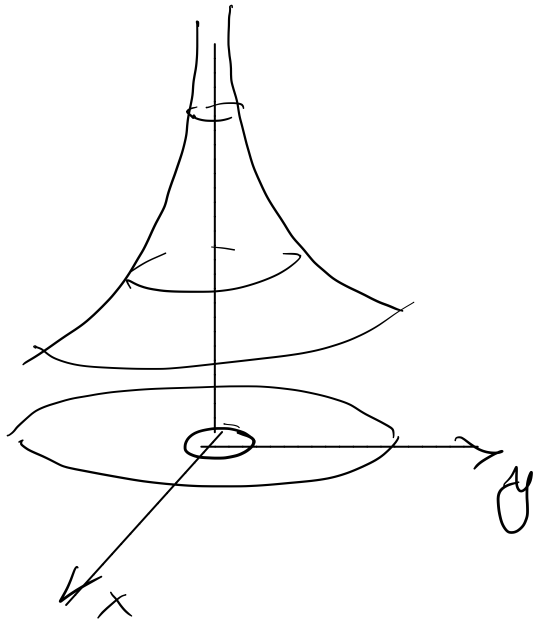


Cenno agli integrali multipli impropri.

$$\iint_{B_1} \frac{dx\,dy}{\sqrt{x^2+y^2}}$$, dove B_1 è il cerchio unitario.

OSS $\quad f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ è illimitata vicino a $(0,0)$.

$\Rightarrow$ Non è un integrale di Riemann.



Idea:

$$\iint_{B_1 \setminus B_\varepsilon} \frac{dx\,dy}{\sqrt{x^2+y^2}} \longleftarrow \text{è di Riemann}$$

Lo calcolo e mando $\varepsilon \to 0^+$

$$= \int_0^{2\pi} d\theta \int_\varepsilon^1 d\rho \, \cancel{\frac{\rho}{\rho}} = 2\pi(1-\varepsilon) \xrightarrow{\varepsilon \to 0} 2\pi$$

Poniamo $\displaystyle\iint_{B_1} \frac{dx\,dy}{\sqrt{x^2+y^2}} = \lim_{\varepsilon \to 0} \iint_{B_1 \setminus B_\varepsilon} \frac{dx\,dy}{\sqrt{\;--\;}} = 2\pi.$

In generale, se f è a segno costante, e illimitata vicino a un punto dell'insieme E in cui stiamo integrando, si prendono degli insiemi "approssimanti" E_n su cui f è Riemann integrabile $E_n \subset E_{n+1}$

In generale, se f è a segno costante, e illimitata vicino a un punto dell'insieme E in cui stiamo integrando, si prendono degli insiemi "approssimanti" E_n su cui f è Riemann integrabile $E_n \subset E_{n+1}$ e t.c.

$$\text{mis}\left(E \setminus \bigcup_n E_n\right) = 0$$

(nel nostro caso $E_n = B_1 \setminus B_{1/n}$)

e si calcola $\displaystyle\lim_{n\to+\infty} \iint_{E_n} f(x,y)\,dx\,dy$

se tale limite $\exists$ finito, poniamo

$$\iint_E f(x,y)\,dx\,dy = \lim_{n\to+\infty} \iint_{E_n} f(x,y)\,dx\,dy$$

<u>OSS</u> Si può dim. che il risultato di questa procedura non dipende dalla scelta della successione (E_n)

$$\iint_{B_1} \frac{dx\,dy}{\|(x,y)\|^\alpha} = \qquad \alpha > 0.$$

$$= \lim_{n\to+\infty} \iint_{B_1 \setminus B_{1/n}} \frac{dx\,dy}{\| -- \|^\alpha} = \lim_{n\to+\infty} \int_0^{2\pi} d\theta \int_{1/n}^{1} d\rho \, \frac{\rho}{\rho^\alpha} =$$

$$= 2\pi \lim_{n\to+\infty} \int_{1/n}^{1} \frac{d\rho}{\rho^{\alpha-1}} = 2\pi \int_0^1 \frac{d\rho}{\rho^{\alpha-1}}$$

converge sse $\alpha - 1 < 1$

$\alpha < 2.$

$$\iiint_{B_1} \frac{dx\,dy\,dz}{\|(x,y,z)\|^\alpha} =$$

$\alpha > 0$

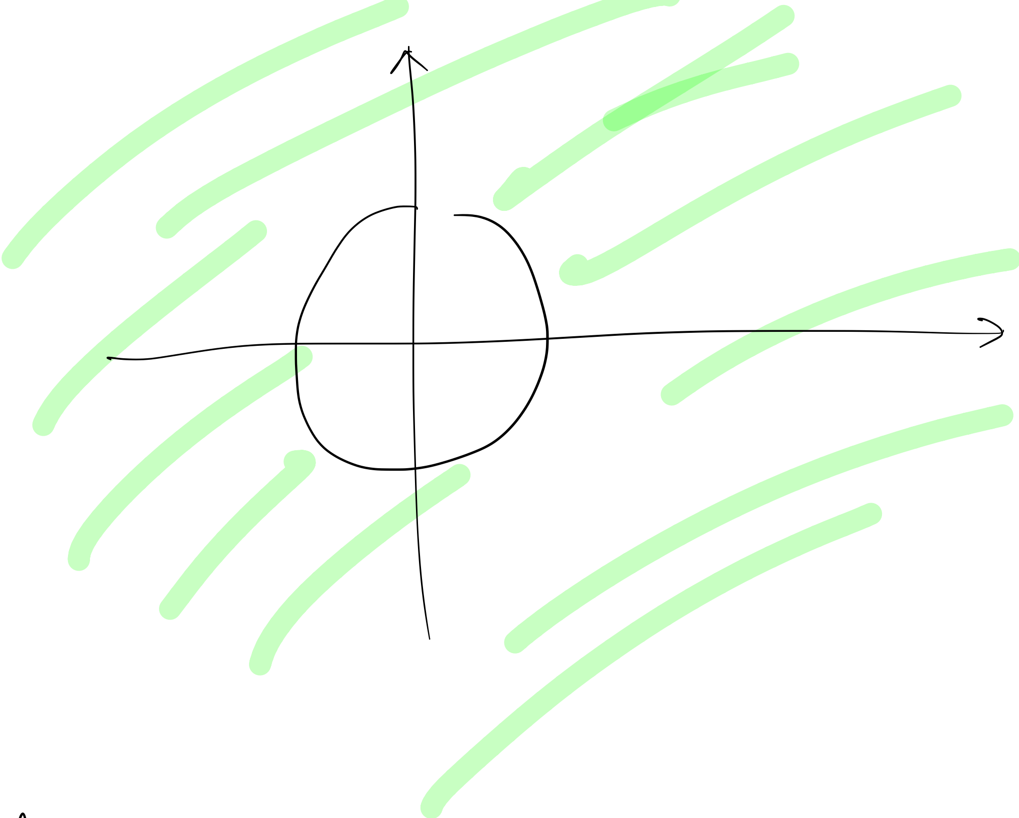
$B_1 = \{ x^2+y^2+z^2 \leq 1 \}$

$$= \lim_{n\to+\infty} \iiint_{B_1 \setminus B_{1/n}} \cdots$$

$$= \lim_{n\to+\infty} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_{1/n}^1 d\rho \; \frac{\rho^2 \operatorname{sen}\theta}{\rho^\alpha} =$$

$$= 2 \cdot 2\pi \int_0^1 \frac{d\rho}{\rho^{\alpha-2}} \quad \text{converge sse} \quad \alpha < 3$$

$$\iint_{\mathbb{R}^2 \setminus B_1} \frac{dx\,dy}{\|(x,y)\|^\alpha}$$



$$\|$$

$$\lim_{n\to+\infty} \iint_{B_n \setminus B_1} \frac{dx\,dy}{\|(x,y)\|^\alpha} = \text{completare.}$$