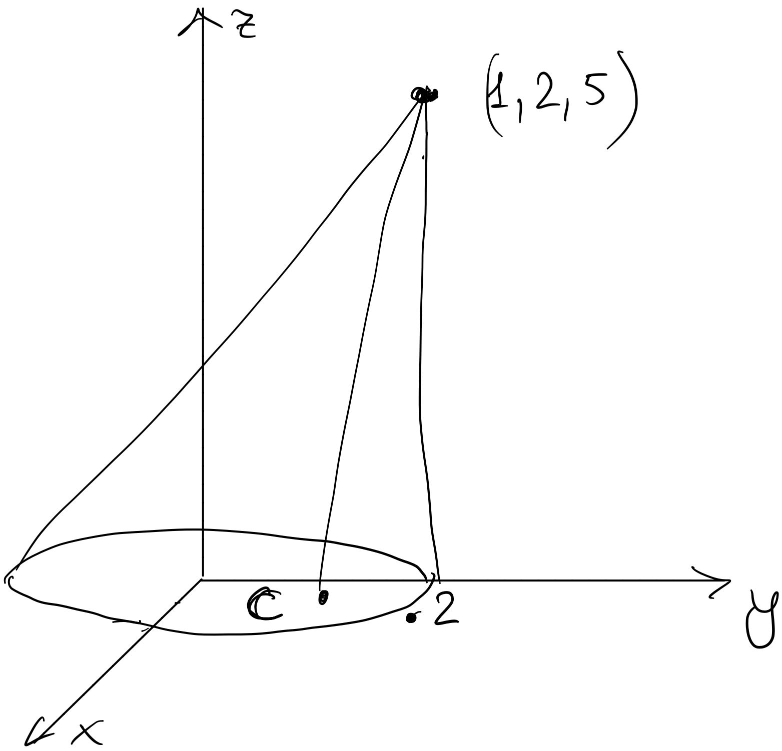
Coordinate "su misura"

Calcolare $\iiint_E x \, dx\,dy\,dz$, dove E è il cono che ha per base il cerchio $C = \{ (x,y,z) : \ z = 0 \ , \ x^2+y^2 \leq 4 \}$ e per vertice il pto $V(1,2,5)$.



Descrizione dei pti di C.

$$\bar{x} = \rho\cos\theta \qquad 0 \leq \rho \leq 2$$
$$\bar{y} = \rho\,\text{sen}\,\theta \qquad 0 \leq \theta \leq 2\pi.$$
$$\bar{z} = 0$$

Segmento che collega $(\bar{x}, \bar{y}, \bar{z})$ a $V(1,2,5)$:

$$x = (1-t)1 + t\,\bar{x}$$
$$y = (1-t)2 + t\,\bar{y} \qquad t \in [0,1]$$
$$z = (1-t)5 + \underbrace{t\,\bar{z}}_{=0}$$

$$\begin{cases} x = (1-t)1 + t\,\bar{x} \\ y = (1-t)2 + t\,\bar{y} \\ z = (1-t)5 + \underbrace{t\,\bar{z}}_{=0} \end{cases}$$

Eq^{ni} parametriche che descrivono il cono

$$\begin{cases} x = (1-t) + t\rho\cos\theta \\ y = (1-t)2 + t\rho\,\text{sen}\theta \\ z = (1-t)5 \end{cases} \qquad 0 \le \rho \le 2 \quad 0 \le \theta \le 2\pi. \qquad 0 \le t \le 1$$

$$\left|\det \frac{\partial(x,y,z)}{\partial(t,\rho,\theta)}\right| = \left|\det \begin{bmatrix} -1+\rho\cos\theta & t\cos\theta & -t\rho\,\text{sen}\theta \\ -2+\rho\,\text{sen}\theta & t\,\text{sen}\theta & t\rho\cos\theta \\ -5 & 0 & 0 \end{bmatrix}\right| =$$

$$= +5t^2\rho$$

$$\iiint_E x\,dx\,dy\,dz = \int_0^{2\pi} d\theta \int_0^2 d\rho \int_0^1 dt\,(1-t+\cancel{t\rho\cos\theta})\,5t^2\rho =$$

$$= 5\cdot 2\pi \int_0^2 d\rho\,\rho \int_0^1 dt\,(t^2-t^3) =$$

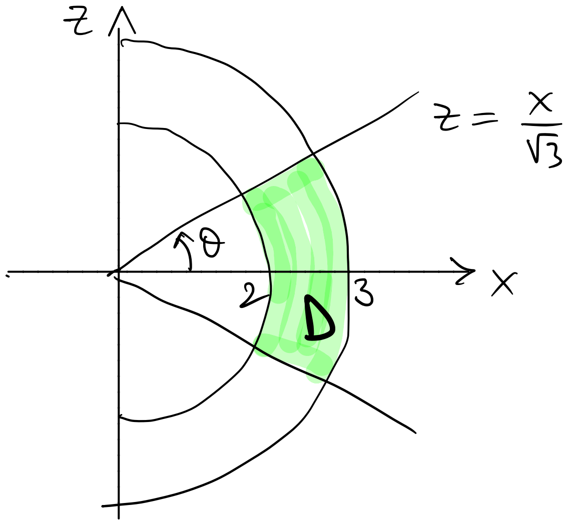
$$= 10\pi \cdot 2 \cdot \left(\frac{1}{3} - \frac{1}{4}\right)$$

Data la regione del piano xz

$$D = \{ (x,z) \in \mathbb{R}^2 : \quad x \geq 0 , \quad 4 \leq x^2 + z^2 \leq 9 , \ 3z^2 \leq x^2 \}$$

Disegnare il solido E, contenuto nel semispazio $y \geq 0$, ottenuto ruotando D di un angolo piatto attorno all'asse z.

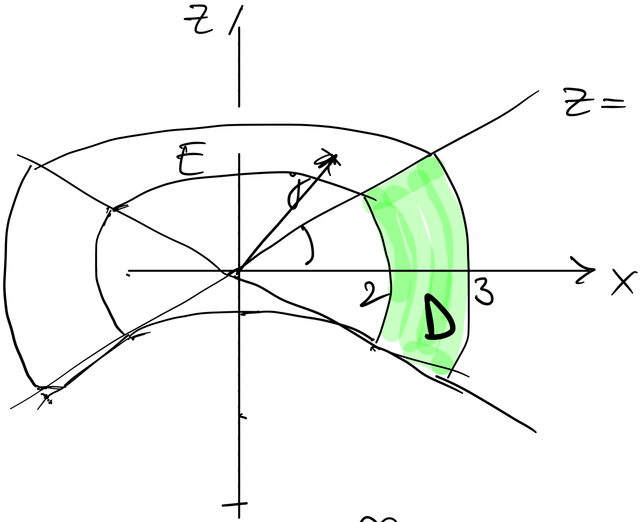
Calcolare $\mathrm{vol}(E)$ e le coord. del suo baricentro.

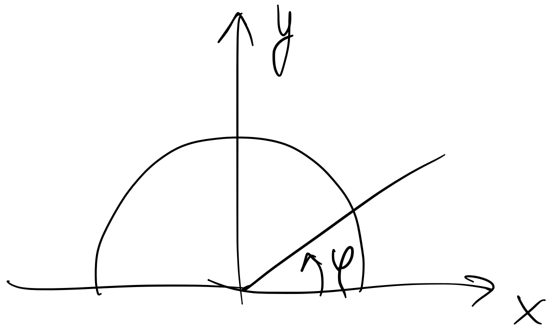


$$3z^2 \leq x^2$$

$$|z| \leq \frac{|x|}{\sqrt{3}} = \frac{x}{\sqrt{3}}$$

$$-\frac{x}{\sqrt{3}} \leq z \leq \frac{x}{\sqrt{3}}$$





coord. polari

$$\begin{cases} x = \rho\cos\theta \\ z = \rho\,\mathrm{sen}\,\theta \end{cases}$$

$$\mathrm{vol}\,E = \pi \iint_D x\,dx\,dz = 2\pi \int_0^{\pi/6} d\theta \int_2^3 d\rho\; \rho^2\cos\theta =$$

Guldino

$$= 2\pi \int_0^{\pi/6} \cos\theta\,d\theta \int_2^3 \rho^2\,d\rho =$$

$$= 2\pi \cdot \frac{1}{2} \quad \frac{1}{3}\;19 = \frac{19\pi}{3}$$

$x_B\ ,\ y_B\ ,\ z_B$

$$x_B = \frac{1}{\mathrm{vol}\,E} \iiint_E x \, dx\,dy\,dz = z_B = 0$$

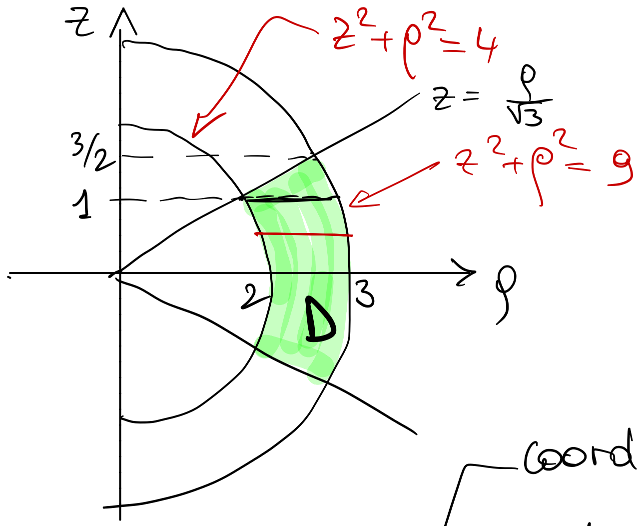
$$y_B = \frac{1}{\mathrm{vol}\,E} \iiint_E y \, dx\,dy\,dz .$$

$$\iiint_E y \, dx\,dy\,dz = 2 \int_{\pi/3}^{\pi/2} d\theta \int_0^{\pi} d\varphi \int_2^3 d\rho \;\; \rho \,\mathrm{sen}\theta \,\mathrm{sen}\varphi \, \rho^2 \,\mathrm{sen}\theta =$$

coord. sferiche

$$= 2 \int_{\pi/3}^{\pi/2} d\theta \,\mathrm{sen}^2\theta \cdot \int_0^{\pi} d\varphi \,\mathrm{sen}\varphi \cdot \int_2^3 d\rho \, \rho^3 = \text{etc...}$$

Solo a titolo di esercizio (perché in pratica non è comodo), impostiamo l'integrale in coord. cilindriche.



coord. cilindriche.

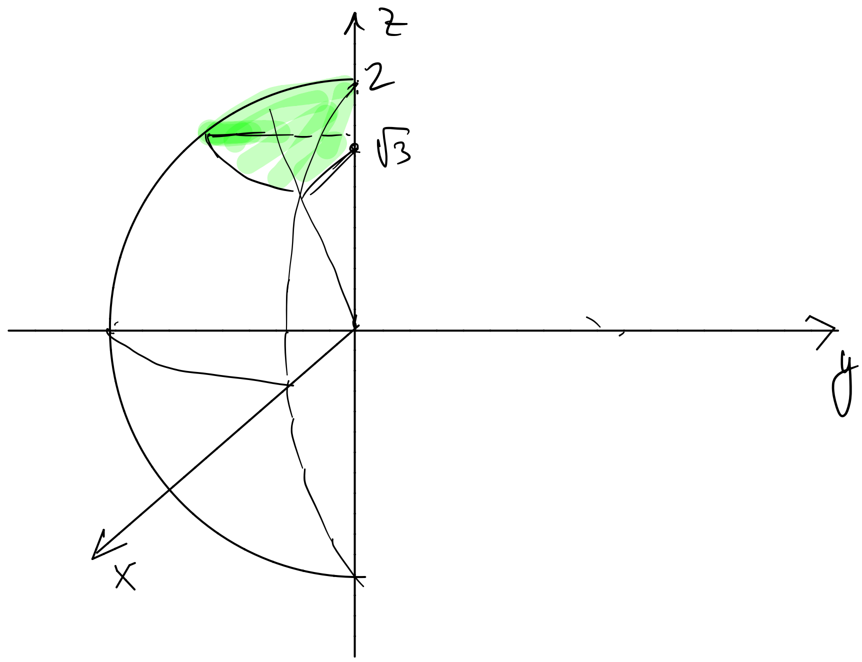
$$\iiint_E y\,dx\,dy\,dz = 2\int_0^1 dz \int_{\sqrt{4-z^2}}^{\sqrt{9-z^2}} d\rho \int_0^{\pi} d\theta \; \rho^2 \operatorname{sen}\theta \; +$$

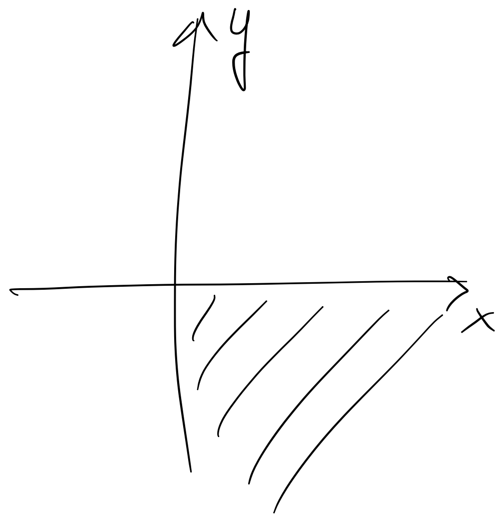
$$+\; 2\int_1^{3/2} dz \int_{\sqrt{3}z}^{\sqrt{9-z^2}} d\rho \int_0^{\pi} d\theta \; \rho^2 \operatorname{sen}\theta$$

Corretto ma complicato ....

Calcolare $\displaystyle\iiint_E \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^3}$

$E = \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2+z^2 \leq 4, \ \ x \geq 0, \ \ y \leq 0, \ z \geq \sqrt{3}\}$



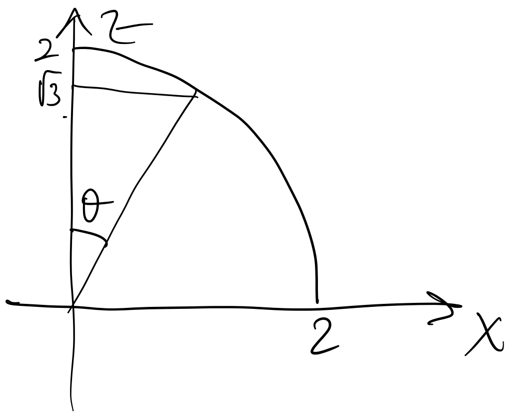


in Coord sferiche E diventa

$\tilde{E} = \{(\rho, \theta, \varphi) : \ \rho \leq 2 \quad -\frac{\pi}{2} \leq \varphi \leq 0 \quad \rho \geq \frac{\sqrt{3}}{\cos\theta}\}$

$z \geq \sqrt{3} \iff \rho\cos\theta \geq \sqrt{3}$

$\tilde{E} = \{(\rho, \theta, \varphi) : -\frac{\pi}{2} \leq \varphi \leq 0 \ , \ 0 \leq \theta \leq \frac{\pi}{6} \ , \ \frac{\sqrt{3}}{\cos\theta} \leq \rho \leq 2\}$



$$\iiint_E \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^3} = \underbrace{\int_{-\pi/2}^{0} d\varphi}_{\frac{\pi}{2}} \int_0^{\pi/6} d\theta \int_{\frac{\sqrt{3}}{\cos\theta}}^{2} d\rho \; \frac{\rho^2 \operatorname{sen}\theta}{\rho^6} =$$

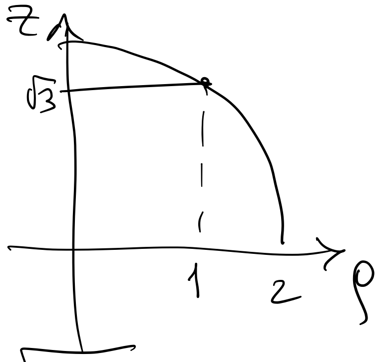
$$= \frac{\pi}{2} \int_0^{\pi/6} d\theta \; \operatorname{sen}\theta \left(-\frac{1}{3}\right) \frac{1}{\rho^3}\Bigg|_{\rho=\frac{\sqrt{3}}{\cos\theta}}^{\rho=2} =$$

$$= \frac{\pi}{6} \int_0^{\pi/6} d\theta \; \operatorname{sen}\theta \left[\frac{\cos^3\theta}{3\sqrt{3}} - \frac{1}{8}\right] =$$

$$= \frac{\pi}{18\sqrt{3}} \int_{\frac{\sqrt{3}}{2}}^{1} t^3\,dt - \frac{\pi}{48} \int_0^{\pi/6} d\theta \operatorname{sen}\theta = \text{etc...}$$

Se l'avessi fatto in coordinate cilindriche.

E diventa $\tilde{E} = \left\{ (\rho, \theta, z) : -\frac{\pi}{2} \le \theta \le 0 \quad \sqrt{3} \le z \le 2 \quad 0 \le \rho \le \sqrt{4-z^2} \right\}$



$$\int_{-\pi/2}^{0} d\theta \int_{\sqrt{3}}^{2} dz \int_{0}^{\sqrt{4-z^2}} d\rho \, \frac{2\rho}{2(\rho^2+z^2)^3} = \frac{\pi}{4} \int_{\sqrt{3}}^{2} dz \, \frac{1}{2} \, \frac{1}{(\rho^2+z^2)^2} \Bigg|_{\rho=\sqrt{4-z^2}}^{\rho=0}$$

$$= \frac{\pi}{8} \int_{\sqrt{3}}^{2} dz \left[\frac{1}{z^4} - \frac{1}{(4-\cancel{z^2}+\cancel{z^2})^2} \right] = \text{immediato}$$