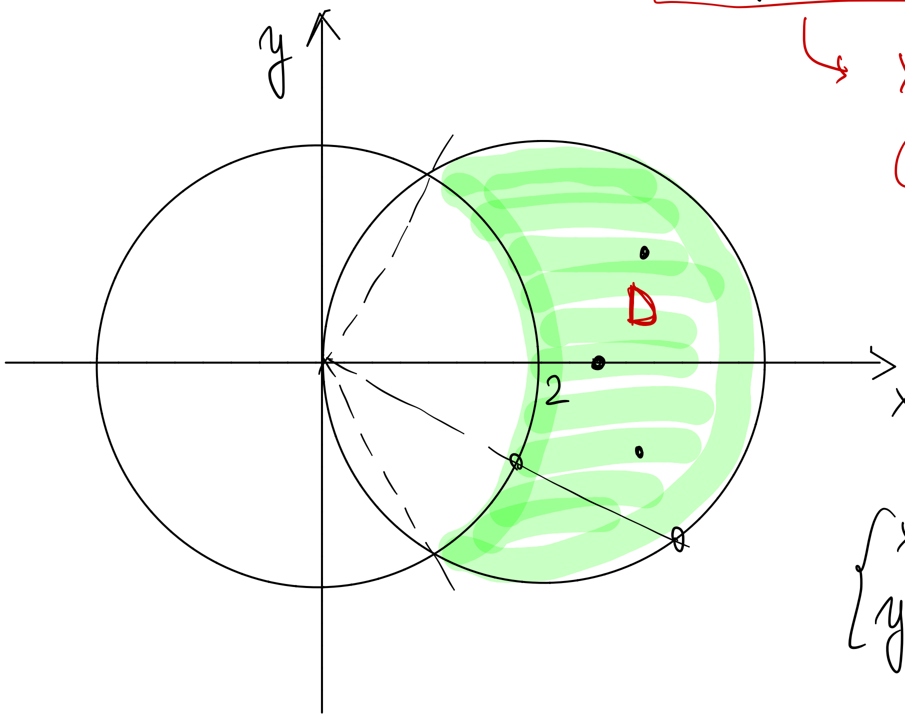
ESERCIZIO Calcolare il baricentro della regione piana

$$D = \{(x,y) \in \mathbb{R}^2 : \ 4 \le \underbrace{x^2+y^2 \le 4x}\}$$

$x^2-4x+4+y^2 \le 4$

$(x-2)^2+y^2 \le 4.$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}$$

La circonferenza $x^2+y^2=4$ si scrive, in coordin. polari

$$\rho = 2.$$

La circonferenza $x^2+y^2=4x$ si scrive come

$$\rho^{\cancel{2}} = 4\cancel{\rho}\cos\theta$$

$$\boxed{\rho = 4\cos\theta}$$

Eq.^{ne} in coord. polari della circonf. $x^2+y^2=4x$.

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Per descrivere il dominio D,

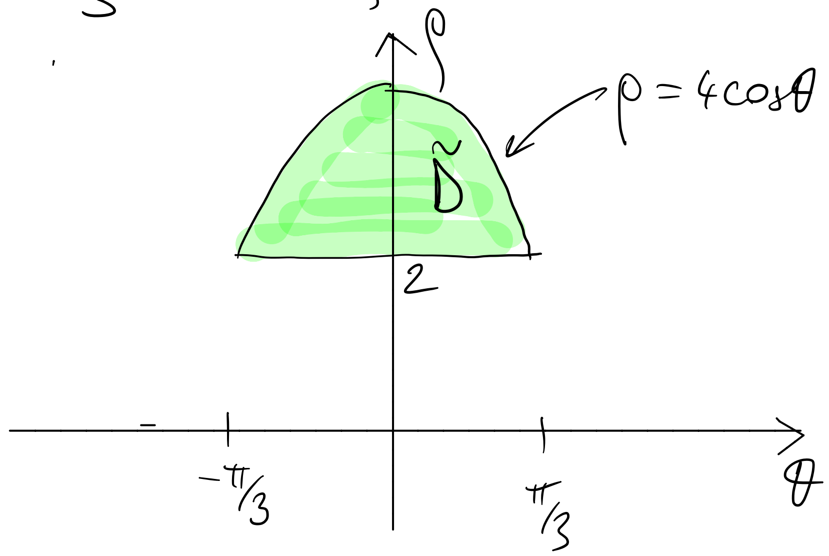
$$2 \le \rho \le 4\cos\theta \qquad \boxed{-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}}$$

Si vede anche così: sono i valori θ t.c.

$$2 \le 4\cos\theta.$$

Quindi, passando a coordinate polari, il dominio D "diventa" il dominio

$$\tilde{D} = \left\{(\rho, \theta) : -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}, \quad 2 \le \rho \le 4\cos\theta\right\}.$$



$$X_B = \frac{1}{\text{area}(D)} \iint_D x \, dx\,dy$$

$$\text{area}(D) = \iint_D 1 \, dx\,dy = \iint_{\tilde{D}} 1 \; \rho \; d\rho\, d\theta =$$

$$= \int_{-\pi/3}^{\pi/3} d\theta \int_2^{4\cos\theta} d\rho \; \rho = 2\int_0^{\pi/3} d\theta \int_2^{4\cos\theta} d\rho \; \rho =$$

$$= \int_0^{\pi/3} d\theta \left(16\cos^2\theta - 4\right) = 16\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) - \frac{4\pi}{3} = \frac{4\pi}{3} + 2\sqrt{3}$$

$$\int \cos^2\theta \, d\theta = \int \frac{1 + \cos 2\theta}{2} = \frac{\theta}{2} + \frac{\text{sen}(2\theta)}{4} + c$$

$$\iint_D x \, dx\,dy = 2 \int_0^{\pi/3} d\theta \int_2^{4\cos\theta} d\rho \; \rho^2 \cos\theta = (*)$$

oss: x è una funzione "pari" rispetto alla riflessione attraverso l'asse x, e il dominio è simmetrico

$$(*) = \frac{2}{3} \int_0^{\pi/3} d\theta \, \cos\theta \left(64 \cos^3\theta - 8 \right) =$$

$$= \frac{128}{3} \int_0^{\pi/3} \underbrace{\cos^4\theta}_{\shortparallel} \, d\theta - \frac{16}{3} \int_0^{\pi/3} \cos\theta \, d\theta = (*)$$

$$\frac{(1+\cos 2\theta)^2}{2^2} = \frac{1}{4}\left[1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\shortparallel \; \frac{1+\cos 4\theta}{2}} \right] =$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{\cos 4\theta}{8}$$

$$(*) = \frac{128}{3} \left[\frac{\cancel{3}}{8} \cdot \frac{\pi}{\cancel{3}} + \left.\frac{\operatorname{sen} 2\theta}{4}\right|_0^{\pi/3} + \left.\frac{\operatorname{sen} 4\theta}{32}\right|_0^{\pi/3} \right] - \frac{16}{3} \frac{\sqrt{3}}{2}$$

$$= \frac{16\pi}{3} + \frac{32}{3} \cdot \frac{\sqrt{3}}{2} - \frac{4}{3} \frac{\sqrt{3}}{2} - \frac{8}{3} \sqrt{3} = \ldots$$

$$\iint_D y \, dx\,dy = 0 \quad \text{per simmetria.}$$

Controlliamo

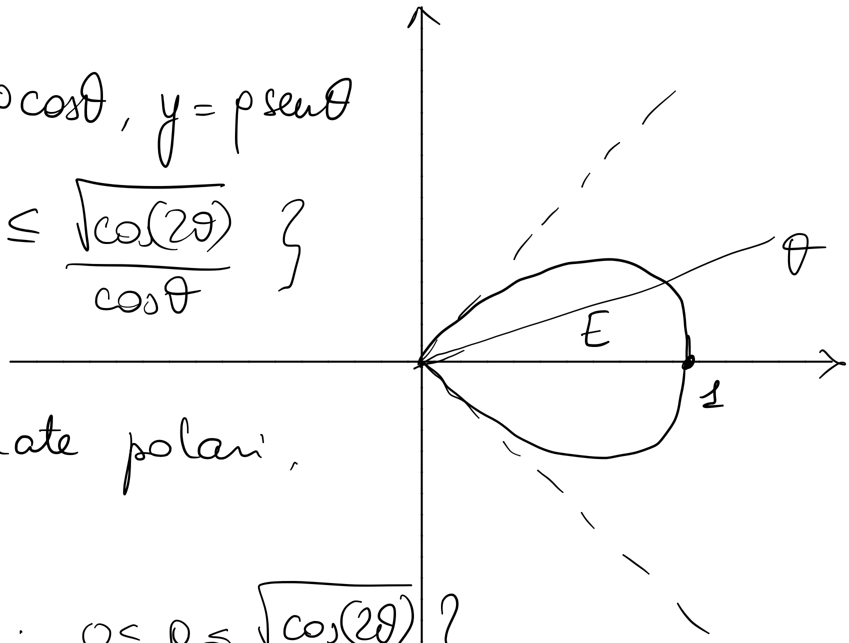
$$\iint_D y \, dx\,dy = \int_{-\pi/3}^{\pi/3} d\theta \int_2^{4\cos\theta} d\rho \; \rho^2 \operatorname{sen}\theta =$$

$$= \frac{1}{3}\int_{-\pi/3}^{\pi/3} d\theta \; \operatorname{sen}\theta \left(64\cos^3\theta - 8\right) = 0$$

ESERCIZIO Calcolare $\iint_E x^2\,dx\,dy$, dove E è la porzione di piano chiusa dalla curva $\rho = \frac{\sqrt{\cos(2\theta)}}{\cos\theta}$ $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

(osserviamo che $\frac{\sqrt{\cos(2\theta)}}{\cos\theta}$ è ben definito per questi θ).

$$E = \left\{ (x,y) \in \mathbb{R}^2 :\ x = \rho\cos\theta,\ y = \rho\,\mathrm{sen}\,\theta \right.$$
$$\left. -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ 0 \le \rho \le \frac{\sqrt{\cos(2\theta)}}{\cos\theta} \right\}$$



ovviamente, in coordinate polari, E diventa

$$\tilde{E} = \left\{ (\rho,\theta) :\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4};\ 0 \le \rho \le \frac{\sqrt{\cos(2\theta)}}{\cos\theta} \right\}.$$

$$\iint_E x^2\,dx\,dy = \int_{-\pi/4}^{\pi/4} d\theta \int_0^{\frac{\sqrt{\cos(2\theta)}}{\cos\theta}} d\rho\ \rho^2\cos^2\theta\ \rho =$$

$$= \frac{2}{4}\int_0^{\pi/4} d\theta\ \cancel{\cos^2\theta}\ \frac{\cos^2(2\theta)}{\cos^{\cancel{4}\,2}\theta} =$$

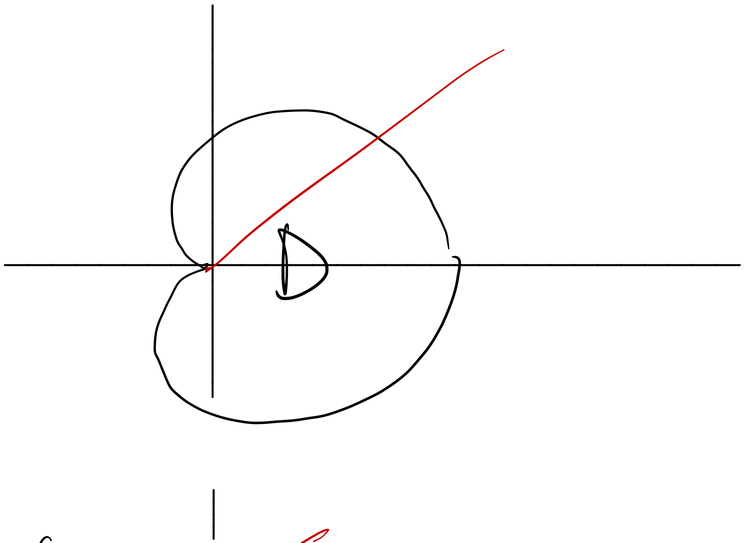
$[\cos(2\theta) = 2\cos^2\theta - 1]$

$[\cos^2(2\theta) = 4\cos^4\theta + 1 - 4\cos^2\theta]$

$$= \frac{1}{2}\int_0^{\pi/4} d\theta \left[4\cos^2\theta + \frac{1}{\cos^2\theta} - 4 \right] = \text{immediato.}$$

Area della regione di piano contenuta nella cardioide

$$\rho = 1 + \cos\theta$$



$$\text{Area } D = \iint_D dx\,dy =$$

$$= 2\int_0^{\pi} d\theta \int_0^{1+\cos\theta} d\rho \; \rho =$$

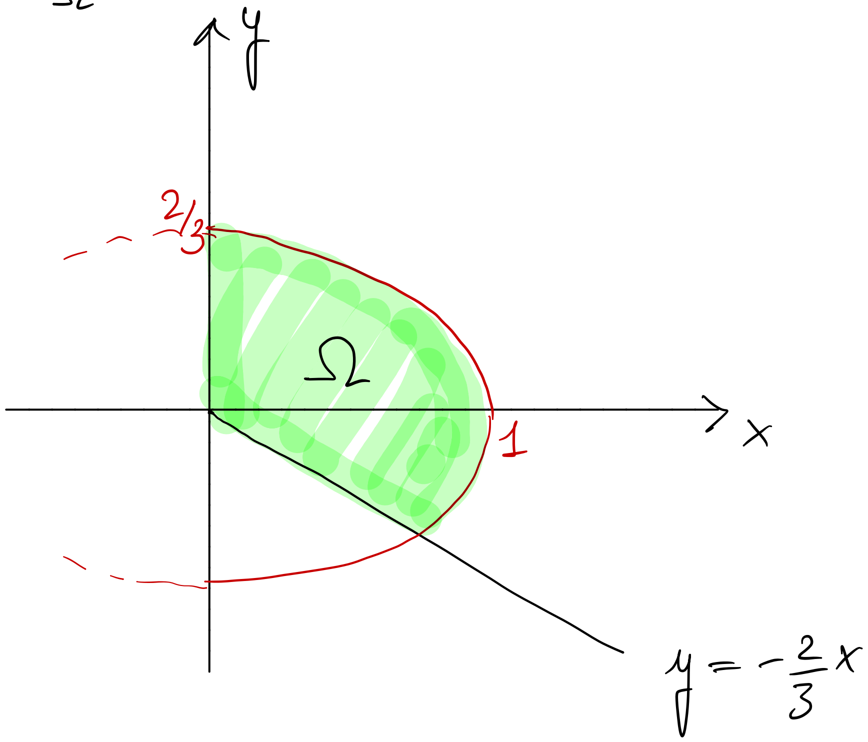
$$= \int_0^{\pi} d\theta \, (1+\cos\theta)^2 = \int_0^{\pi} d\theta \left(1 + \cancel{2\cos\theta} + \cos^2\theta\right) =$$

$$= \pi + \frac{\pi}{2} = \frac{3}{2}\pi$$

$$\iint_\Omega \sqrt{4x^2+9y^2}\,dx\,dy \qquad \Omega=\{(x,y):\ x\geq 0,\ y\geq -\tfrac{2}{3}x,\ \underbrace{4x^2+9y^2\leq 4}\}.$$

$$x^2+\frac{9}{4}y^2\leq 1$$

$$x^2+\left(\frac{y}{2/3}\right)^2\leq 1$$



Coordinate "polari ellittiche".

$$\boxed{\begin{cases} x=\rho\cos t \\ y=\frac{2}{3}\rho\,\mathrm{sen}\,t \end{cases}}$$

$x^2+\frac{9}{4}y^2\leq 1$ diventa

$$\rho^2\cos^2 t+\cancel{\frac{9}{4}}\,\cancel{\frac{4}{9}}\rho^2\,\mathrm{sen}^2 t\leq 1$$

$$\rho^2\leq 1$$

$$0\leq\rho\leq 1$$

Il semiasse positivo delle y diventa $t=\pi/2$.

La retta $y=-\frac{2}{3}x$

$$\cancel{\frac{2}{3}}\cancel{\rho}\,\mathrm{sen}\,t=-\cancel{\frac{2}{3}}\cancel{\rho}\cos t \qquad \mathrm{tg}\,t=-1,\quad t=-\frac{\pi}{4}$$

Con il cambio di variabile indicato, Ω diventa

$$\tilde{\Omega}=\left\{(\rho,t):\ -\frac{\pi}{4}\leq t\leq\frac{\pi}{2},\ 0\leq\rho\leq 1\right\}$$

$$\iint_{\Omega} \sqrt{4x^2+9y^2}\, dx\, dy = \iint_{\tilde{\Omega}} d\rho\, d\vartheta \;\; 2\rho \cdot \frac{2}{3}\rho = (*)$$

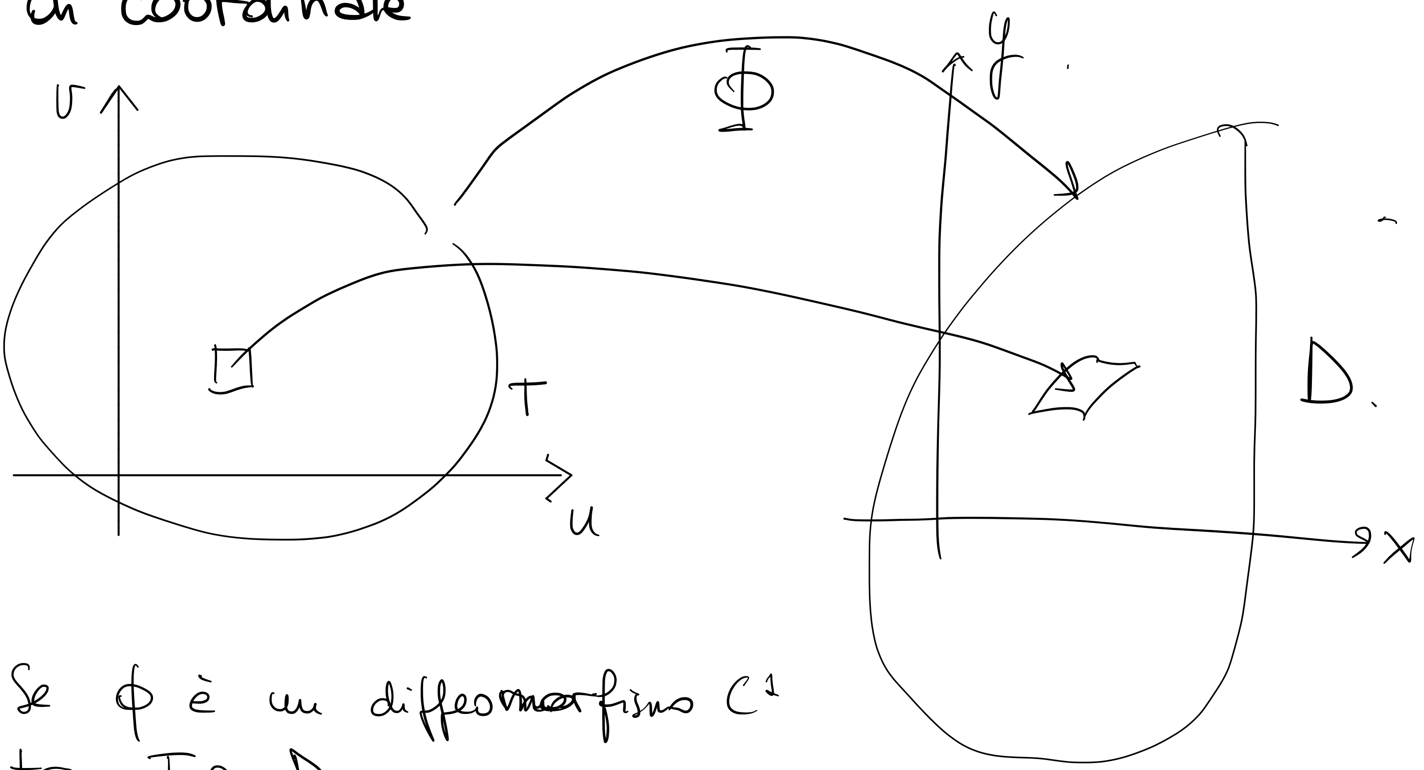
$$4x^2+9y^2 = 4\rho^2\cos^2 t + \cancel{9}\cdot\frac{4}{\cancel{9}}\rho^2 \operatorname{sen}^2 t = 4\rho^2$$

$$\boxed{\begin{cases} x = \rho\cos t \\ y = \frac{2}{3}\rho \operatorname{sen} t \end{cases}}$$

$$\det \frac{\partial(x,y)}{\partial(\rho,t)} = \det\begin{pmatrix} \cos t & -\rho \operatorname{sen} t \\ \frac{2}{3}\operatorname{sen} t & \frac{2}{3}\rho\cos t \end{pmatrix} = \frac{2}{3}\rho$$

$$(*) = \frac{4}{3}\int_{-\pi/4}^{\pi/2} dt \int_0^1 d\rho\; \rho^2 = \frac{4}{3}\cdot\frac{3}{4}\pi\,\frac{1}{3} = \frac{\pi}{3}$$

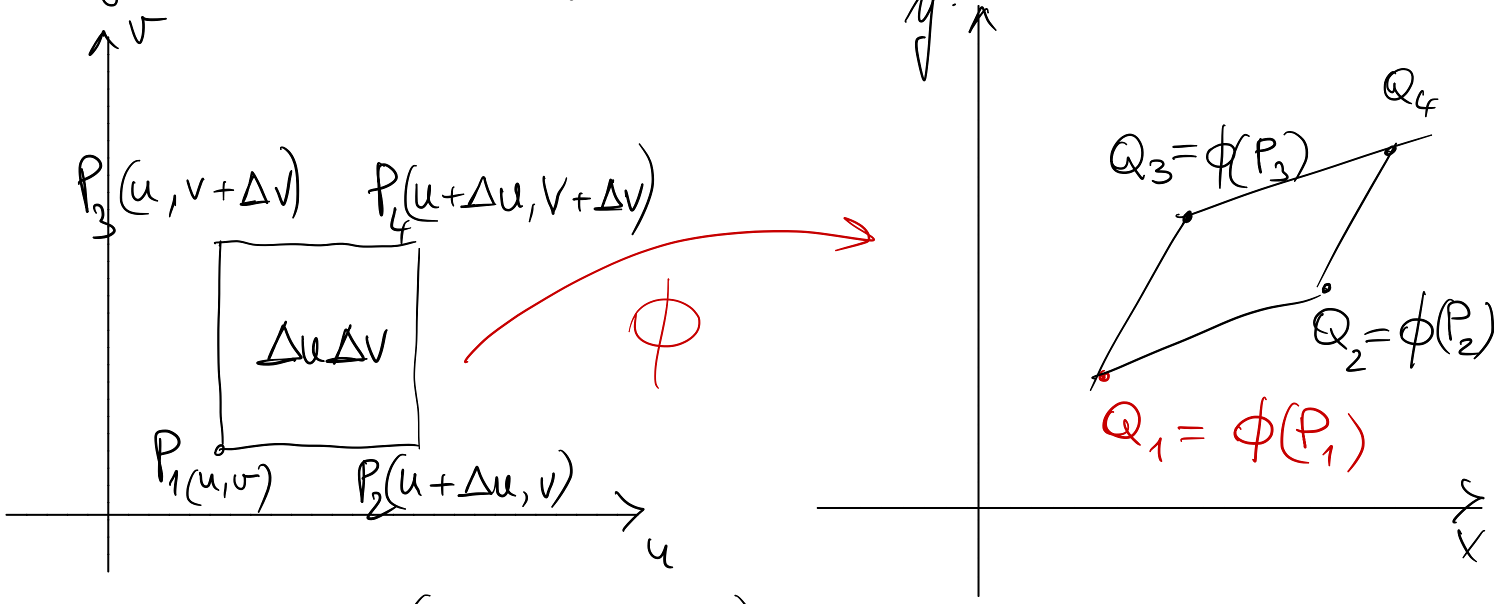
Giustificazione della formula di cambiamento di coordinate



Se ϕ è un diffeomorfismo C^1 tra T e D.

$$\iint_D f(x,y)\,dx\,dy = \iint_T f(\phi(u,v)) \underbrace{\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{giustificare questo}} du\,dv.$$

Ingrandiamo il disegno



$$Q_1 = \phi(P_1) = (x(u,v), y(u,v))$$

$$Q_2 = \Phi(P_2) = (x(u+\Delta u, v), y(u+\Delta u, v))$$

se Δu è piccolo,

$$x(u+\Delta u, v) = x(u,v) + x_u(u,v)\Delta u + o(|\Delta u|) \cong$$

$$\cong x(u,v) + x_u(u,v)\Delta u$$

$$Q_2 \cong (x(u,v) + x_u(u,v)\Delta u, y(u,v) + y_u(u,v)\Delta u) =$$

$$Q_1 + (x_u(u,v), y_u(u,v))\Delta u$$

Analogamente

$$Q_3 = \Phi(P_3) \cong Q_1 + (x_v(u,v), y_v(u,v))\Delta v.$$

$$Q_4 = \Phi(P_4) = (x(u+\Delta u, v+\Delta v), y(u+\Delta u, v+\Delta v)) = \ldots$$

$$x(u+\Delta u, v+\Delta v) = x(u,v) + x_u(u,v)\Delta u + x_v(u,v)\Delta v + \boxed{+\, o\left(\sqrt{(\Delta u)^2 + (\Delta v)^2}\right)}$$

$$\cong x(u,v) + x_u(u,v)\Delta u + x_v(u,v)\Delta v$$

$$Q_4 \cong Q_1 + \left(x_u(u,v), y_u(u,v)\right)\Delta u + \left(x_v(u,v), y_v(u,v)\right)\Delta v.$$