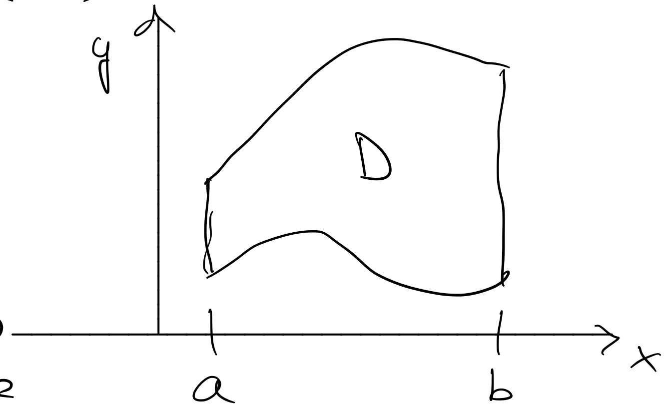
CAMBIO DI VARIABILI PER INTEGRALI DOPPI

Def Dominio normale regolare è un dominio della forma

$$D = \{(x,y): \ a \leq x \leq b, \quad \alpha(x) \leq y \leq \beta(x)\}$$

dove $\alpha(x), \beta(x)$ sono funzioni di classe $C^1[a,b]$ e t.c. $\alpha(x) < \beta(x) \quad \forall x \in (a,b)$.

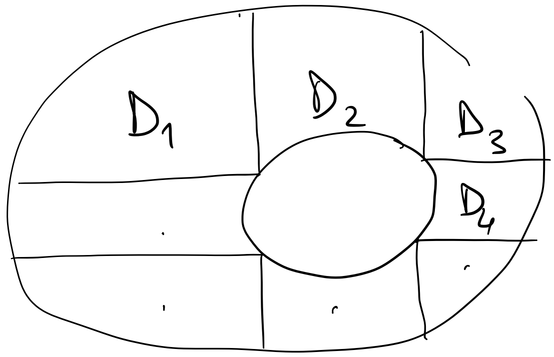
(oppure a variabili scambiate).



DEF. Dominio regolare

E' l'unione di un numero finito di domini normali regolari a due a due privi di pti interni in comune.

<u>OSS</u> Un dominio regolare ha la frontiera costituita da un numero finito di curve regolari.



TEOREMA (Cambiamento di variabili negli $\iint$ doppi).

Siano T, D due domini regolari di $\mathbb{R}^2$, e sia

$$\underline{\phi} : T \longrightarrow D$$
$$(u,v) \longmapsto \underline{\Phi}(u,v) = (x(u,v), y(u,v))$$

una funzione t.c.

1) $\underline{\phi}$ biettiva tra T e D

2) $\underline{\phi} \in C^1(T; D)$

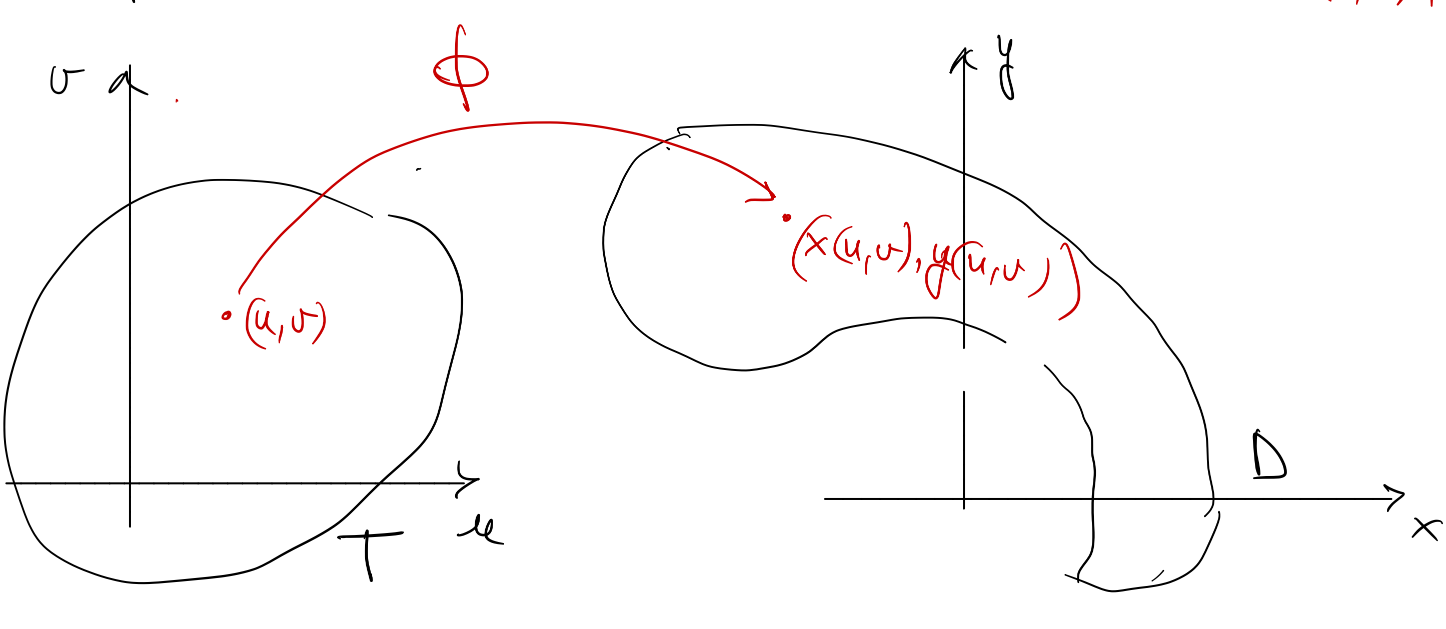
3) il det. jacobiano di ϕ $\quad J(u,v) = \det \frac{\partial(x,y)}{\partial(u,v)}(u,v) =$

$$= \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \neq 0 \quad \forall (u,v) \in T.$$

OSS Una tale $\underline{\Phi}$ si dice diffeomorfismo C^1.

Allora, $\forall$ funzione $f(x,y)$ continua in D si ha.

$$\iint_{\substack{D \\ \shortparallel \\ \phi(T)}} f(x,y)\,dx\,dy = \iint_T \underbrace{f(\underline{\phi}(u,v))}_{\substack{\shortparallel \\ f(x(u,v),\, y(u,v))}} \underbrace{|J(u,v)|}_{\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right|} du\,dv$$



<u>OSS</u> Si può dim. che se $\phi : T \to D$ è un diffeomorf. C^1, anche $\phi^{-1} : D \to T$ è un diffeomorfismo C^1.

$$D(\phi^{-1})(x,y) = [D\Phi(u,v)]^{-1}$$

dove $(u,v) = \phi^{-1}(x,y)$

matrice inversa

in particolare per i determinanti si ha

$$J_{\phi^{-1}}(x,y) = \frac{1}{J_\phi(u,v)\Big|_{(u,v)=\phi^{-1}(x,y)}}$$

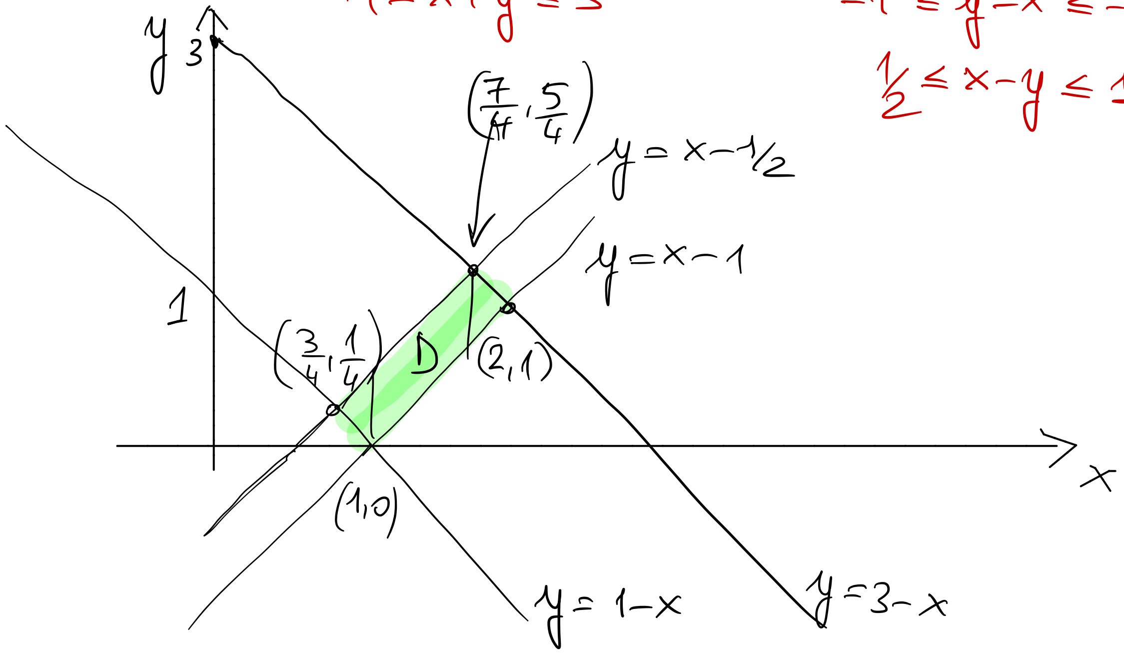
Esercizio 1.

$$\iint_D (x+y)\log(x-y)\,dx\,dy.$$

$D = \{(x,y) : \quad 1-x \le y \le 3-x \,; \quad x-1 \le y \le x-\frac{1}{2}\}.$

$1 \le x+y \le 3$ $\qquad -1 \le y-x \le -1/2$

$1/2 \le x-y \le 1.$



In coordinate standard (x,y) sarebbe difficile.

Viene naturale porre $\quad u = x+y, \quad v = x-y.$

Il dominio D si trasforma nel dominio $\tilde{D} = [1,3] \times \left[\frac{1}{2}, 1\right]$

L'integrale sembrerebbe trasformarsi in

$$\iint_{\tilde{D}} u \log v \,(?)\, du\, dv$$

Pb: $\quad dx\,dy = (??)\; du\,dv$

cosa devo mettere qui?

Poniamo
$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$
Sarebbe ϕ^{-1}.

$$\phi: \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases} \qquad \begin{cases} x(u,v) = \dfrac{u+v}{2} \\ y(u,v) = \dfrac{u-v}{2} \end{cases}$$

ϕ e ϕ^{-1} sono biettive da $\mathbb{R}^2$ a $\mathbb{R}^2$

$$\phi : T \longrightarrow D$$

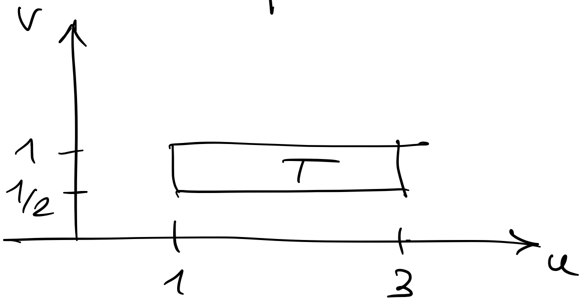
dove D è quello dato nel testo e
$$T = [1,3] \times \left[\tfrac{1}{2}, 1\right].$$

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$$

Teorema $\Longrightarrow$

$$\iint_D (x+y)\log(x-y)\,dx\,dy = \iint_T u \log v \,\frac{1}{2}\,du\,dv =$$

$$T = [1,3] \times \left[\tfrac{1}{2}, 1\right] =$$



$$= \frac{1}{2}\int_1^3 du \int_{1/2}^1 dv \; u \log v = \frac{1}{2} \; \frac{u^2}{2}\Big|_1^3$$

$$= \frac{1}{2}\int_1^3 du \int_{1/2}^1 dv\, u \log v = \frac{1}{2} \quad \frac{u^2}{2}\Bigg|_1^3 \; v(\log v - 1)\Bigg|_{v=1/2}^{v=1} = (*)$$

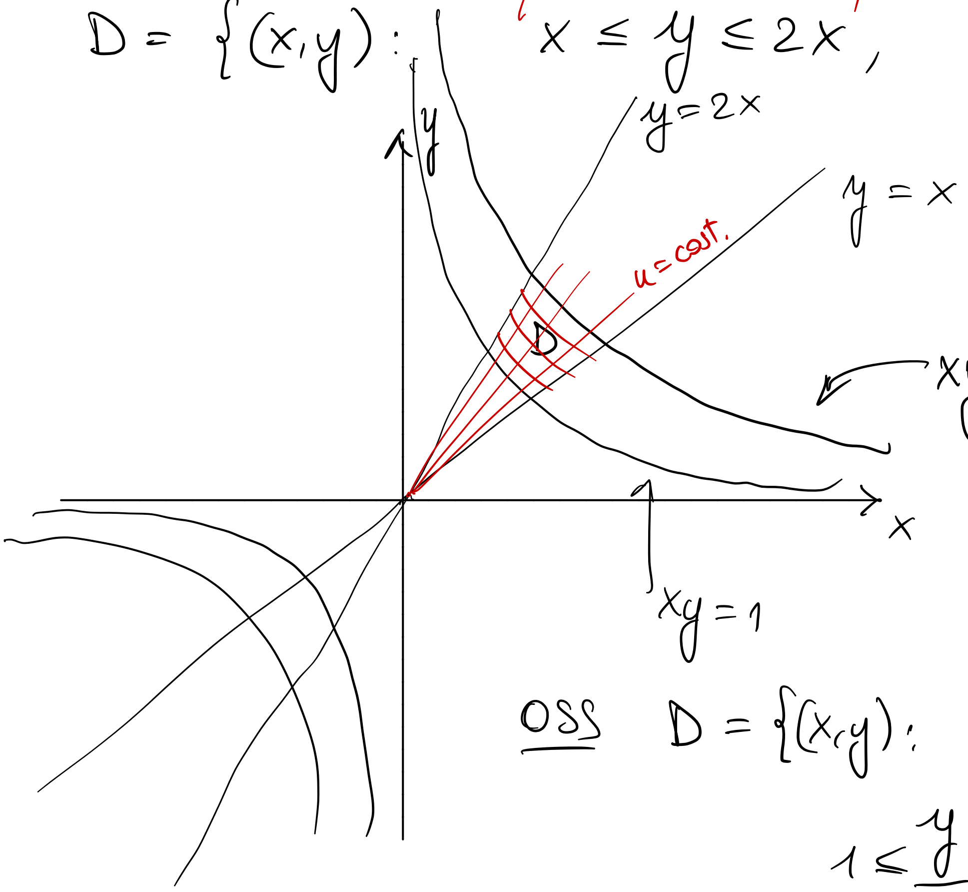
$$\int \log v \, dv = v \log v - \int \frac{v}{v}\, dv = v(\log v - 1) + c$$

$$(*) = \frac{1}{2}\cdot\frac{1}{2}(9-1)\left[1\cdot(-1) - \frac{1}{2}\left(-\log 2 - 1\right)\right]$$

ESERCIZIO $\displaystyle\iint_D \frac{x^3}{y} \operatorname{sen}(xy)\, dx\,dy$

$\Rightarrow x \geq 0$

$D = \{(x,y) : \overbrace{x \leq y \leq 2x}, \quad 1 \leq xy \leq 2\}$

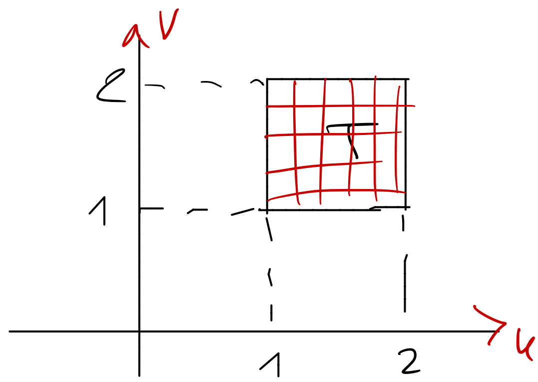


OSS $D = \{(x,y) : x > 0, y > 0$

$1 \leq \frac{y}{x} \leq 2, \quad 1 \leq xy \leq 2\}$

Viene naturale porre $\begin{cases} u = \frac{y}{x} \\ v = xy \end{cases}$ $(*)$

$(u,v) \in T = [1,2]^2$



Invertiamo $(*)$

$y^2 = uv \Rightarrow y = \sqrt{uv}$

$x = \frac{v}{y} = \frac{v}{\sqrt{uv}} = \sqrt{\frac{v}{u}}$

$\Phi \begin{cases} x = \sqrt{\frac{v}{u}} \\ y = \sqrt{uv} \end{cases}$

$\Phi : T \longrightarrow D$

$$\Phi \begin{cases} x = \sqrt{\dfrac{v}{u}} \\ y = \sqrt{uv} \end{cases} \qquad x_u = -\frac{1}{2}\frac{\sqrt{v}}{u\sqrt{u}} \qquad x_v = \frac{1}{2\sqrt{u}\sqrt{v}}$$

$$y_u = \frac{\sqrt{v}}{2\sqrt{u}} \qquad y_v = \frac{\sqrt{u}}{2\sqrt{v}}$$

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} -\frac{1}{2}\frac{\sqrt{v}}{u\sqrt{u}} & \frac{1}{2}\frac{1}{\sqrt{u}\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{pmatrix} = \frac{1}{4}\left(-\frac{1}{u} - \frac{1}{u}\right) = -\frac{1}{2u}$$

$$\iint_D \frac{x^3}{y} \underbrace{\operatorname{sen}(xy)}_{\operatorname{sen} v} \,dx\,dy = \iint_T \frac{v}{u^2} \operatorname{sen} v \,\frac{1}{2u}\,du\,dv =$$

$$\frac{x^3}{y} = \frac{x^2}{y^2}\,xy = \frac{v}{u^2}$$

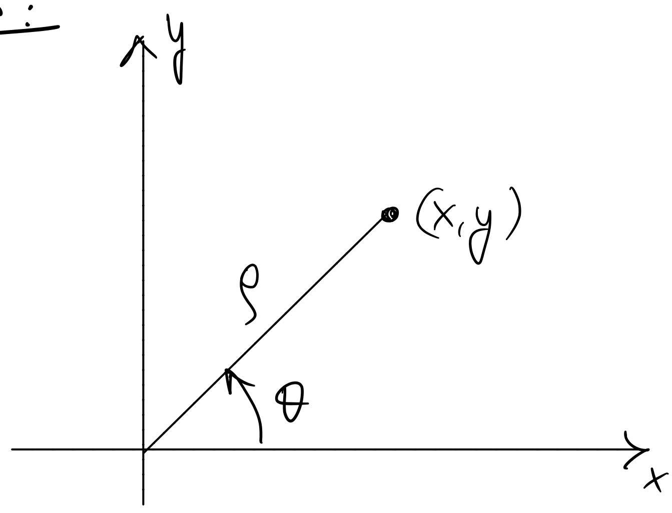
$$= \frac{1}{2}\int_1^2 du \int_1^2 dv \,\frac{v \operatorname{sen} v}{u^3} = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{u^2}\Big|_2^1 \Big(-v\cos v + \operatorname{sen} v\Big)\Big|_1^2 =$$

$$\boxed{\int v \operatorname{sen} v\,dv = -v\cos v + \int \cos v\,dv = -v\cos v + \operatorname{sen} v}$$

$$= \frac{1}{4}\left(1 - \frac{1}{4}\right)\left(-2\cos 2 + \operatorname{sen} 2 + \cos 1 - \operatorname{sen} 1\right)$$

Passaggio a coordinate polari:

u, v si chiamano ρ, θ.



$$\Phi : \begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$$

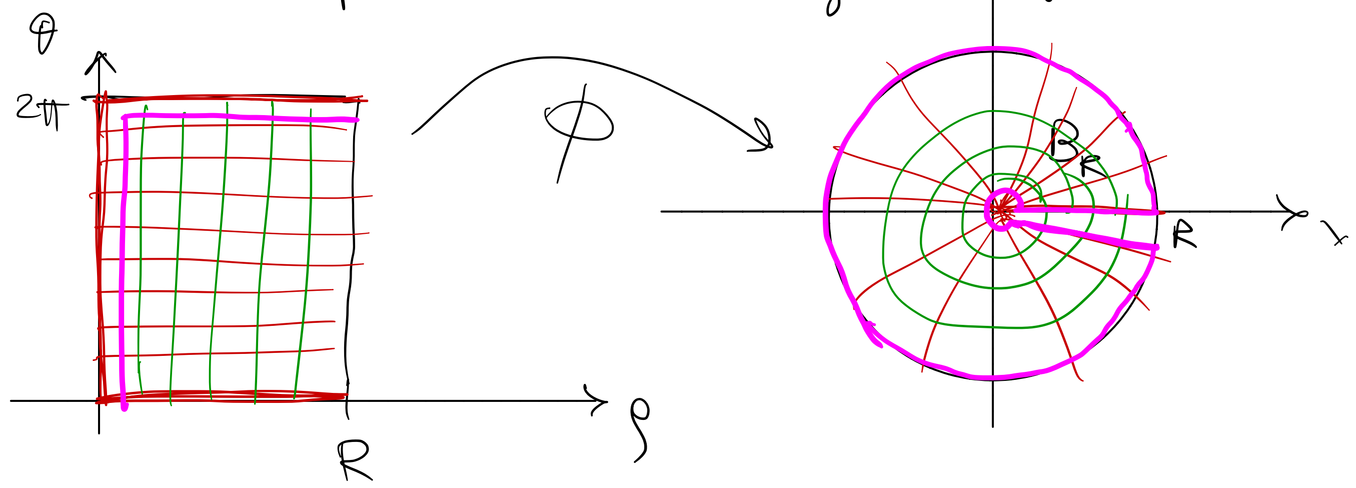
(x,y) variano in $\mathbb{R}^2$

(ρ,θ) in $[0,+\infty) \times [0,2\pi]$

Ad esempio se volessi fare un integrale sul cerchio di centro l'origine e raggio R

$$B_R = \{(x,y) : x^2+y^2 \leq R^2\}$$

In coordinate polari "diventa" un rettangolo



OSS Questa trasformazione ha dei piccoli pb. di biettività

I punti $(\rho,0)$ e $(\rho,2\pi)$ vanno a finire nelle stesse (x,y)

Tutti i punti $(0,\theta)$ vanno " " nell'origine.

Non si applica il teorema così com'è.

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}$$

$$\det \frac{\partial(x,y)}{\partial(\rho,\theta)} = \det \begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \rho\cos\theta \end{pmatrix} =$$

$$= \rho\cos^2\theta + \rho\,\mathrm{sen}^2\theta = \rho .$$

TEOREMA Siano T, D due domini regolari risp di $[0,+\infty)\times[0,2\pi]$ e $\mathbb{R}^2$ t.c. la trasformazione $\Phi(\rho,\theta) = (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$ verifichi $\Phi(T) = D$.

Allora, $\forall$ funzione $f(x,y)$ continua in D si ha

$$\iint_D f(x,y)\,dx\,dy = \iint_T f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\,\rho\,d\rho\,d\theta .$$

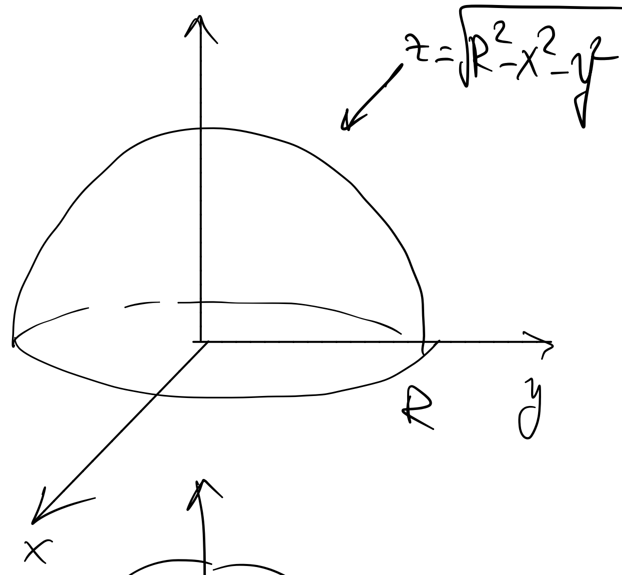
ESEMPIO

$$B_R = \{(x,y,z) : x^2+y^2+z^2 \leq R^2\}$$

Volume della palla

$$\mathrm{Vol}(B_R) = 2 \iint_{C_R} \sqrt{R^2-x^2-y^2}\, dx\,dy = (*)$$

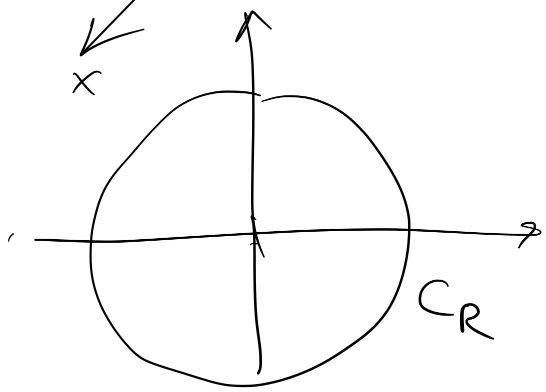
dove $C_R = \{(x,y) : x^2+y^2 \leq R^2\}$



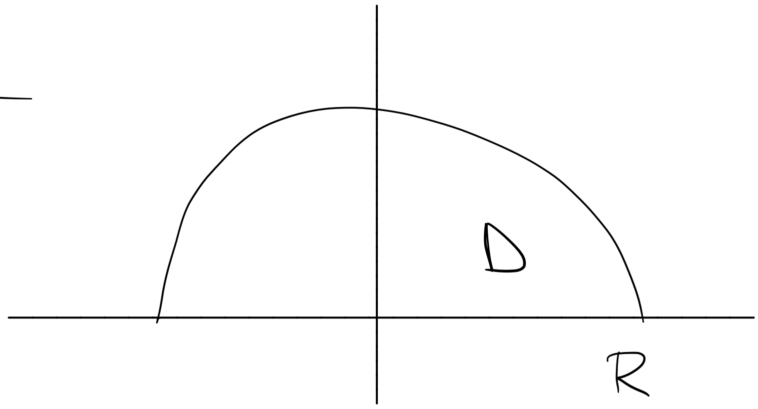
$$(*) = 2 \int_0^{2\pi} d\theta \int_0^R d\rho \sqrt{R^2-\rho^2}\, \rho =$$

$$= 2\pi \cdot \frac{2}{3} (R^2-\rho^2)^{3/2} \Big|_{\rho=R}^{\rho=0} =$$

$$= \frac{4\pi}{3} R^3$$



Baricentro del semicerchio



$$y_B = \frac{2}{\pi R^2} \iint_D y \, dx\,dy =$$

$$= \frac{2}{\pi R^2} \int_0^{\pi} d\theta \int_0^{R} d\rho \;\; \rho^2 \operatorname{sen}\theta =$$

$$= \frac{2}{\pi R^2} \cdot 2 \; \frac{R^3}{3} = \frac{4}{3\pi} R .$$