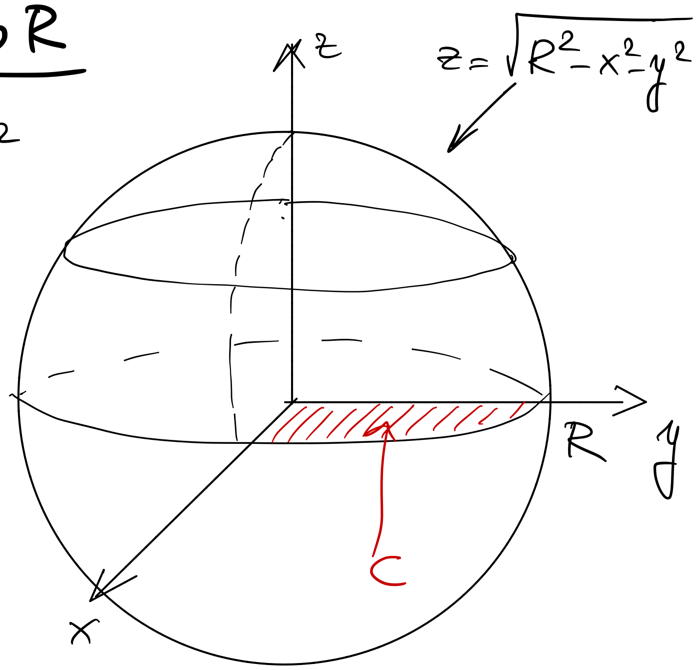
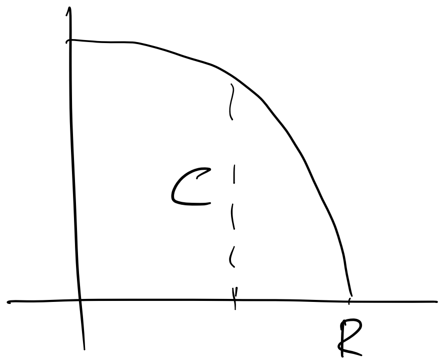
Volume di una palla di raggio R

$$B_R = \{ (x,y,z) : x^2+y^2+z^2 \le R^2 \}$$

$$\text{Vol } B_R = 8 \iint_C \sqrt{R^2-x^2-y^2}\, dx\, dy = (*)$$





$$C = \{ (x,y) : x \ge 0,\ y \ge 0,\ x^2+y^2 \le R^2 \} =$$

$$= \{ (x,y) : 0 \le x \le R,\ 0 \le y \le \sqrt{R^2-x^2} \}$$

$$(*) = 8 \int_0^R dx \left(\int_0^{\sqrt{R^2-x^2}} dy \sqrt{R^2-x^2-y^2} \right)$$

$$\int dy\ 1 \sqrt{a^2-y^2} = \qquad\qquad a^2 = R^2-x^2$$

$$= y\sqrt{a^2-y^2} + \int \frac{(y^{2}\cancel{2} - a^2) + a^2}{\cancel{2}\sqrt{a^2-y^2}}\, dy =$$

$$= \text{"} \quad - \int \sqrt{a^2-y^2} + \int \frac{a^2}{\sqrt{a^2-y^2}}$$

$$\int dy \sqrt{a^2-y^2} = \frac{1}{2} y \sqrt{a^2-y^2} + \frac{1}{2} \int \frac{a^2}{\sqrt{a^2-y^2}}\, dy .$$

$$\int dy\,\sqrt{a^2-y^2} = \frac{1}{2}\,y\sqrt{a^2-y^2} + \frac{1}{2}\int \frac{a^2}{\sqrt{a^2-y^2}}\,dy =$$

$$= \quad '' \quad + \frac{a}{2}\int \frac{dy}{\sqrt{1-\frac{y^2}{a^2}}}$$

$$\boxed{a=\sqrt{R^2-x^2}}$$

$$= \frac{1}{2}\,y\sqrt{a^2-y^2} + \frac{a^2}{2}\,\mathrm{arcsen}\left(\frac{y}{a}\right)$$

$$(*) = 8\int_0^R dx \left(\int_0^{\sqrt{R^2-x^2}} dy\, \sqrt{R^2-x^2-y^2} \right) =$$

$$= 4\int_0^R dx \left(\cancel{y\sqrt{R^2-x^2-y^2}} + (R^2-x^2)\,\mathrm{arcsen}\,\frac{y}{\sqrt{R^2-x^2}} \right)\Bigg|_{y=0}^{y=\sqrt{R^2-x^2}} =$$

$$= 4\int_0^R dx \left((R^2-x^2)\frac{\pi}{2} \right) = 2\pi\left(R^3-\frac{R^3}{3}\right) = 2\pi R^3\,\frac{2}{3} = \frac{4}{3}\pi R^3 \qquad \square$$

Modo alternativo.

$$\mathrm{Vol}(B_R) = \int_{-R}^{R} dz\;\mathrm{area}(C_z) = \int_{-R}^{R} dz\,\pi\,(R^2-z^2) = (*)$$

$$C_z = \{(x,y):\; x^2+y^2+z^2 \leq R^2\} = \{(x,y):\; x^2+y^2 \leq R^2-z^2\}$$

= cerchio di centro $(0,0)$ e raggio $\sqrt{R^2-z^2}$

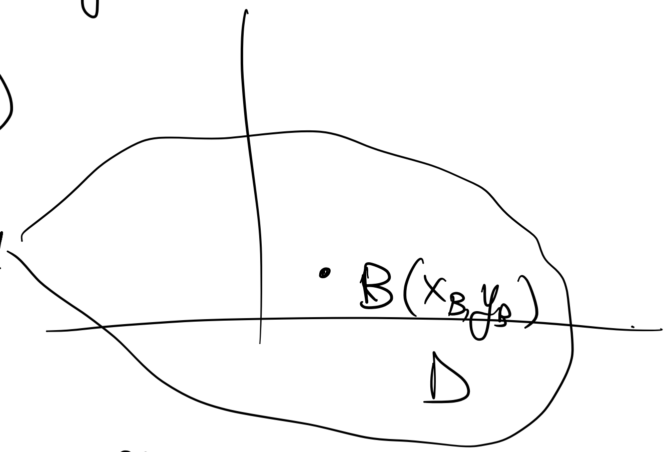
$$(*) = 2\pi\int_0^R dz\,(R^2-z^2) = 2\pi\left(R^3-\frac{R^3}{3}\right) = \frac{4\pi R^3}{3}$$

Baricentro di domini piani

D = dominio normale = lamina metallica piana con densità superficiale $\mu(x,y)$

Baricentro di D $\quad B(x_B, y_B)$

$$x_B = \frac{1}{\underbrace{\iint_D \mu(x,y)\,dx\,dy}_{\text{massa della lamina}}} \iint_D x\,\mu(x,y)\,dx\,dy$$

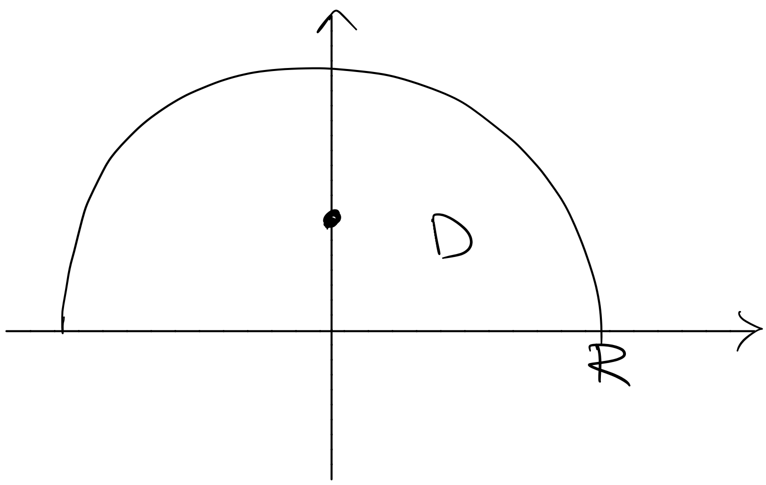


Se $\mu \equiv$ costante $\qquad x_B = \frac{1}{\text{area } D} \iint_D x\,dx\,dy.$

y_B = idem.

Baricentro di un semicerchio $(\mu = \text{costante})$

$$\text{area } D = \frac{\pi R^2}{2}$$



$$x_B = \frac{2}{\pi R^2} \iint_D x\,dx\,dy = 0$$

$$y_B = \frac{2}{\pi R^2} \iint_D y\,dx\,dy =$$

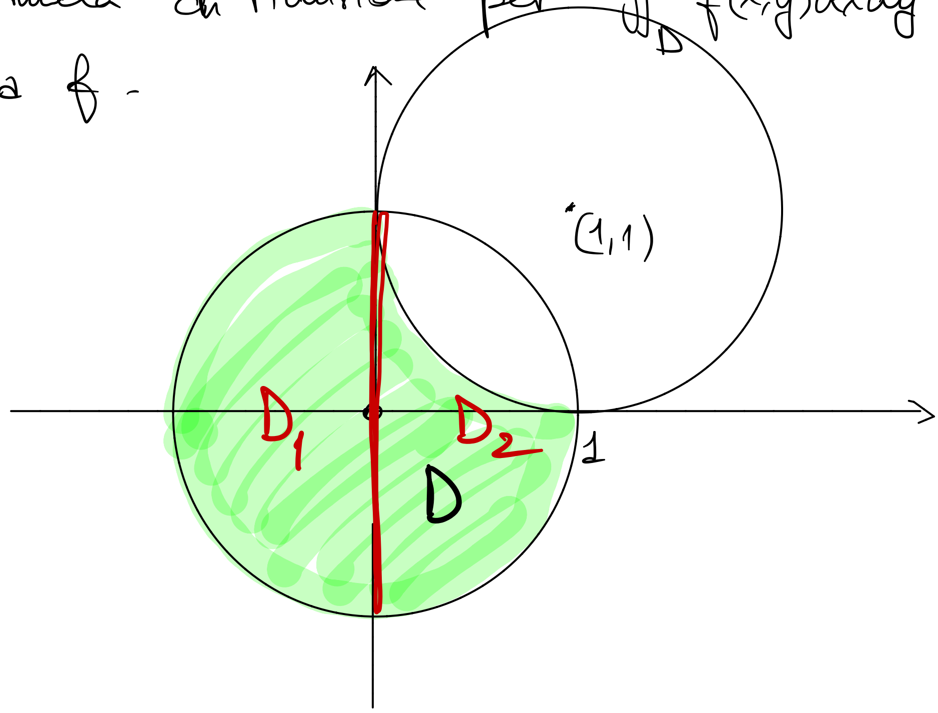
$$= \frac{2}{\pi R^2} \cdot 2 \cdot \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy\; y = \frac{2}{\pi R^2} \int_0^R dx\,(R^2 - x^2) =$$

$$= \frac{2}{\pi R^2}\left(R^3 - \frac{R^3}{3}\right) = \frac{2R^{\cancel{3}}}{\pi \cancel{R^2}}\,\frac{2}{3} = \frac{4R}{3\pi}$$

ESERCIZIO Disegnare l'insieme

$$D = \{ (x,y) \in \mathbb{R}^2 : \ x^2+y^2 \le 1, \ (x-1)^2+(y-1)^2 \ge 1 \}$$

e scrivere una formula di riduzione per $\iint_D f(x,y)\,dx\,dy$ $\forall$ funzione continua f.



$$\iint_D f(x,y)\,dx\,dy =$$

$$= \iint_{D_1} \dots + \iint_{D_2} \dots = \int_{-1}^{0} dx \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \ f(x,y) \right) + \int_{0}^{1} dx \left(\int_{-\sqrt{1-x^2}}^{1-\sqrt{2x-x^2}} dy \ f(x,y) \right)$$

$$(x-1)^2+(y-1)^2 = 1$$

$$x^2-2x+1+y^2-2y+\cancel{1} = \cancel{1}$$

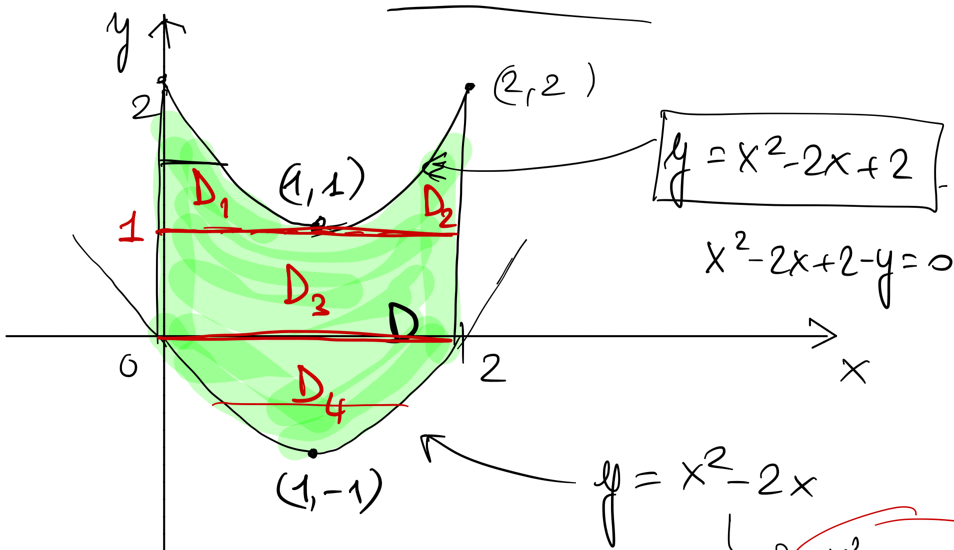
$$y^2-2y+x^2-2x+1 = 0$$

$$y = 1 \pm \sqrt{\cancel{1}-x^2+2x-\cancel{1}} = 1 \pm \sqrt{2x-x^2}$$

ESERCIZIO Disegnare l'insieme D t.c.

$$\iint_D f(x,y)\,dx\,dy = \int_0^2 \left(\int_{x^2-2x}^{x^2-2x+2} f(x,y)\,dy \right) dx$$

$\forall f$ funzione continua, e scrivere la formula per invertire l'ordine di integrazione delle variabili.



$x^2-2x+2-y=0$

$x = 1 \pm \sqrt{1-2+y}$

$x = 1 \pm \sqrt{y-1}$

$x^2-2x-y=0$

Invertiamo l'ordine di integrazione

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} \dots + \iint_{D_2} \dots + \iint_{D_3} \dots + \iint_{D_4} \dots =$$

$$\boxed{x = 1 \pm \sqrt{1+y}}$$

$$= \int_1^2 dy \left(\int_0^{1-\sqrt{y-1}} dx\, f(x,y) \right) + \int_1^2 dy \left(\int_{1+\sqrt{y-1}}^2 dx\, f(x,y) \right) +$$

$$+ \int_0^1 dy \left(\int_0^2 dx\, f(x,y) \right) + \int_{-1}^0 dy \left(\int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dx\, f(x,y) \right)$$

Cambiamenti di variabile

Caso 1-dimensionale:

$$\int_a^b f(g(t))\,g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx$$

$$\boxed{\begin{aligned} g(t) &= x \\ g'(t)\,dt &= dx \end{aligned}}$$

f continua nell'intervallo di estremi $g(a)$, $g(b)$

$g \in C^1([a,b])$

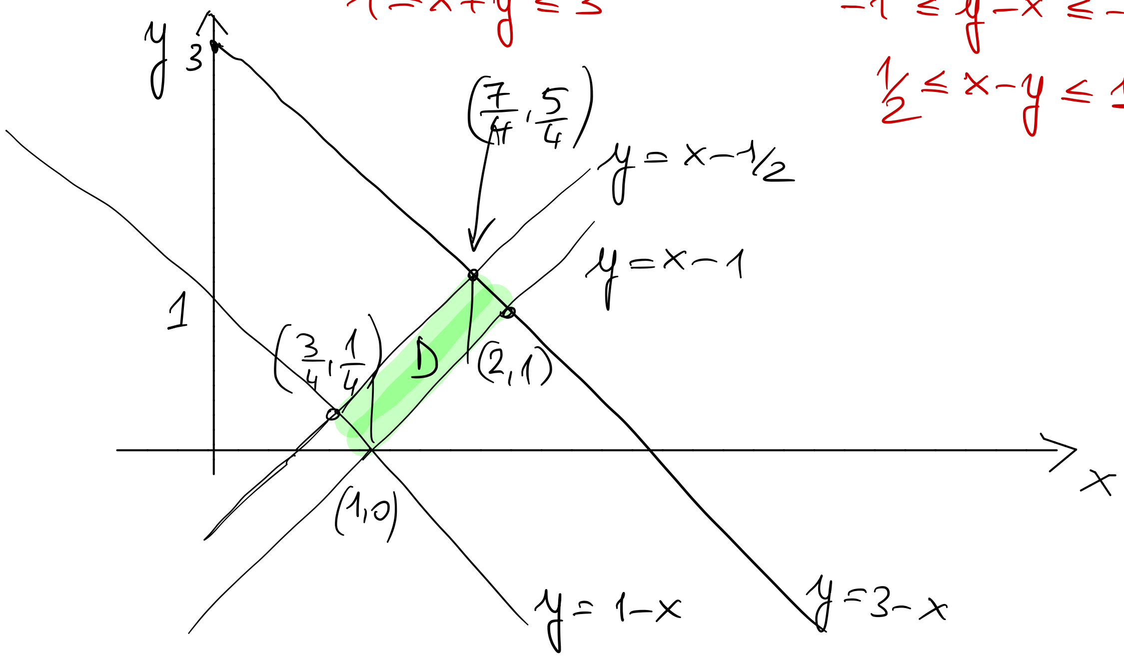
Esercizio 1.

$$\iint_D (x+y)\log(x-y)\,dxdy.$$

$$D = \left\{ (x,y) : \quad 1-x \leq y \leq 3-x\ ; \quad x-1 \leq y \leq x-\tfrac{1}{2} \right\}.$$

$$1 \leq x+y \leq 3 \qquad -1 \leq y-x \leq -\tfrac{1}{2}$$

$$\tfrac{1}{2} \leq x-y \leq 1.$$



In coordinate standard (x,y) sarebbe difficile.

Viene naturale porre $u=x+y$, $v=x-y$.

Il dominio D si trasforma nel dominio $\tilde{D} = [1,3]\times\left[\frac{1}{2},1\right]$

L'integrale sembrerebbe trasformarsi in

$$\iint_{\tilde{D}} u \log v\ (?)\ du\,dv$$

Pb: $dxdy = (??)\ du\,dv$

↑ cosa devo mettere qui?