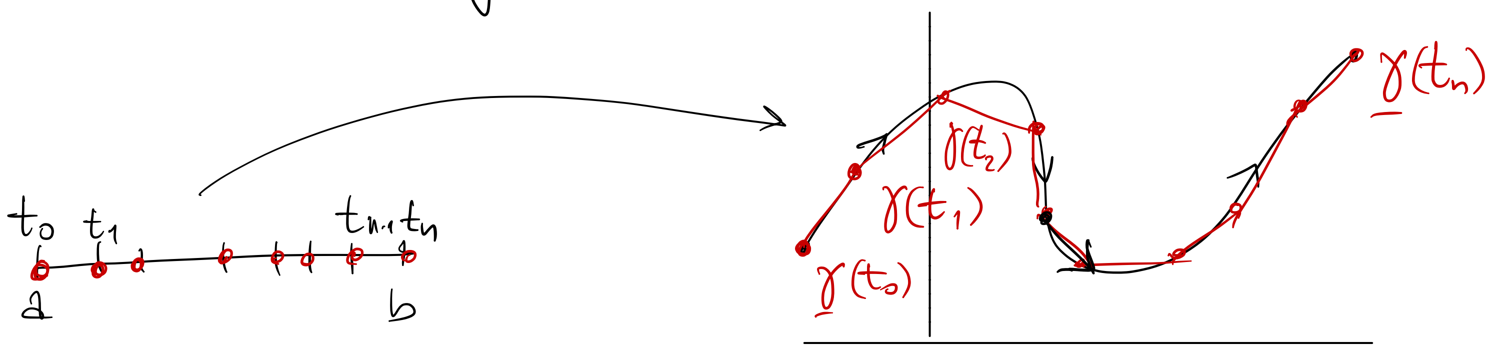
Lunghezza di una curva.

$$\underline{\gamma}: [a,b] \longrightarrow \mathbb{R}^2 \quad (\mathbb{R}^3, \mathbb{R}^N)$$

$$t \longmapsto \underline{\gamma}(t) = (x(t), y(t)).$$

$$\gamma \in C^1([a,b])$$

$$\|\underline{\gamma}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2} \neq 0 \quad \forall t \in (a,b).$$



Sia $D = \{t_0, t_1, \dots t_n\}$, con $t_0 = a < t_1 < t_2 < \dots < t_n = b$ una partizione di $[a,b]$.

Considero $L_D(\gamma)$ la lunghezza della "spezzata" di estremi $\underline{\gamma}(t_i)$, $i = 0, \dots, n$

$$L_D(\underline{\gamma}) = \sum_{i=1}^{n} \|\underline{\gamma}(t_i) - \underline{\gamma}(t_{i-1})\|$$

è un' approssimazione per difetto della "lunghezza vera" della curva

E' naturale definire la lunghezza della curva

$$L(\gamma) = \sup_{D \text{ partiz. di } [a,b]} L_D(\underline{\gamma})$$

TEOREMA Se $\underline{\gamma} \in C^1([a,b];\mathbb{R}^N)$, allora

$L(\underline{\gamma}) < \infty$, e si ha

$$L(\underline{\gamma}) = \int_a^b \|\underline{\gamma}'(t)\|\, dt \overset{N=2}{=}$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

s.d.

ESEMPIO. Lunghezza di una circonferenza.

$$\underline{\gamma}(t) \begin{cases} x(t) = R\cos t \\ y(t) = R\,\mathrm{sen}\, t \end{cases} \qquad t \in [0,2\pi]$$

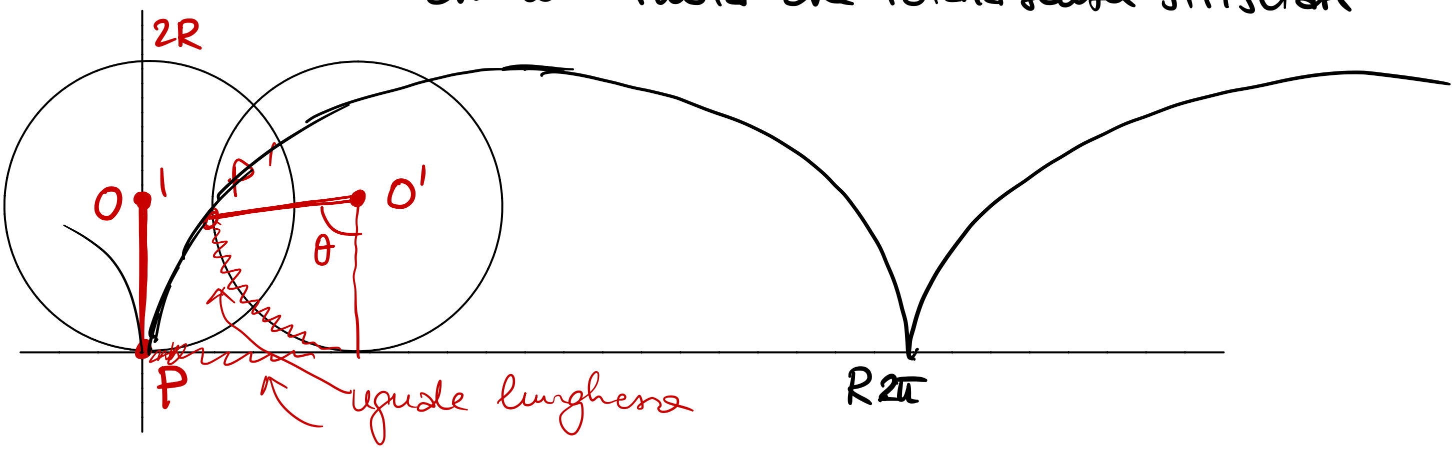
$R>0$

$$\begin{aligned} x'(t) &= -R\,\mathrm{sen}\, t \\ y'(t) &= R\cos t \end{aligned} \implies \|\underline{\gamma}'(t)\| = \sqrt{R^2\mathrm{sen}^2 t + R^2\cos^2 t} = R$$

$$L(\gamma) = \int_0^{2\pi} R\, dt = 2\pi R.$$

CICLOIDE

Curva percorsa da un punto del battistrada di una ruota che rotola senza strisciare



Dopo aver ruotato di un angolo θ, il mozzo si trova in posizione

$O'(R\theta, R)$

Le coordinate di P' sono $P'(R\theta - R\,\text{sen}\,\theta,\ R - R\cos\theta)$.

<u>cicloide</u> $\gamma(\theta) = \begin{cases} x(\theta) = R(\theta - \text{sen}\,\theta) \\ y(\theta) = R(1-\cos\theta) \end{cases}$ $\quad \theta \in [0, 2\pi]$

$\gamma'(\theta) \quad \begin{cases} x'(\theta) = R(1-\cos\theta) \\ y'(\theta) = R\,\text{sen}\,\theta \end{cases}$

$$\|\gamma'(\theta)\| = R\sqrt{(1-\cos\theta)^2 + (\text{sen}\,\theta)^2} = R\sqrt{2 - 2\cos\theta} = \sqrt{2}\,R\sqrt{1-\cos\theta}$$

$$L(\gamma) = \sqrt{2}\,R\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta = \sqrt{2}\,R\int_{-\pi}^{\pi}\sqrt{1-\cos\theta}\,d\theta =$$

f è 2π-periodica $\qquad$ f è pari

$$= 2\sqrt{2}\,R\int_0^{\pi}\sqrt{1-\cos\theta}\,d\theta = 2\sqrt{2}\,R\int_0^{\pi}\frac{\sqrt{1-\cos\theta}\,\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\,d\theta =$$

$$= 2\sqrt{2}\,R\int_0^{\pi}\frac{\sqrt{1-\cos^2\theta}}{\sqrt{1+\cos\theta}}\,d\theta = 2\sqrt{2}\,R\int_0^{\pi}\frac{\text{sen}\,\theta}{\sqrt{1+\cos\theta}}\,d\theta =$$

$\frac{1-\cos\theta}{2} = \text{sen}^2\frac{\theta}{2} \Longleftarrow$ altro modo di fare l'integrale.

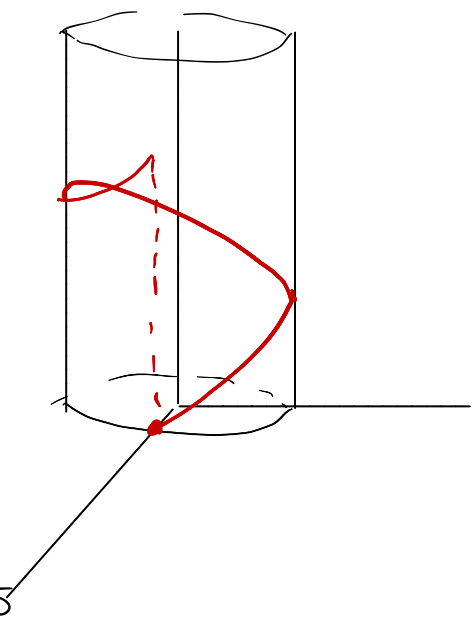
$$= 2\sqrt{2}R \int_0^{\pi} \frac{\operatorname{sen}\theta}{\sqrt{1+\cos\theta}}\, d\theta =$$

$1+\cos\theta = u$ $\quad -\operatorname{sen}\theta\, d\theta = du$

$$= 2\sqrt{2}R \int_0^2 \frac{du}{\sqrt{u}} = 2\sqrt{2}R \; 2\sqrt{u}\Big|_0^2 = 8R.$$

Lunghezza dell'elica cilindrica

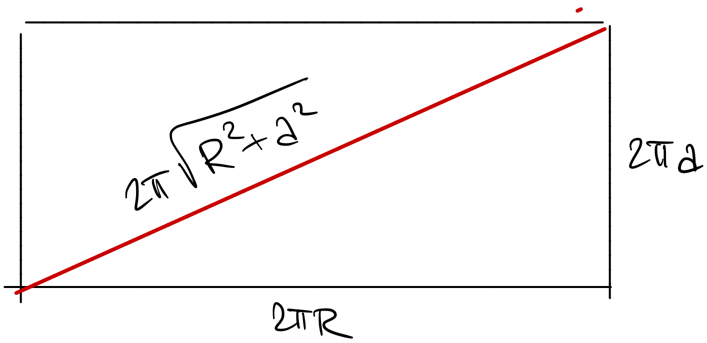
$$\gamma \quad \begin{cases} x(t) = R\cos t \\ y(t) = R\operatorname{sen} t \\ z(t) = a\,t \end{cases} \qquad t \in [0, 2\pi] \qquad a, R > 0.$$



$$\|\gamma'(t)\| = \sqrt{R^2\operatorname{sen}^2 t + R^2\cos^2 t + a^2} = \sqrt{R^2+a^2}$$

$$L(\gamma) = \int_0^{2\pi} dt \sqrt{R^2+a^2} = 2\pi\sqrt{R^2+a^2}$$

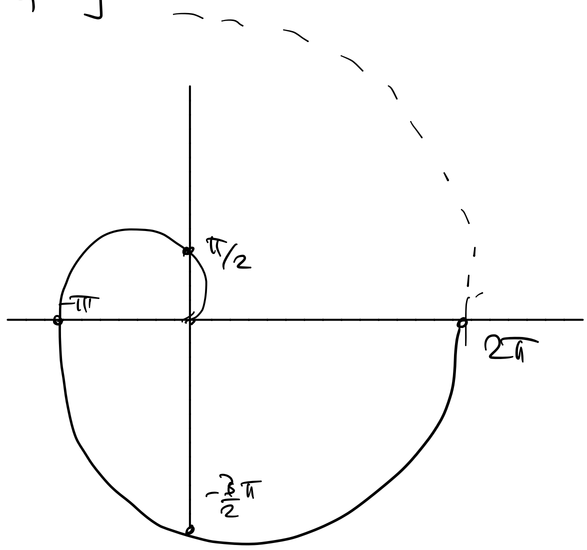
Infatti,
Srotolando il cilindro, si ottiene questo.



Curve in coordinate polari

$$\rho = \rho(\theta) \in C^1([a,b])$$

$$\begin{cases} x(\theta) = \rho(\theta)\cos\theta \\ y(\theta) = \rho(\theta)\,\mathrm{sen}\,\theta \end{cases} \qquad \theta \in [a,b]$$



$$\|\gamma'(\theta)\| = \sqrt{\rho(\theta)^2 + \rho'(\theta)^2}$$

1) $\rho(\theta) = \theta$ spirale di Archimede

$$L(\gamma) = \int_0^{2\pi} \sqrt{\theta^2+1}\, d\theta = (\text{per parti})$$

$$= \theta\sqrt{\theta^2+1}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{(\theta^2+1)-1}{\sqrt{\theta^2+1}}\, d\theta =$$

$$= 2\pi\sqrt{4\pi^2+1} - \underbrace{\int_0^{2\pi}\sqrt{\theta^2+1}\, d\theta}_{\text{a } 1^\circ \text{ membro}} + \int_0^{2\pi} \frac{1}{\sqrt{\theta^2+1}}\, d\theta$$

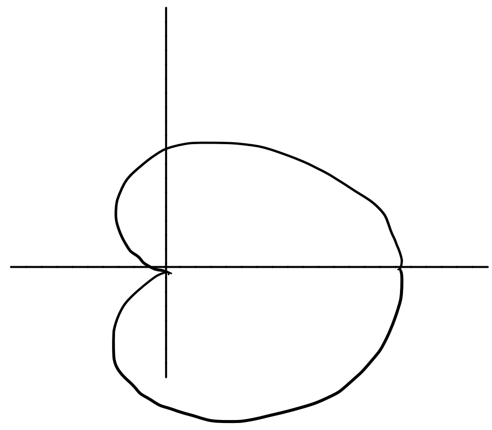
$$\Rightarrow L(\gamma) = \pi\sqrt{4\pi^2+1} + \frac{1}{2}\,\mathrm{settsh}\,\theta\Big|_0^{2\pi} =$$

$$= \text{"} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)$$

Esercizio spirale logaritmica $\rho = e^{\theta}$

Cardioide $\qquad \rho(\theta) = 1 + \cos\theta$

$$-\pi \le \theta \le \pi$$



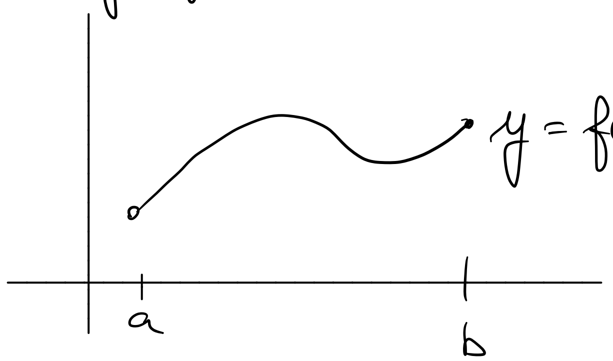
$$\rho'(\theta) = -\operatorname{sen}\theta$$

$$L(\gamma) = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + \operatorname{sen}^2\theta} =$$

$$= 2\int_0^{\pi} \sqrt{2}\,\sqrt{1+\cos\theta}\, d\theta = (\text{come prima}) = 8$$

Lunghezza di una curva grafico. $\qquad y = f(x) \quad x \in [a,b]$.

$f \in C^1([a,b])$



$$\gamma(t) \begin{cases} x = t \\ y = f(t) \end{cases}$$

$$\gamma'(t) \quad \begin{cases} x'(t) = 1 \\ y'(t) = f'(t) \end{cases}$$

$$\|\gamma'(t)\| = \sqrt{1 + f'(t)^2}$$

$y = x^2 \quad x \in [0, b]$ lunghezza di un arco di parabola.

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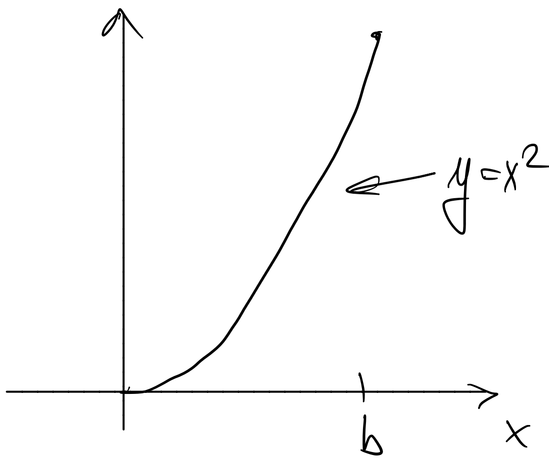


$$L(\gamma) = \int_0^b \sqrt{1+4x^2}\, dx = \left[2x = u\right]$$

$$= \frac{1}{2}\int_0^{b/2} \sqrt{1+u^2}\, du =$$

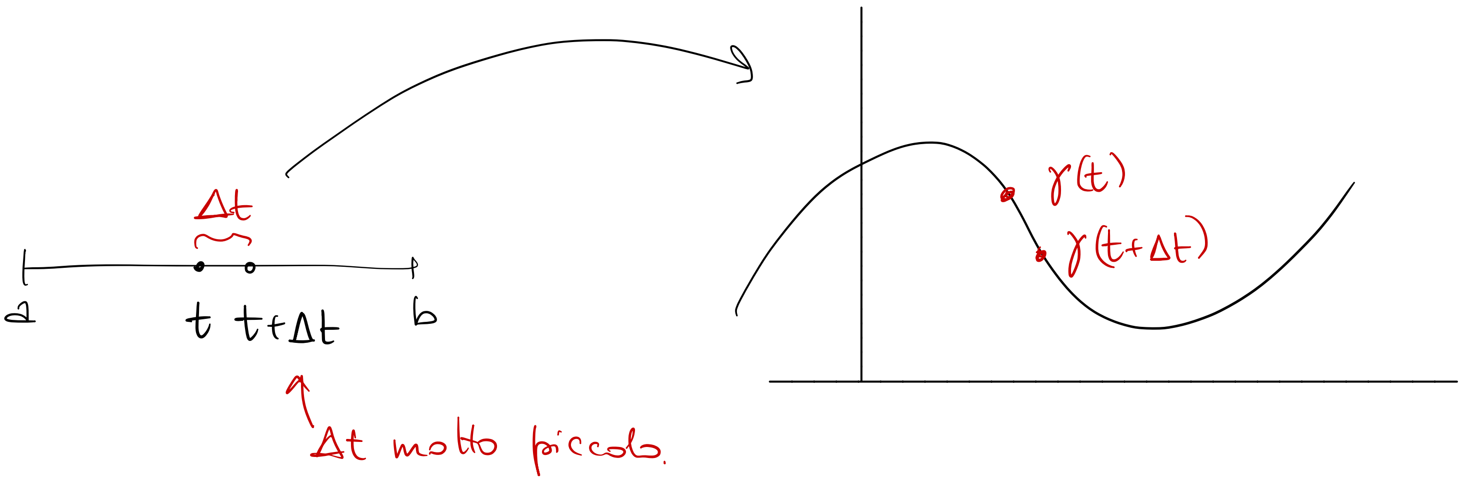
$$= \frac{1}{4} \quad \frac{b}{2}\sqrt{1+\frac{b^2}{4}} + \frac{1}{4}\, \mathrm{settsh}\left(\frac{b}{2}\right)$$

<u>Ellisse</u>

$$x(t) = a\cos t \qquad t \in [0, 2\pi].$$
$$y(t) = b \sin t$$

$$L(\gamma) = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\; dt \qquad \text{(integrale ellittico)}$$

Giustificazione per la formula di $L(\gamma)$.



$$\underline{\gamma}(t+\Delta t) = (x(t+\Delta t), y(t+\Delta t))$$

$$x(t+\Delta t) = x(t) + x'(t)\,\Delta t + o(\Delta t)$$

$$x(t+\Delta t) \underset{=}{\sim} x(t) + x'(t)\,\Delta t \qquad \text{se } |\Delta t| << 1$$

$$\underline{\gamma}(t+\Delta t)-\underline{\gamma}(t) = \left(x(t+\Delta t)-x(t),\ y(t+\Delta t)-y(t)\right) \cong$$

$$\cong \left(x'(t)\Delta t,\ y'(t)\Delta t\right)$$

$$\|\underline{\gamma}(t+\Delta t)-\underline{\gamma}(t)\| \cong \Delta t\sqrt{x'(t)^2+y'(t)^2}$$

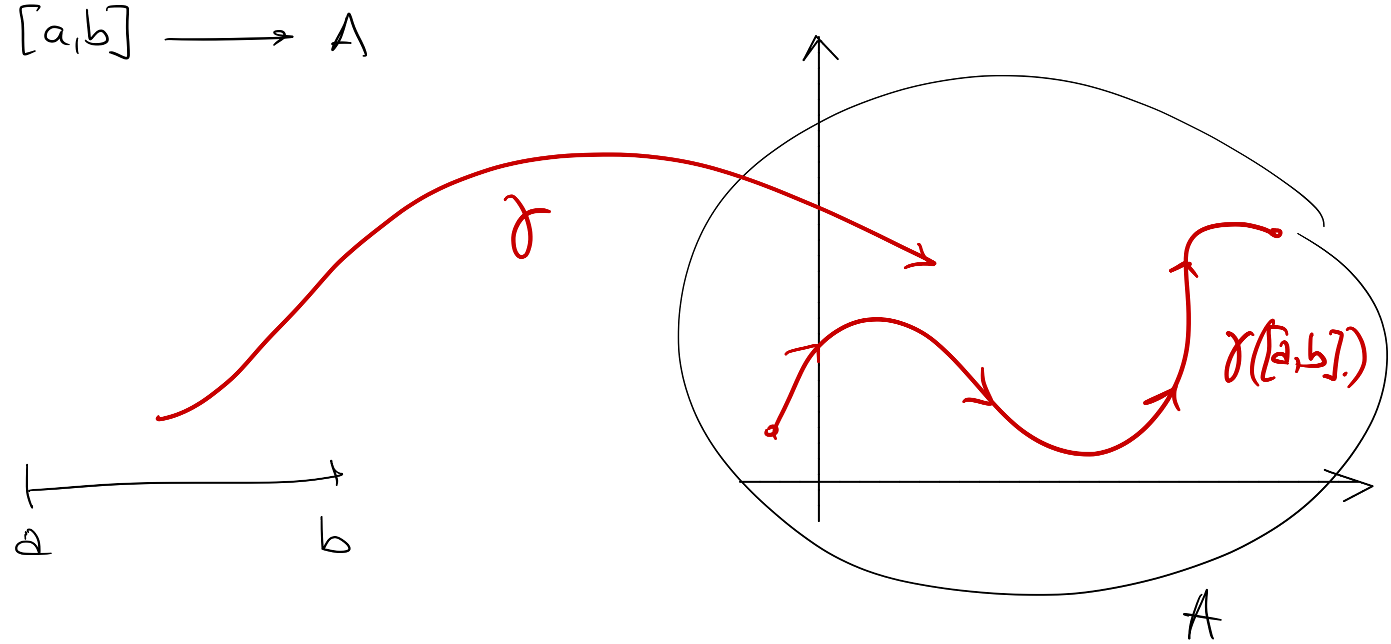
Facendo $\Delta t \downarrow 0$, e sommando, si ottiene

$$\int_a^b dt\sqrt{x'(t)^2+y'(t)^2}$$

Integrali curvilinei di funz. scalari (Int. curvilinei di 1ª specie)

$f(x,y): A \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ A aperto, f continua

$\underline{\gamma}: [a,b] \longrightarrow A$



Vogliamo definire

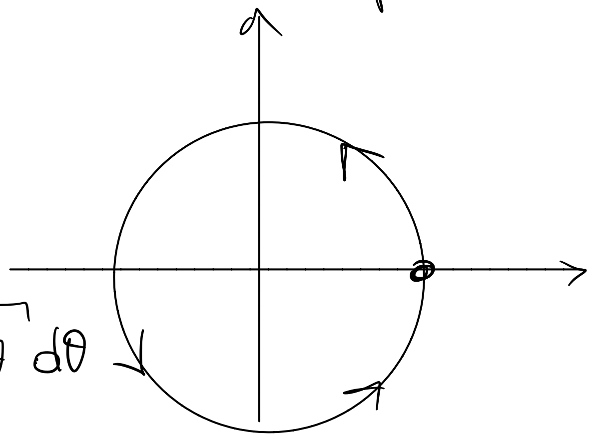
$$\int_\gamma f \, \underbrace{ds}_{} = \int_a^b f(\underline{\gamma}(t)) \, \|\underline{\gamma}'(t)\| \, dt$$

elementino di lunghezza della curva

$$ds = \|\gamma'(t)\| \, dt$$

Calcolare $\int_\gamma x^2y^2\,ds$ dove γ è la circonf. unitaria

(Non viene specificato il verso di percorrenza, perché gli integrali curvilinei di 1ª specie non dipendono dal verso di percorrenza)



$$\begin{cases} x(\theta) = \cos\theta \\ y(\theta) = \operatorname{sen}\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

$$\int_\gamma x^2y^2\,ds = \int_0^{2\pi} \cos^2\theta\,\operatorname{sen}^2\theta\,\sqrt{\operatorname{sen}^2\theta + \cos^2\theta}\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\,\operatorname{sen}^2\theta\,d\theta = \frac{1}{4}\int_0^{2\pi} \operatorname{sen}^2(2\theta)\,d\theta =$$

$$= \frac{1}{4}\int_0^{2\pi} \frac{1 - \cancel{\cos 4\theta}}{2}\,d\theta = \frac{1}{8}\cdot 2\pi = \frac{\pi}{4}.$$

Interpretazione "fisica" di $\int_\gamma f\,ds$.

Interpretiamo γ come un filo metallico ed f come la densità lineare in $\left(\frac{g}{cm}\right)$ (può variare da pto a pto.)

$\int_\gamma f\,ds$ si interpreta come la massa complessiva del filo.