Avvisi

- Oggi tutoraggio ore 17:30 → 19:00 Aula 16
- Oggi ricevim. studenti anticipato
  15:00 → 16:30.

Oscillatore smorzato. (Damped oscillator)

$x(t)$ posizione del pto all' istante t.

$$x''(t) = \underbrace{-K\,x(t)}_{\text{molla}} \underbrace{-c\,x'(t)}_{\text{termine di attrito}}$$

$$\begin{cases} x''(t) + 2a\,x'(t) + b\,x(t) = 0 \\ x(0) = 1 \\ x'(0) = 0 \end{cases} \qquad a, b > 0.$$

E.D.O. a coeff.ti costanti, omogenea

(EA) $\lambda^2 + 2a\lambda + b = 0$

$$\lambda = -a \pm \sqrt{a^2 - b}$$

1) $a^2 - b < 0$ (underdamped)

$$\lambda = -a \pm i\,\underbrace{\sqrt{b-a^2}}_{\beta}$$

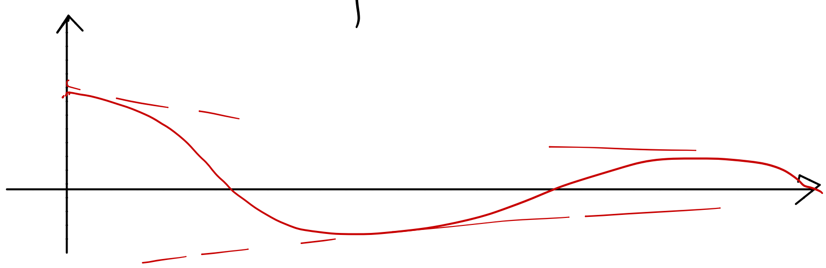
$$x(t) = e^{-at}\left(\cancel{c_1}^{\,1}\cos(\beta t) + c_2\,\text{sen}(\beta t)\right)$$

$$x'(t) = e^{-at}\left(-a c_1 \cos\beta t - a c_2\,\text{sen}\,\beta t - \beta c_1\,\text{sen}\,\beta t + \beta c_2 \cos\beta t\right).$$

$x(0) = 1 \qquad c_1 = 1$

$x'(0) = 0 \qquad -a + \beta c_2 = 0 \Rightarrow c_2 = \dfrac{a}{\beta}$

$$x(t) = e^{-at}\left(\cos(\beta t) + \frac{a}{\beta}\,\text{sen}\,\beta(t)\right)$$

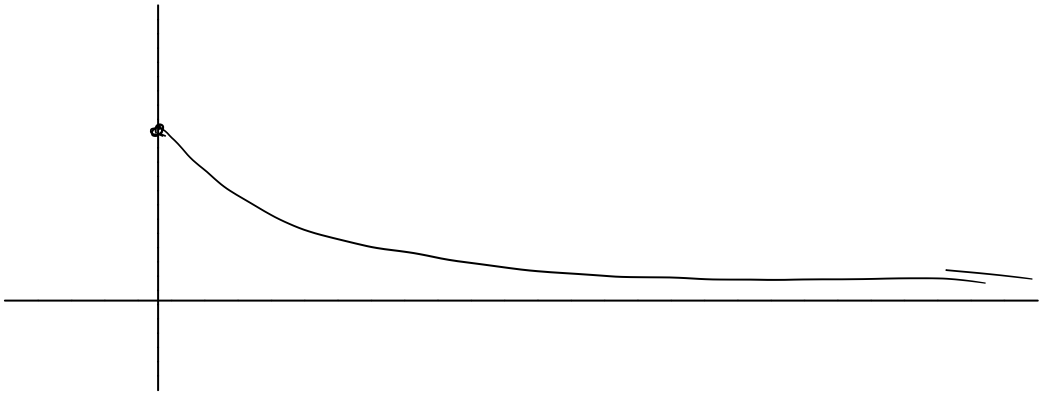


3) $a^2 - b > 0$ (overdamped)

$$\lambda_{1,2} = -a \pm \sqrt{a^2 - b} < 0 \qquad \lambda_1 < \lambda_2 < 0.$$

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \underset{t \to +\infty}{\sim} c_2 e^{\lambda_2 t}$$

Con le cond. iniziali si trovano c_1, c_2.

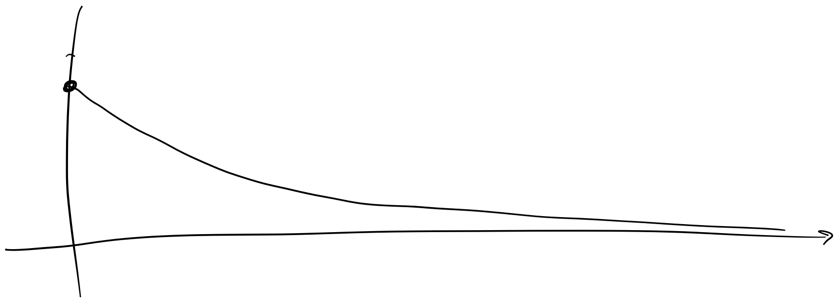


2) $a^2 - b = 0$ (critically damped)

$$\lambda_1 = \lambda_2 = -a.$$

$$x(t) = e^{-at}(c_1 + c_2 t) \underset{t \to +\infty}{\sim} c_2 e^{-at} t$$

(Si trovano c_1 e c_2).



Oscillatore forzato

$$\begin{cases} x''(t) + x(t) = \operatorname{sen}(\alpha t) \\ x(0) = x'(0) = 0 \end{cases} \qquad (E) \qquad \alpha > 0$$

$$x''(t) + x(t) = 0 \qquad (E_0)$$

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$x(t) = c_1 \cos t + c_2 \operatorname{sen} t$$

cerco

$$x_p(t) = A \cos \alpha t + B \operatorname{sen} \alpha t$$

$$x_p'(t) = -\alpha A \operatorname{sen} \alpha t + \alpha B \cos \alpha t$$

$$x_p''(t) = -\alpha^2 A \cos \alpha t - \alpha^2 B \operatorname{sen} \alpha t$$

$$-\alpha^2 A \underbrace{\boxed{\cos \alpha t}}_{c(t)} - \alpha^2 B \underbrace{\boxed{\operatorname{sen} \alpha t}}_{s(t)} + A\, c(t) + B\, s(t) = s(t)$$

$$\begin{cases} -\alpha^2 A + A = 0 \\ -\alpha^2 B + B = 1 \end{cases}$$

$\alpha \neq 1.$

$$\begin{cases} B = \dfrac{1}{1-\alpha^2} \\ A = 0 \end{cases}$$

$$x(t) = \cancel{c_1 \cos t} + c_2 \operatorname{sen} t + \frac{1}{1-\alpha^2} \operatorname{sen}(\alpha t)$$

Imponendo le cond. iniziali si trova $c_1 = 0$

$$x'(t) = c_2 \cos t + \frac{\alpha}{1-\alpha^2} \cos(\alpha t)$$

$$x'(0) = 0 \iff c_2 = -\frac{\alpha}{1-\alpha^2}$$

$\underline{\alpha = 1}$ caso di risonanza

$$x_p(t) = At\cos t + Bt\,\mathrm{sen}\,t$$

$$x_p'(t) = A\cos t - At\,\mathrm{sen}\,t + B\,\mathrm{sen}\,t + Bt\cos t$$

$$x_p''(t) = \underbrace{-A\,\mathrm{sen}\,t - A\,\mathrm{sen}\,t}\; - At\cos t + \underbrace{B\cos t + B\cos t}\; - Bt\,\mathrm{sen}\,t$$

$$x'' + x = \mathrm{sen}\,t$$

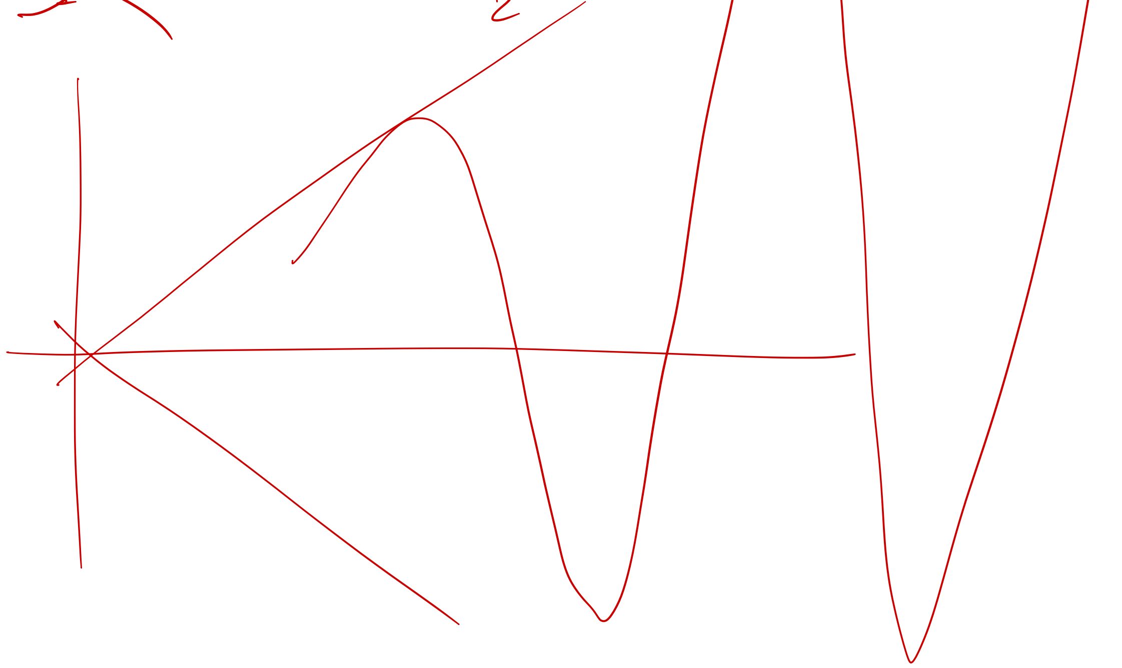
$$-2A\,\mathrm{sen}\,t - \cancel{At\cos t} + 2B\cos t - \cancel{Bt\,\mathrm{sen}\,t} + \cancel{At\cos t} + \cancel{Bt\,\mathrm{sen}\,t} = \mathrm{sen}\,t$$

$$A = -\frac{1}{2} \qquad B = 0$$

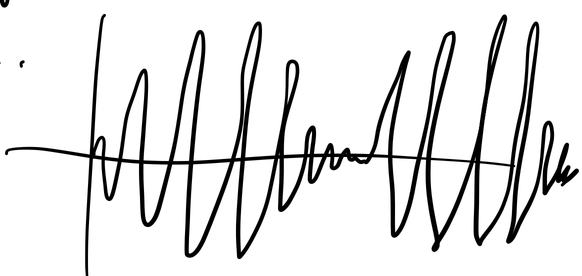
$$x_p(t) = -\frac{1}{2}t\cos t$$

Int. generale

$$x(t) = \cancel{c_1\cos t} + c_2\,\mathrm{sen}\,t - \frac{1}{2}t\cos t.$$



$\underline{\text{Esercizio:}}$ Plottare i grafici di $x(t)$ quando α è vicino a 1.



Riepilogo sulle eq^{ni} lineari a coeff^{ti} costanti:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(x) \qquad (E)$$

$$- - - - - - - - = 0 \qquad (E_0)$$

Si cercano n sol^{ni} lin. indip. di (E_0), nella forma $e^{\lambda t}$

$$\Longrightarrow \quad \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0 \qquad (EA)$$

eq^{ne} algebrica di grado n. $\Rightarrow$ ammette sempre n radici (event. complesse e contate con la loro molt.)

Per ogni λ_j radice semplice (molt. 1) $\in \mathbb{R}$ di (EA)

$\Longrightarrow$ una sol^{ne} $\quad y(x) = e^{\lambda_j x}$

Per ogni λ_j radice reale con molt. k

(cioè $(\lambda - \lambda_j)^k$ divide il polinomio in (EA))

$\Longrightarrow$ k sol^{ni} $\quad e^{\lambda_j x}, \; x e^{\lambda_j x}, \; x^2 e^{\lambda_j x}, \; \dots, \; x^{k-1} e^{\lambda_j x}$

Per ogni coppia $\lambda_j, \bar{\lambda}_j$ della forma

$$\lambda_j, \bar{\lambda}_j = \alpha \pm i\beta \quad \text{con } \alpha, \beta \text{ reali}$$

e queste radici sono semplici

cioè il polinomio $\left((\lambda - \alpha)^2 + \beta^2\right)$ divide 1 volta sola il polinomio in (EA)

$$e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$

$\Longrightarrow$ 2 sol^{ni} $\quad e^{\alpha x}\cos\beta x, \; e^{\alpha x}\operatorname{sen}\beta x$

Infine, se $\lambda_j, \bar{\lambda}_j = \alpha \pm i\beta$
sono radici complesse coniugate di molt k.
(cioè $((\lambda-\alpha)^2+\beta^2)^k$ divide il polinomio in (EA))

$$\Longrightarrow \quad e^{\alpha x}\cos\beta x, \quad x e^{\alpha x}\cos\beta x \quad \dots \quad x^{k-1}e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\operatorname{sen}\beta x, \quad x e^{\alpha x}\operatorname{sen}\beta x \quad \dots \quad x^{k-1}e^{\alpha x}\operatorname{sen}\beta x$$

<u>Esempio:</u>

$$y'''' + 2y'' + y = 0 \qquad (E_0)$$

$$\lambda^4 + 2\lambda^2 + 1 = 0 \qquad (EA)$$

$$(\lambda^2+1)^2 = 0$$

$\lambda = \pm i$ con molt. 2.

$\Longrightarrow$ Int. generale di (E_0)

$$y(x) = c_1\cos x + c_2\operatorname{sen} x + c_3 x\cos x + c_4 x\operatorname{sen} x$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(x) \qquad (E)$$

$$\text{— — — — — — —} = 0 \qquad (E_0)$$

Trovato l'int. generale di (E_0), devo cercare una sol^{ne} particolare $y_p(x)$ di (E).

Se $f(x) = P_k(x) e^{\alpha x}$ dove $P_k(x)$ è un polinomio di grado k.

$\Rightarrow$ cerco $y_p(x)$ della forma

$$y_p(x) = Q_k(x) e^{\alpha x}$$

dove $Q_k(x)$ è un polinomio (da trovare)

$$f(x) = 5x^2 e^{\alpha x} \Rightarrow y_p(x) = (Ax^2 + Bx + C) e^{\alpha x}$$

A, B, C da det. tramite l'eq.

A MENO CHE... α sia una radice di (EA) di molt. $h \geq 1$.

In tal caso cerchiamo $y_p(x)$ della forma

molteplicità della radice α.

$$y_p(x) = x^h Q_k(x) e^{\alpha x}$$

Se $\quad f(x) = P_k(x)\, e^{\alpha x} \cos(\beta x)$ oppure

$\qquad\qquad\qquad$ " $\quad$ " $\quad \operatorname{sen}(\beta x)$

cerco $y_p(x)$ nella forma

$$y_p(x) = Q_k(x)\, e^{\alpha x} \cos(\beta x) + R_k(x)\, e^{\alpha x} \operatorname{sen}(\beta x)$$

dove $Q_k(x)$ e $R_k(x)$ sono polinomi (da determinare)

A MENO CHE .. $\lambda = \alpha \pm i\beta$ sia radice di molt h di EA

In tal caso si cerca

$$y_p(x) = \left(Q_k(x)\, e^{\alpha x} \cos(\beta x) + R_k(x)\, e^{\alpha x} \operatorname{sen}(\beta x) \right) x^h$$

Al variare di $\alpha \in \mathbb{R}$, calcolare l'int. generale di:

$$y''' - 3y'' + 2y' = x e^{\alpha x} \qquad (E)$$

$$y''' - 3y'' + 2y' = 0 \qquad (E_0)$$

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0 \qquad (EA)$$

$$\lambda(\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda(\lambda-1)(\lambda-2) = 0$$

Sol^{ni} di (EA) $\quad \lambda = 0, \ \lambda = 1, \ \lambda = 2$

Int. generale di (E_0)

$$z(x) = c_1 + c_2 e^{x} + c_3 e^{2x}$$

Cerco una sol^{ne} della non omogenea

$$y_p(x) = (Ax+B)e^{\alpha x}$$ purché $\alpha \neq 0, 1, 2$.

$$y_p'(x) = (A + \alpha A x + \alpha B)e^{\alpha x}$$

$$y_p''(x) = (\alpha A + \alpha A + \alpha^2 A x + \alpha^2 B)e^{\alpha x}$$

$$y_p'''(x) = (\alpha^2 A + 2\alpha^2 A + \alpha^3 A x + \alpha^3 B)e^{\alpha x}$$

$$\underbrace{3\alpha^2 A} + \alpha^3 A x + \underbrace{\alpha^3 B} - \underbrace{6\alpha A} - 3\alpha^2 A x \underbrace{- 3\alpha^2 B} + \underbrace{2A} + 2\alpha A x + \underbrace{2\alpha B} = x$$

$$\begin{cases} 3\alpha^2 A + \alpha^3 B - 6\alpha A - 3\alpha^2 B + 2A + 2\alpha B = 0 \\ \alpha^3 A - 3\alpha^2 A + 2\alpha A = 1 \end{cases}$$

$$A = \frac{1}{\alpha^3 - 3\alpha^2 + 2\alpha} \qquad B(\alpha^3 - 3\alpha^2 + 2\alpha) = -3\alpha^2 A + 6\alpha A - 2A$$

$$B = \dots$$

A e B sono stati determinati

$$y(x) = c_1 + c_2 e^{x} + c_3 e^{2x} + (Ax+B)e^{\alpha x} \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

Inf. cercato.

$\alpha = 0, \quad \alpha = 1, \quad \alpha = 2.$

$$\Rightarrow \quad y_p(x) = (Ax^2 + Bx)\, e^{\alpha x}$$

Per es se $\alpha = 0$

$$y_p(x) = Ax^2 + Bx = \frac{x^2+3x}{4}$$

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

$$y_p'''(x) = 0$$

$$0 - 6A + 4Ax + 2B = x$$

$$A = \frac{1}{4}$$

$$B = 3A = \frac{3}{4}$$

$$y(x) = c_1 + c_2 e^{x} + c_3 e^{2x} + \frac{x^2+3x}{4}$$

$$y''' - 3y'' + 2y' = x e^{2x} \cos(3x)$$

(E_0) come prima

$$y_p(x) = (Ax+B)e^{2x}\cos(3x) + (Cx+D)e^{2x}\operatorname{sen}(3x)$$

$\Longrightarrow$ si trovano A, B, C, D.

E se il "termine noto" $f(x)$ non è di questo tipo?

$$y'' + y = \frac{1}{\operatorname{sen} x} \qquad (E)$$

$$y'' + y = 0 \qquad (E_0)$$

Int. generale di (E_0): $y(x) = c_1 \cos x + c_2 \operatorname{sen} x$

Cerco una sol^{ne} particolare della forma

$$y_p(x) = c_1(x)\cos x + c_2(x) \operatorname{sen} x$$

metodo di variaz. delle costanti

$$y_p'(x) = \underbrace{c_1' \cos x + c_2' \operatorname{sen} x}_{\overset{\text{impongo}}{=0}} - c_1 \operatorname{sen} x + c_2 \cos x$$

$$y_p''(x) = -c_1' \operatorname{sen} x + c_2' \cos x - c_1 \cos x - c_2 \operatorname{sen} x$$

$$-c_1' \operatorname{sen} x + c_2' \cos x - \cancel{c_1 \cos x} - \cancel{c_2 \operatorname{sen} x} + \cancel{c_1 \cos x} + \cancel{c_2 \operatorname{sen} x} = \frac{1}{\operatorname{sen} x}$$

$$\begin{cases} c_1' \cos x + c_2' \operatorname{sen} x \equiv 0 & \cdot \operatorname{sen} x \\ -c_1' \operatorname{sen} x + c_2' \cos x = \dfrac{1}{\operatorname{sen} x} & \cdot \cos x \end{cases}$$

$$c_2' = \frac{\cos x}{\operatorname{sen} x} \Longrightarrow c_2(x) = + \ln|\operatorname{sen} x|$$

$$c_1' = -c_2' \frac{\operatorname{sen} x}{\cos x} = -1 \Longrightarrow c_1(x) = -x$$

$$y_p(x) = -x\cos x + \ln|\operatorname{sen} x| \cdot \operatorname{sen} x$$