CORREZIONE DELL'ESONERO DEL 20.1.2016

$\boxed{1}$ $E = \{(x,y,z) \in \mathbb{R}^3 : 3x^2 + 5y^2 \leq z \leq 1 + 2x^2 + 4y^2\}$

vol E?

Dove vanno x e y? nell'insieme D t.c.

$$3x^2 + 5y^2 \leq 1 + 2x^2 + 4y^2$$

$$D = \{x^2 + y^2 \leq 1\}.$$

$$E = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, \quad 3x^2 + 5y^2 \leq z \leq 1 + 2x^2 + 4y^2\}$$

$$\text{vol } E = \iint_D (1 + 2x^2 + 4y^2 - 3x^2 - 5y^2)\, dx\,dy = \iint_D (1 - x^2 - y^2)\, dx\,dy =$$

$$= \int_0^{2\pi} d\theta \int_0^1 d\rho\, (1-\rho^2)\rho = \frac{2\pi}{2} \left. \frac{(1-\rho^2)^2}{2} \right|_{\rho=1}^{\rho=0} =$$

$$= \frac{\pi}{2}$$

N.B. E non è di rotazione!!

$\boxed{2}$ Trovare $\alpha \in \mathbb{R}$ in modo che il campo

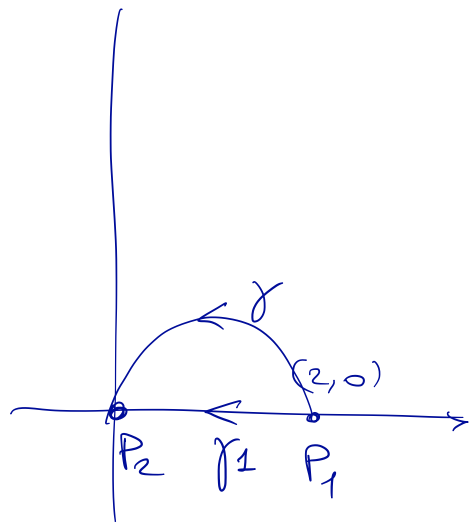
$$\underline{F}(x,y) = \left(-\frac{\alpha}{1+y^2} + 4x \,,\; \frac{2xy}{(1+y^2)^2} \right)$$

sia conservativo nel suo dominio. Per tali α, calcolare il lavoro lungo l'arco $\gamma(x(t),y(t)) = (\cos(t)+1, \text{sen } t)$ $t \in [0,\pi]$

Dominio $\mathbb{R}^2$ $\Rightarrow$ [conservativo $\Leftrightarrow$ irrotazionale]

$$\frac{\partial}{\partial y}\left(-\frac{\alpha}{1+y^2} + 4x\right) = \frac{\partial}{\partial x}\frac{2xy}{(1+y^2)^2}$$

$$\frac{\alpha 2y}{(1+y^2)^2} \overset{?}{=} \frac{2y}{(1+y^2)^2} \qquad \text{ok sse } \alpha = 1.$$



γ ha per estremi $(2,0)$ e $(0,0)$

1) γ_1^- $\begin{cases} x = t \\ y = 0 \end{cases}$ $t \in [0,2]$.

Campo conservativo $\Rightarrow$ $\int_\gamma \underline{F}\cdot\underline{T}\, ds = \int_{\gamma_1} \underline{F}\cdot\underline{T}\, ds =$

$$= -\int_{\gamma_1^-} \underline{F}\cdot\underline{T}\, ds = -\int_0^2 dx\,(-1+4x) = \int_0^2 dx\,(1-4x) = 2 - \frac{4}{2}\cdot 4 = 2-8 = -6$$

2) Calcolo il potenziale

$$V(x,y) = \int dx \left(-\frac{1}{1+y^2} + 4x\right) = -\frac{x}{1+y^2} + 2x^2 + g(y)$$

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$$V_y(x,y) = \cancel{\frac{2xy}{(1+y^2)^2}}$$

$$\parallel$$

$$\cancel{\frac{x\,2y}{(1+y^2)^2}} + g'(y) \Rightarrow g'(y) \equiv 0 \Rightarrow g(y) = c.$$

$$V(x,y) = -\frac{x}{1+y^2} + 2x^2 + c$$

$$\int_\gamma \underline{F} \cdot \underline{T}\, ds = V(0,0) - V(2,0) =$$

$$= 2 - 2 \cdot 4 = -6$$

$\boxed{3}$ $\begin{cases} u''(t) = -u'(t)\,u(t) \\ u(0) = 0 \\ u'(0) = 1/2 \end{cases}$ (P)

1) Provare che $\underbrace{2u'(t) + u^2(t)}_{\varphi(t)} = C_1$ (e calcolare C_1)

$$\varphi'(t) = 2u''(t) + 2u(t)\,u'(t) = 0 \quad \text{per l'eq}^{ne}.$$

$$\Rightarrow \varphi(t) \equiv C = \varphi(0) = 2u'(0) + (u(0))^2 = 1$$

$$\Rightarrow u \text{ risolve } 2u' + u^2 = 1.$$

Risolvendo l'eq^{ne} in (P) come eq^{ne} autonoma

$$u'(t) = v(u) \qquad u''(t) = \dot v(u)\, v(u) \Longrightarrow$$

$$\dot v\, v = -vu \qquad v(\dot v + u) = 0$$

$$v = 0 \Longrightarrow u' = 0.$$

$$\dot v = -u \Rightarrow \dot v(u) = -u \Longrightarrow \underset{u'}{v(u)} = -\frac{u^2}{2} + c$$

$$2u' + u^2 = 2c$$

ii) ricavare un pb. di Cauchy del 1° ordine equivalente a (P)

$\begin{cases} u''(t) = -u'(t)\,u(t) \\ u(0) = 0 \\ u'(0) = 1/2 \end{cases}$ (P) $\qquad \begin{cases} 2u' + u^2 = 1 \\ u(0) = 0 \end{cases}$ $(\tilde P)$

$$\begin{cases} u''(t) = -u'(t)\,u(t) \\ u(0) = 0 \\ u'(0) = 1/2 \end{cases} \qquad (P)$$

$$\begin{cases} 2u' + u^2 = 1 \\ u(0) = 0 \end{cases} \qquad (\tilde{P})$$

u sol^{ne} di (P) $\Rightarrow$ u sol^{ne} di $(\tilde{P})$

u sol^{ne} di $(\tilde{P})$ $\Rightarrow$ derivando si ottiene l'eq^{ne} di (P)

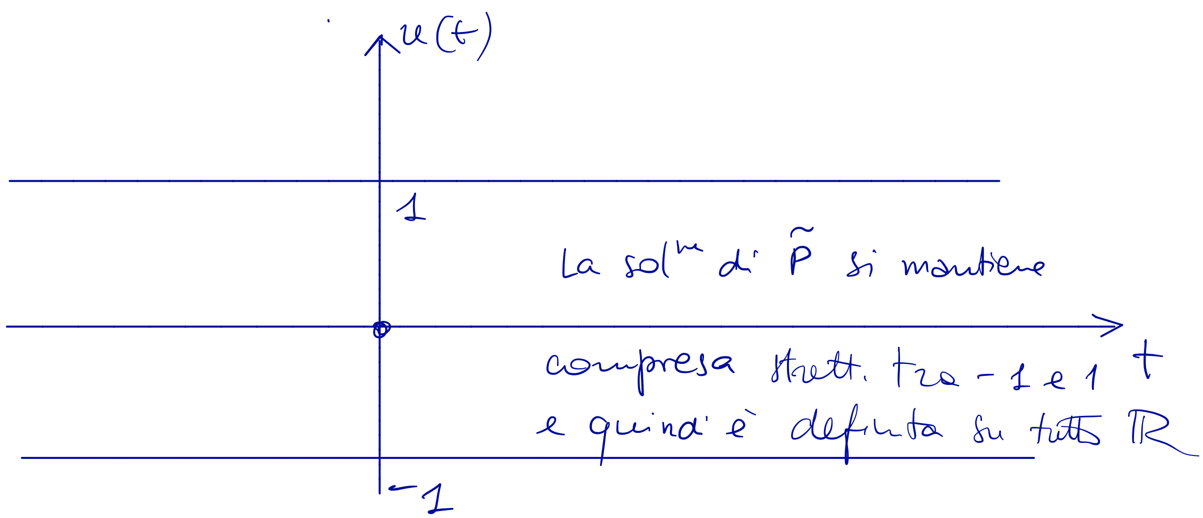
$$u(0) = 0 \quad , \quad 2u'(0) + \underbrace{(u(0))^2}_{=0} = 1 \Rightarrow u'(0) = \frac{1}{2}.$$

$\Rightarrow$ u sol^{ne} di (P)

Il pb. $(\tilde{P})$ ammette (localmente) una unica soluzione perché l'eq^{ne} $u' = \frac{1-u^2}{2}$ è della forma $u' = f(t,u)$ con f di classe C^1 che è una condizione sufficiente per il teorema di Cauchy in piccolo

(iii) verificare che la sol^{ne} di $(\tilde{P})$ è definita su tutto $\mathbb{R}$.

$u' = \frac{1-u^2}{2}$ non verifica le condizioni per l'esistenza globale ma esistono le due sol^{ni} costanti $u(t) \equiv 1$ e $u(t) \equiv -1$



La sol^{ne} di $\tilde{P}$ si mantiene compresa strett. tra -1 e 1 e quindi è definita su tutto $\mathbb{R}$

(iv) Risolviamo $\tilde{P}$.

$$\frac{2u'}{1-u^2} = 1$$

$$\int \frac{2u'\,dt}{1-u^2} = \int 1\,dt = t + c$$

$$\| $$

$$\int \frac{2}{1-u^2}\,du = \int\left(\frac{A}{1-u} + \frac{B}{1+u}\right) du = \int\left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$$

$$\|$$

$$\ln\left|\frac{1+u}{1-u}\right| = \ln\frac{1+u}{1-u}$$

$$\frac{2}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$2 = A(1+u) + B(1-u)$$

$$\begin{cases} A - B = 0 \\ 2 = A + B \end{cases} \Rightarrow A = B = 1$$

$$\ln\frac{1+u}{1-u} = t + c$$ $\Rightarrow$ impongo cond^{ne} iniziale $u(0)=0$

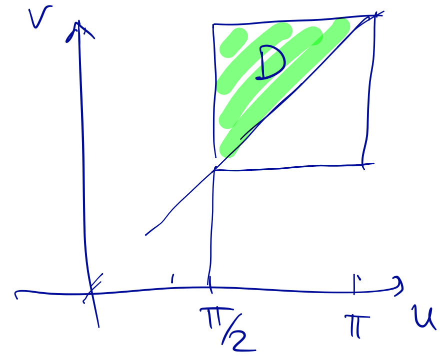
$$\ln 1 = 0 + c \Rightarrow c = 0$$

$$\frac{1+u}{1-u} = e^t \Rightarrow 1+u = e^t(1-u) \Rightarrow$$

$$\Rightarrow u(e^t+1) = e^t - 1 \Rightarrow u(t) = \frac{e^t-1}{e^t+1} \quad \blacksquare$$

$\boxed{4a}$ Considerare la superficie Σ di eqⁿⁱ

$$\psi \begin{cases} x(u,v) = \operatorname{sen}(uv) \\ y(u,v) = \cos(uv) \\ z(u,v) = u \end{cases} \qquad (u,v) \in D = \left\{ \frac{\pi}{2} \le u \le v \le \pi \right\}.$$



i) dim. che Σ è una superf. regolare.

1) $\psi \in C^1$.

dim. iniettività

$(u_1,v_1), (u_2,v_2) \in D$

$$\left.\begin{aligned} \operatorname{sen}(u_1 v_1) &= \operatorname{sen}(u_2 v_2) \\ \cos(u_1 v_1) &= \cos(u_2 v_2) \\ u_1 &= u_2 \end{aligned}\right| \Longrightarrow \begin{aligned} &u_1 = u_2 \quad \text{ovvio.} \\ &v_1 = v_2 \end{aligned}$$

$$u_1 v_1 = u_1 v_2 + 2k\pi \Rightarrow u_1(v_1 - v_2) = 2k\pi \Rightarrow k = 0$$

$$\Rightarrow v_1 = v_2.$$

$$\psi_u \wedge \psi_v \neq 0$$

$$\psi_u = \left(v\cos(uv), -v\operatorname{sen}(uv), 1\right)$$

$$\psi_v = \left(u\cos(uv), -u\operatorname{sen}(uv), 0\right)$$

$$E = \|\psi_u\|^2 = v^2 + 1$$

$$G = \|\psi_v\|^2 = u^2$$

$$F = uv\cos^2(uv) + uv\operatorname{sen}^2 uv = uv$$

$$EG - F^2 = (v^2+1)u^2 - u^2v^2 = u^2$$

$$\|\varphi_u \wedge \varphi_v\| = \sqrt{EG - F^2} = \sqrt{u^2} = |u| = u \neq 0$$

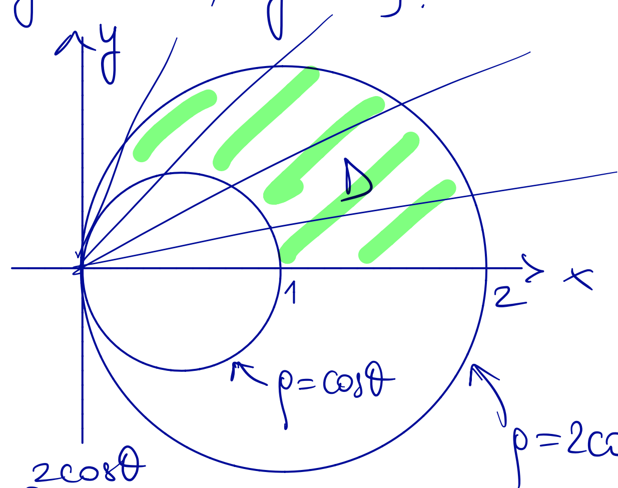
$$\text{Area di } \Sigma = \iint_D u \, du\, dv = \int_{\pi/2}^{\pi} du\, u \int_u^{\pi} dv = \int_{\pi/2}^{\pi} u(\pi - u)\, du =$$

$$= \int_{\pi/2}^{\pi} (\pi u - u^2)\, du = \ldots$$

$\boxed{4b}$ Flusso di $F(x,y) = (x^2 - y, xy)$ uscente da ∂D

$$D = \{(x,y) : \quad x \le x^2 + y^2 \le 2x, \; y \ge 0\}.$$

$$\operatorname{div} F = 2x + x = 3x$$



$$\int_{\partial D} F \cdot N_e \, ds = \iint_D \operatorname{div} F \, dx\, dy =$$

$$= 3 \iint_D x \, dx\, dy = 3 \int_0^{\pi/2} d\theta \int_{\cos\theta}^{2\cos\theta} d\rho\, \rho^2 \cos\theta = (*)$$

coord. polari

$$x^2 + y^2 = x \iff \rho^2 = \rho\cos\theta \iff \rho = \cos\theta$$

$$x^2 + y^2 = 2x \iff \rho = 2\cos\theta$$

$$(*) = \frac{3}{3} \int_0^{\pi/2} d\theta \cos\theta \; \rho^3 \Big|_{\rho=\cos\theta}^{\rho = 2\cos\theta} = \int_0^{\pi/2} d\theta \cos\theta \left(8\cos^3\theta - \cos^3\theta\right) =$$

$$= \int_0^{\pi/2} 7\cos^4\theta \, d\theta = 7 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta =$$

$$= 7 \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{7}{4} \int_0^{\pi/2} \left(1 + \cos^2 2\theta + \cancel{2\cos 2\theta}\right) d\theta =$$

$$= \frac{7}{4} \int_0^{\pi/2} \left(1 + \frac{1+\cancel{\cos 4\theta}}{2}\right) d\theta = \frac{7}{4} \, \frac{3}{2} \cdot \frac{\pi}{2}$$

$$\int \cos^4\theta \, d\theta = \int \cos\theta \, \cos^3\theta \, d\theta = \operatorname{sen}\theta \cos^3\theta + 3 \int \cos^2\theta \, \underbrace{\operatorname{sen}^2\theta}_{= 1-\cos^2\theta} \, d\theta =$$

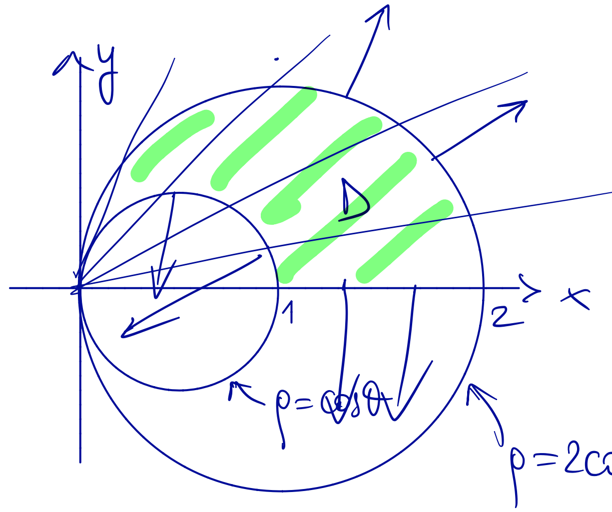
$$= \quad '' \quad + 3 \int \cos^2\theta - 3 \int \cos^4\theta \, d\theta$$

$$\cancel{4}\int \cos^4\theta \, d\theta = \frac{1}{4} \operatorname{sen}\theta \cos^3\theta + \frac{3}{\cancel{2}\,4} \int \cos^2\theta \, d\theta$$

si può riintegrare per parti.

Altro modo:

$$3\iint_D x \, dx\,dy = (*)$$



Osservo che $D = C_2 \setminus C_1$ dove $C_2 =$ semicerchio di centro $(1,0)$ e raggio 1

$C_1 =$ semicerchio di centro $(\frac{1}{2}, 0)$ e raggio $\frac{1}{2}$.

$$(*) = 3\left(\iint_{C_2} x\,dx\,dy - \iint_{C_1} x\,dx\,dy\right)$$

$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases} \quad 0 \le \theta \le \pi, \quad 0 \le \rho \le 1$$

$$\begin{cases} x = \frac{1}{2} + \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases} \quad 0 \le \theta \le \pi, \quad 0 \le \rho \le \frac{1}{2}.$$

Ma anche senza fare integrali:

$$\iint_{C_2} x\,dx\,dy = \text{Area } C_2 \underbrace{\left(\frac{1}{\text{area } C_2}\iint_{C_2} x\,dx\,dy\right)}_{=1} = \text{area } C_2 = \frac{\pi}{2}$$

$$\iint_{C_1} x\,dx\,dy = \text{Area } C_1 \underbrace{\left(\frac{1}{\text{area } C_1}\iint_{C_2} x\,dx\,dy\right)}_{=1/2} = \frac{1}{2}\cdot\frac{\pi}{4}$$

$$(*) = 3\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \frac{9}{8}\pi.$$

ESERCIZIO Dato il campo $F(x,y,z) = (y^2 e^z, y^3, 3x^2 z)$

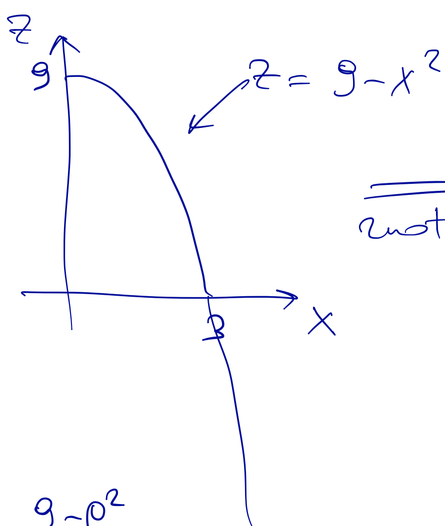
calcolare il flusso di F uscente da S, frontiera del dominio delimitato dal paraboloide $z = 9 - x^2 - y^2$, dal piano xz e dal piano xy e contenuto nel semispazio $y \geq 0$.

Teorema della divergenza

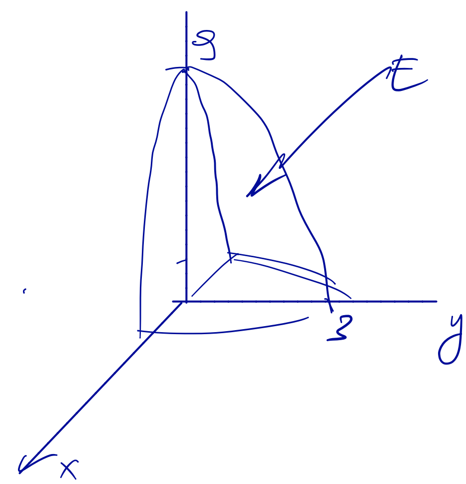
$$\operatorname{div} F = 0 + 3y^2 + 3x^2 = 3(x^2+y^2)$$

$$\iint_S F \cdot \nu \, d\sigma = \iiint_E 3(x^2+y^2)\, dx\, dy\, dz = (*)$$

$z = 9 - x^2 - y^2$



ruotando



$$(*) = 3\int_0^{\pi} d\theta \int_0^3 d\rho \int_0^{9-\rho^2} dz \; \rho^2 \cdot \rho = 3 \cdot \pi \int_0^3 d\rho \; \rho^3 (9-\rho^2) =$$

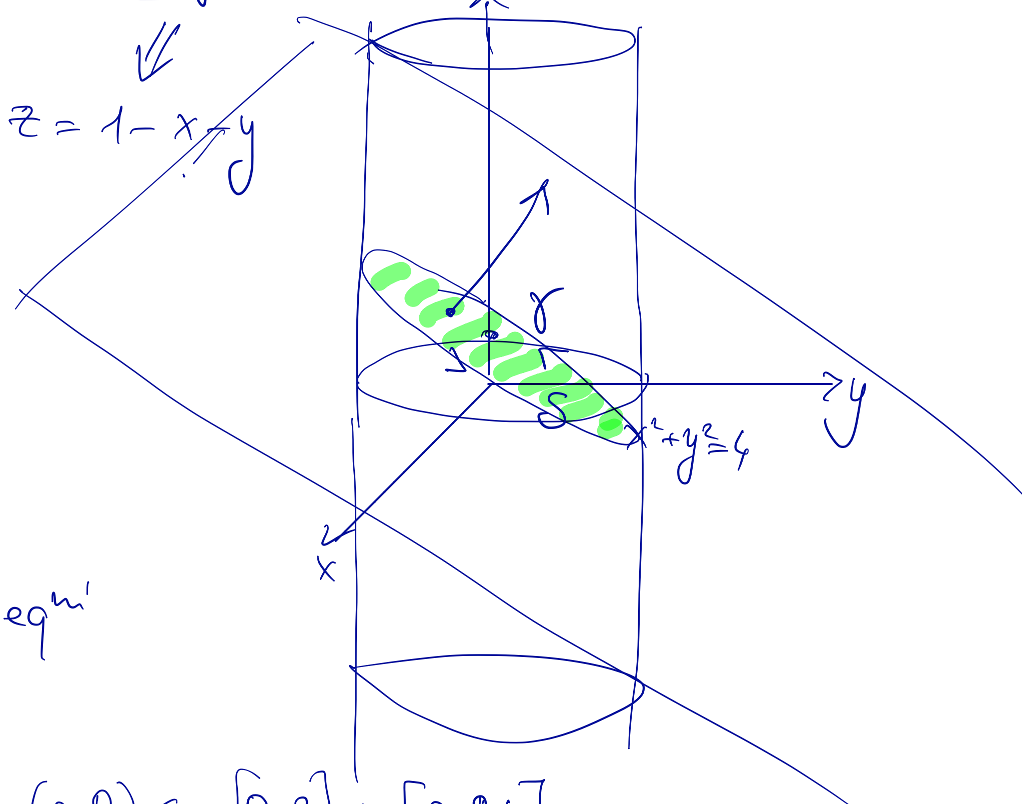
$$= 3\pi \int_0^3 (9\rho^3 - \rho^5)\, d\rho = \dots$$

ESERCIZIO $\quad \underline{F}(x,y,z) = (2y^3, -x^3, 2z^3)$.

Calcolare il lavoro di F lungo γ intersezione del cilindro $x^2+y^2=4$ con il piano $x+y+z=1$, orientata (vista dall'alto) in senso antiorario.

$\Downarrow$

$z = 1-x-y$



1° modo: Stokes

γ è il bordo di S di eq.ni

$$\psi \begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\theta \\ z = 1-\rho\cos\theta-\rho\,\mathrm{sen}\theta \end{cases} \qquad (\rho,\theta)\in[0,2]\times[0,2\pi]$$

$$F = (2y^3, -x^3, 2z^3)$$

$$\mathrm{rot}\,F = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2y^3 & -x^3 & 2z^3 \end{pmatrix} = (0, 0, -3x^2-6y^2)$$

$$\psi_\rho = (\cos\theta, \mathrm{sen}\theta, -\cos\theta-\mathrm{sen}\theta)$$

$$\psi_\theta = (-\rho\,\mathrm{sen}\theta, \rho\cos\theta, \rho\,\mathrm{sen}\theta-\rho\cos\theta)$$

$c(\rho,\theta) = \rho \qquad$ orientazione corretta perché $c>0$

$$\int_\gamma F\cdot T\,ds = \iint_S \operatorname{rot} F\cdot N\,d\sigma =$$

$$\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\operatorname{sen}^2\theta\,d\theta = \pi$$

$$= \int_0^{2\pi} d\theta \int_0^2 d\rho\,\left(-3\rho^2\cos^2\theta - 6\rho^2\operatorname{sen}^2\theta\right)\rho =$$

$$= -3\pi \left(\frac{\rho^4}{4} + \frac{2\rho^4}{4}\right)\Bigg|_0^2 = -3\pi\,\frac{3}{4}\cdot 16 = -36\pi.$$

Calcolo diretto

$$\gamma \quad \begin{cases} x = 2\cos\theta \\ y = 2\operatorname{sen}\theta \\ z = 1 - x - y = 1 - 2\cos\theta - 2\operatorname{sen}\theta \end{cases}$$

orientaz. corretta

$$\int_\gamma F\cdot T\,ds = \int_0^{2\pi}\Big[2\cdot(2\operatorname{sen}\theta)^3(-2\operatorname{sen}\theta) - (2\cos\theta)^3\,2\cos\theta +$$

$$+ \underbrace{2\,(\underbrace{1-2\cos\theta-2\operatorname{sen}\theta}_{z})^3\,(\underbrace{2\operatorname{sen}\theta-2\cos\theta}_{dz})}_{}\Big]\,d\theta$$

$$\parallel$$

$$2\,\left(1-2\cos\theta-2\operatorname{sen}\theta\right)^4\Big|_0^{2\pi} = 0$$